Debt Collateralization, Capital Structure, and Maximal Leverage

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Abstract

We study the effects of allowing risky debt to be used as collateral in a general equilibrium model with heterogeneous agents and collateralized financial contracts. With debt collateralization, investors switch to using exclusively high-leverage contracts for every investment they choose (issuing risky debt when possible). High-leverage positions maximize the ability of contracts to serve as collateral, expanding the set of state-contingencies created from collateralized debt. We provide conditions under which debt collateralization will increase the price of the underlying asset. Our results also apply to variations in capital structure since many capital structures implicitly provide the ability to use debt contracts as collateral.

Keywords: Leverage, incomplete markets, asset prices, default, securitized markets, asset-backed securities, collateralized debt obligations.

JEL classification: D52, D53, G11, G12.

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1 Introduction

An essential feature of many securitized markets is the explicit or implicit ability to use debt contracts as collateral to issue new financial promises. In using debt as collateral, risky assets can be tranched into securities with state contingencies quite different from the underlying asset or from simple debt and equity. Such features of securitized markets significantly contributed to the growth of the market for leveraged buyouts (Shivdasani and Wang, 2011) and subprime mortgages (via Asset-backed securities (ABS) and collateralized debt obligations (CDOs)). We argue that one reason for this expansion is that these securitized markets, by using debt as collateral to issue other promises, vastly increased the set of state-contingent payoffs available to trade. These innovations allowed investors, explicitly or implicitly, to choose leverage decisions that would maximize the ability for assets to serve as collateral for multiple levels of promises. We show that allowing debt to be used as collateral endogenously increases leverage in the economy as investors switch to issuing exclusively high-leverage risky contracts.

We use a general equilibrium model featuring heterogeneous agents and collateralized financial contracts following Geanakoplos (1997, 2003). Our analysis considers the interaction of two key frictions. First, we suppose that collateral is the only means of enforcing promises, with lenders seizing collateral that has been agreed upon in advance by contract. Second, we suppose that investors are limited to making non-contingent promises, so markets are incomplete. The interaction of the two frictions is crucial for studying the use of debt as collateral: with sufficiently rich state-contingent contracts, there is no room to use contracts as collateral to issue other promises (i.e., collateralization is redundant). We consider a model with multiple states of uncertainty so that in an economy with debt contracts, agents trade risky and risk-free debt in equilibrium. Following Geerolf (2015), we allow agents to use debt contracts as collateral to back new financial contracts, a process we call *debt collateralization*. In equilibrium agents use *risky* debt as collateral to issue new promises, which changes the state-contingent properties of risky debt.

While it is well understood that default can create state-contingent securities when incomplete markets restrict contracts to non-contingent promises (Zame, 1993), debt collateralization does not *merely* mechanically expand the set of contingencies via default. Instead, in equilibrium investors make decisions to isolate only a subset of contingent payoffs rather than capturing the full set of contingent payoffs. We show that with debt collateralization investors switch to using only the highest-leverage promises available for the assets or contracts in which they invest. Using maximal leverage creates new securities that can be further collateralized (i.e., leveraged) by "downstream investors" to the maximal degree; investing otherwise creates securities with fewer opportunities for collateralization and also fewer opportunities to create state-contingencies. Thus, only those state-contingent payoffs that maximize further collateralization "downstream" occur in equilibrium, and payoffs created by issuing risk-free promises on "upstream" assets do not occur. Allowing debt to back debt (to back debt, *ad infinitum*) increases collateral values, increasing leverage in each contract; each "level of debt collateralization" reinforces these effects. With complete collateralization, equilibrium features a "pyramiding arrangement" of investors lending to downstream investors by issuing promises which are used as collateral to issue further promises. Nonetheless, debt collateralization does not complete markets because the set of contingencies remains limited (i.e., does not recover Arrow-Debreu securities) and the set of fundamental assets that can be used to issue contracts remains limited.

Since with debt collateralization investors choose positions that create the greatest set of state-contingent securities backed by collateral, even the original risky assets, which can back the same contracts as before, now have enhanced collateral value. In addition, we show that equilibrium with debt collateralization also corresponds to equilibrium with contingent claims defined by senior-subordinated capital structures. Our results suggest that one motivating factor for senior-subordinated capital structures is to provide a way to stretch scarce collateral.

We show that debt collateralization has important implications for risk premia, debt prices, and asset prices. First, increases in economy-wide leverage on the original risky asset can be driven by financial innovations in debt collateralization, and not only by changes in fundamental risk or beliefs (Fostel and Geanakoplos, 2012b; Simsek, 2013). Second, we show that the prices of risky debt always increase (risk premia decrease) because debt contracts now have collateral value. Third, under certain conditions debt collateralization increases the price of the underlying risky asset.

1.1 Related Literature

Our paper follows the model of collateral equilibrium developed in Geanakoplos (1997, 2003) and Geanakoplos and Zame (2014), and is closely related to the literature on collateral and financial innovation (Fostel and Geanakoplos, 2008, 2012a,b, 2015, 2016). This literature uses binomial models to explain asset prices and investment, and defines the financial environment as the set of assets that can serve as collateral and the set of promises that can be made with existing collateral. Debt collateralization, or "pyramiding" to use the term introduced by Geanakoplos (1997), expands the set of assets that can be used as collateral, fitting directly into this definition of financial environment. Our main contribution is characterizing the equilibrium pyramiding structure, together with asset pricing implications, in the model of Fostel and Geanakoplos (2012b).

Geanakoplos and Zame (2013, 2014) discuss how using promises to back further promises (what they call pyramiding and what we are calling debt collateralization given our restriction to debt) can potentially allow the market to achieve efficient allocations, though the central finding of Geanakoplos and Zame (2013) is that even with pyramiding, equilibrium is robustly inefficient. The central result of our analysis is that, when investors are restricted to debt contracts, the set of state-contingent payoffs that arise in equilibrium are those created when investors use maximum leverage for their investments. Thus, not all possible state-contingencies are traded, but only those that correspond to maximal leverage because these trades maximize the collateral value of all assets and derivative debt contracts.

Few papers study debt collateralization, or pyramiding, in equilibrium. Gottardi and Kubler (2015) implicitly assume that all financial securities serve as collateral. Provided the financial markets are sufficiently rich in terms of the specification of payoffs and of collateral requirements, any Arrow-Debreu equilibrium allocation with limited pledgeability can also be attained at a collateral-constrained financial market equilibrium and debt pyramiding can replicate tranching. In contrast to this rich environment, we limit our analysis to non-contingent debt and show that similar results emerge when state-contingencies must be created via default. Geerolf (2015) studies an economy with a continuum of states and a continuum of agents with differing point-beliefs about the asset payoff. A continuum of contracts are traded in equilibrium, and with pyramiding the asset price increases with each layer of pyramiding, the measure of contracts traded decreases, and the distribution of leverage changes.

While these results are closely related to ours, there are important distinctions. In Geerolf (2015), agents' disagreements are of the form of point-expectations about the asset's value, implying that agents trade debt they perceive to be risk-free. With pyramiding, agents switch to making larger promises, which are perceived to be risk-free by the buyers, and interest rates adjust to clear supply and demand, not to compensate for risk ("risk-free" promises are collateral for other risk-free promises). In our setting, interest rates compensate for default risk because agents use risky debt as collateral. We prove that with debt collateralization agents use maximal leverage on the assets in which they invest—agents switch to using contracts with the highest possible level of risk—and economy-wide margins decrease because the composition of leverage changes as more investors issue risky contracts. Critically, in our setting agents make larger promises because the downstream valuation of risk changes, precisely because buyers of risky debt can leverage their debt position to create objectively risk-free debt for investors who demand it.

Several papers study collateral equilibrium with multiple states. Simsek (2013) uses a model with a continuum of states to study belief disagreements, and conjectures that equilibrium in multi-state models will feature a pyramiding arrangement when debt contracts can be used as collateral. We prove that this conjecture holds only when the *maximum* level of securitization has been reached. Toda (2018) shows that demand for safe assets, to hedge and insure idiosyncratic risks, lead investors to take maximum leverage when collateralized loans are securitized into pools of ABS, and Phelan and Toda (2019) study the consequences of cross-country margin heterogeneity for international capital flows and risk sharing. These papers focus on the welfare consequences of maximum leverage and securitization. Araujo et al. (2012) examine the effects of default and collateral on risk sharing. Gong and Phelan (2017) study how expanding the sets of assets that can serve as collateral affects the basis between risky bonds and credit default swaps.

Our results relate to the literature on how securitized markets create safe and liquid assets (see Gorton and Metrick, 2009), and we show that this process increases the supply of both risky and safe debt and the overall level of leverage and volatility increase. Cao (2010, 2017) and Cao and Nie (2017) study how collateral constraints and incomplete markets affect asset price volatility and amplification (see also Brumm et al., 2015). Shen et al. (2014) propose a collateral view of financial innovation driven by the cross-netting friction. In our model, debt collateralization and innovative capital structures are ways of stretching collateral, which is similar to their insight that financial innovation is a response to scarce collateral. Dang et al. (2011) study how debt collateralization can alleviate asymmetric information problems by creating information-insensitive securities, and they show that the optimal financial instrument is debt backed by debt. Finally, Rampini and Viswanathan (2013) also argue that asset tangibility and collateral requirements determine firms capital structure, and their analysis focuses on firm decisions to lease versus buying capital, with implications for investment and risk management.

2 General Equilibrium Model with Collateral

This section presents the basic general equilibrium model with collateralized borrowing and characterizes the potential contracts traded in equilibrium in a general setting.

2.1 The Model

To simplify the analysis and the exposition, we consider a multi-state extension of Geanakoplos (2003) as found in Fostel and Geanakoplos (2012b).

Time, Assets, and Households

We begin by considering a two-period, N-state general equilibrium model with time $t = 0, 1$. Uncertainty is represented by a tree with a node s_0 at $t = 0$ and N states $n \in S = \{1, \ldots, N\}$ at $t = 1$.

There are two fundamental assets (risk-free and risky), denoted by X and Y respectively, which produce dividends of the consumption good at time 1. For a generic asset Z, let d_n^Z be the dividend of asset Z in state n. We normalize $d_n^X = 1$ for all n, and $d_n^Y = s_n$, where $s_1 < s_2 < \ldots < s_N$ (states are ordered so higher n implies higher dividend payout), and we set $s_N = 1$.

We suppose that agents are uniformly distributed on $(0, 1)$, that is they are described by Lebesgue measure. (We will use the terms "agents" and "investors" interchangeably.) Agents are risk-neutral and have linear utility in consumption c at time 1. Each agent $h \in (0, 1)$ assigns subjective probability $\gamma_n(h)$ to the state n, and beliefs $\gamma_n(h)$ are continuous in h. The expected utility of agent h is

$$U^h(c) = \sum_{n=1}^{N} \gamma_n(h) c_n,$$

where c_n is consumption in state n. At time 0, each investor is endowed with one unit of each asset.

To ensure that in equilibrium investors' positions are sorted by their level of optimism, we suppose hazard rate dominance (see also Simsek, 2013; Phelan, 2015):

$$\text{For all } n \in S, \text{ the ratio } \quad \frac{\gamma_n(h)}{\sum_{s=n}^{N} \gamma_s(h)} \quad \text{is strictly decreasing in } h. \tag{A1}$$

This condition implies that more optimistic agents are increasingly optimistic about states above a threshold state n. Investors with higher h have uniformly higher marginal utility for consumption in states in which the asset payoff is higher (i.e., they are uniformly more optimistic). This setup is equivalent to a model with finitely many heterogeneous risk-averse agents, where endowments and preferences are such that marginal utilities or "hedging needs" are monotonic and uniformly increasing by state.

Financial Contracts and Collateral

The heart of our analysis involves contracts and collateral. We explicitly incorporate repayment enforceability problems, and we suppose that collateral acts as the only enforcement mechanism. Agents trade financial contracts at $t = 0$. A financial contract $j = (A^j, C^j)$, consists of a promise $A^j = (A^j_n)_{n \in S}$ of payment in terms of the consumption good at $t = 1$, and an asset C^j serving as collateral backing the promise. The lender has the right to seize as much of the collateral as was promised, but no more. Therefore, upon maturity, the financial contract yields $\min\{A^j_n, d^{C^j}_n\}$ in state n. Agents must own collateral in order to make promises. Let J be the set of all possible financial contracts. Each contract $j \in J$ trades for a price π_j.

We introduce multiple levels of debt collateralization inductively. Level-0 debt collateralization are promises using the risky asset Y as collateral. Without loss of generality we normalize the collateral to one unit of Y, and let J^0 denote the set of promises backed by one unit of Y. These assets are referred to as level-0 debt. A promise $j_n = (s_n, Y) \in J^0$, which promises to pay s_n at time 1 and uses Y as collateral, delivers $\min\{s_n, s_l\}$ in the state l. Note that $j_1 = (s_1, Y)$ is risk-free debt.

We allow level-0 debt contracts in J^0 to be used as collateral to issue further non-contingent promises.

Definition 1. We say the *first level of debt collateralization* is the creation of promises j_n^1 using $j_k \in J^0$ as collateral. Denote the set of contracts at the first level of debt collateralization by J^1. We write $j_n^1(j_k) = (s_n, j_k)$ to denote the *debt contract* that is traded when an agent holds j_k as collateral and promises to pay s_n. We denote the act of holding j_k and selling j_n^1 by j_k/j_n^1.

For a contract j_k to be meaningful collateral for a promise s_n it must be that $s_k > s_n$ because otherwise the payoff to j_k would always be less than the promise (and equality would render the new promise redundant). Thus, in what follows we will only consider when agents use meaningful collateral to make new promises, restricting our attention to contracts $j_n^1(j_k)$ with $k > n$. Given this restriction, the payoffs to $j_n^1(j_k)$ are the same for every $k > n$, and so we can denote the price of a contract $j_n^1(j_k)$ by π_n^1.

In general, level L debt collateralization is to promise a non-contingent payment using a level $L-1$ debt as collateral.

Definition 2. Denote the set of contracts at the L-th level of debt collateralization by J^L. The L-th level of of debt collateralization is the creation of the promises j_n^L, backed by contracts $j_k^{L-1} \in J^{L-1}$, where $1 < n < N-L$ and $1 < k < N-L+1$. In other words, the buyer of the promise j_k^{L-1} is able to sell the promise j_n^L, using j_k^{L-1} as collateral. Again, we must have $n < k$. We denote the promise of j_n^L with j_k^{L-1} as collateral by writing $j_n^L(j_k^{L-1}) = (s_n, j_k^{L-1})$. We denote an agent buying j_k^{L-1} and selling j_n^L by j_k^{L-1}/j_n^L.

With L levels of debt collateralization, the set of financial contracts is given by $J = J^0 \cup J^1 \cup \cdots \cup J^L$. Thus, each additional level of collateralization involves the creation of new bonds and allows all previously existing, risky bonds to be purchased with leverage. So long as the backing collateral is meaningful, given the monotonicity of payoffs for debt contracts, the payoff of any contract is defined by the promise. Since the payoff depends on n and not on k, we use π_n^L to denote the price of any debt security $j_n^L(j_k^{L-1}) \in J^L$ with $k > n$. Note that for all k,l, the contract promising s_1 backed by j_k^{l-1}) delivers s_1 in every state (it is risk-free debt).

We denote contract holdings of $j \in J$ by φ_j, where $\varphi_j > 0$ denote *sales* and $\varphi_j < 0$ denote *purchases*. The sale of a contract corresponds to borrowing the sale price and the purchase of a promise is equivalent to lending the price in return for the promise. A position of $\varphi_j > 0$ units of a contract requires ownership of φ_j units of the collateral, whereas the purchase of such contracts does not require ownership of the collateral.

Budget Set

Without loss of generality, we normalize the price of asset X to be 1 in all states of the world, making X the numeraire good (since there is no consumption in the initial period, the price of X is 1 at 0). We let p denote the price of the risky asset Y. Given asset and contract prices at time 0, each agent decides how much X and Y he holds and trades contracts j to maximize utility, subject to the budget set

$$
\begin{aligned}
B^h(p,\pi) = \{ &(x, y, \varphi, (c_s)_{s\in S}) \in R_+ \times R_+ \times R^J \times R^N_+ : \\
&(x-1) + p(y-1) \leq \sum_{j\in J} \varphi_j \pi^j, && (1)\\
&\sum_{j\in J^0} \max(0, \varphi_j) \leq y, && (2)\\
&\sum_{j=j^l_n(j^{l-1}_k)\in J^l} \max(0, \varphi_{j^l_n(j^{l-1}_k)}) \leq \varphi_{j^{l-1}_k} \forall l \in 1,\ldots,L, && (3)\\
&c_s = x + yd^Y_s - \sum_{j\in J} \varphi_j \min(A^j_s, d^{C^j}_s) \}. && (4)
\end{aligned}
$$

Equation (1) states that expenditures on assets (purchased or sold) cannot be greater than the resources borrowed by selling contracts. Equation (2) is the collateral constraint for debt backed by Y, requiring that agents must hold sufficient assets to collateralize the contracts they sell. Equation (3) is the collateral constraint for contracts backed by the risky asset, and for contracts backed by debt, up to L levels which is a parameter of the financial environment. Equation (4) states that in the final states, consumption must equal dividends of the assets held minus debt repayment. Recall that a positive φ_j denotes that the agent is selling a contract or borrowing π_j, while a negative φ_j denotes that the agent is buying the contract or lending π_j. Thus there is no sign constraint on φ_j. Additionally, short selling of fundamental assets is not possible ($y \geq 0$ and $x \geq 0$).

Collateral Equilibrium

A collateral equilibrium in this economy is a price of asset Y, contract prices, asset purchases, contract trades, and consumption decisions all by agents $((p,\pi),(x^h,y^h,\varphi^h,c^h_{h\in(0,1)})\in(R_+\times R^J_+)\times(R_+\times R_+\times R^J\times R^N_+)^H$ such that

1. $\int_0^1 x^h \mathrm{d}h = 1$,
2. $\int_0^1 y^h \mathrm{d}h = 1$,
3. $\int_0^1 \varphi^h_j \mathrm{d}h = 0\ \forall j \in J$,
4. $(x^h,y^h,\varphi^h,c^h)\in B^h(p,\pi),\forall h$,
5. $(x,y,\varphi,c)\in B^h(p,\pi)\Rightarrow U^h(c)\le U^h(c^h),\forall h.$

Conditions 1 and 2 are the asset market clearing conditions for X and Y at time 0 and condition 3 is the market clearing condition for financial contracts. Condition 4 requires that all portfolio and consumption bundles satisfy agents' budget sets, and condition 5 requires that agents maximize their expected utility given their budget sets. Geanakoplos and Zame (2014) show that equilibrium in this model exists under the assumptions made thus far.

2.2 Discussion of the Financial Environment

For debt collateralization or pyramiding to be meaningful, agents must be restricted to trade contracts that pay in *multiple* states. For example, if agents could trade Arrow-Debreu securities, then there would be no role for debt collateralization in equilibrium (see Lemma 3 in the appendix). A fundamental role of debt collateralization is to create contingencies

The simplest environment (when considering contracts paying in multiple states) is when agents are restricted to non-contingent contracts (debt), but this is not without loss of generality. Indeed, our main results illustrate that the degree to which debt collateralization is redundant or not depends on the degree of state-contingencies for level-0 contracts. When level-0 contracts are non-contingent, debt collateralization has a large role to play in creating contingencies, and multiple level of collateralization can be supported in equilibrium (Proposition 5). In contrast, when level-0 contracts are senior-subordinated tranches, there is no ability to use tranches as collateral for further tranches—the set of senior-subordinated tranches already implement an equilibrium equivalent to complete collateralization (Proposition 6).

There are at least three interpretations of our restriction to non-contingent contracts. First, the environment with debt collateralization is equivalent to allowing agents to issue state-contingent claims defined by senior-subordinated capital structure (Proposition 6). In other words, debt contracts are the building blocks for senior-subordinated state-contingent payoffs. Thus, one can interpret our results with non-contingent contracts and collateralization as a metaphor for environments when agents can trade state-contingent contracts derived from senior-subordinated capital structures.

Second, agents may be restricted to non-contingent promises because of some un-modeled informational friction, or because markets are segmented and some investors are restricted to buying "tier-1" securities.[1] Leverage and debt collateralization are mechanisms that create new state-contingent payoffs from underlying non-contingent contracts *without* violating the informational friction (they depend on collateral seizure and limited repayment enforceability). Thus, our results provide an explanation for how financial markets create state-contingent contracts in the presence of these informational frictions. The severity of these frictions determine the degree to which levels of collateralization are meaningful.

Finally, our results could extend to environments with cross-netting frictions and richer contracts. State-contingent contracts may be available, but agents may not be able to use an asset as collateral to back multiple promises, even when doing so would still guarantee repayment. As shown by Geanakoplos and Zame (2014), equilibrium can be endogenously incomplete when collateral is scarce (agents may trade debt contracts even when Arrow securities are available because debt contracts economize on collateral). Shen et al. (2014) show that financial innovations are likely to occur in such a setting. Senior-subordinated capital structures allow an asset to simultaneously collateralize multiple state-contingent contracts. Thus, our restrictions reflect some combination of informational frictions limiting state-contingencies together with some degree of cross-netting frictions.

The financial environment in our model is the set of assets used as collateral or the permissible promises that can be backed by the same collateral (the set of contracts J). Debt collateralization

[1] For examples relating to securitization see DeMarzo (2005); Pagano and Volpin (2012); Friewald et al. (2015). Mada and Soubra (1991) show that nonextremal securities (debt and equity rather than "Arrow Securities") may be optimal when securities must be marketed at a cost. Lemmon et al. (2014) provide evidence that one value of securitization (for nonfinancial firms) is providing access to segmented markets.

expands the set of contracts in J. We take the financial environment as exogenous (see Dang et al., 2011; Gennaioli et al., 2013; Gorton and Ordoñez, 2014, for informational explanations for why financial markets may *decrease* the available set of assets serving as collateral). The financial structures we assume allow us to focus on the abilities to leverage and securitize assets in the most straightforward setting. Allowing investors to issue contracts directly against assets is without loss of generality, as such trades could also correspond to financial assets producing by financial intermediaries that correspond to these cash flows, or to securities issued by firms as part of their capital structure. Similarly, investors could attain higher leverage through an intermediary when collateral is rehypothecated, as is common with Prime brokerage.

3 A Model with Three States

We now focus on a 3-state economy in order to more carefully characterize the equilibrium and to provide intuition for the economic forces determining investors' positions. Uncertainty is represented by a tree $S = \{0, U, M, D\}$ with a root $s = 0$ at $t = 0$ and three states of nature $s = U, M, D$ at time 1. With a slight abuse of notation we let M, D, be the dividends in states M, D, with $D < M < 1$, and the dividend is 1 in U. Figure 1 shows asset payoffs. Note that assumption A1 on beliefs means that $\gamma_U(h) + \gamma_M(h)$ and $\frac{\gamma_U(h)}{\gamma_U(h)+\gamma_M(h)}$ are increasing in h. High h investors believe that state D is unlikely and that, conditional on the state being at least M, state U is relatively likely.

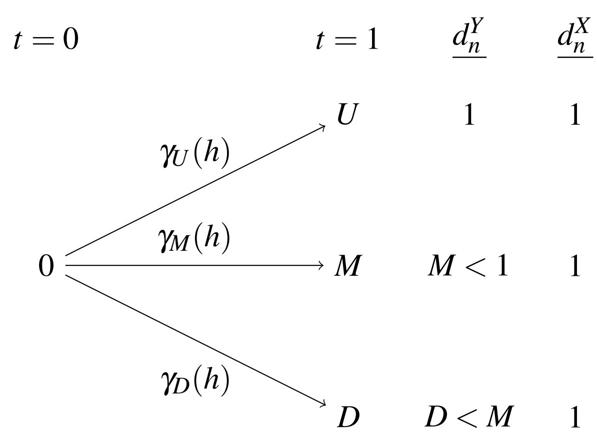

Figure 1: Payoff tree of assets X and Y in three-state world.

We characterize equilibrium with leverage (when agents can trade debt backed by Y only) and with debt collateralization (when agents can also trade debt backed by debt). In the leverage economy, agents can issue non-contingent promises using the asset Y as collateral. With debt collateralization, contracts $j \in J^0$ can also serve as collateral. All proofs are in Appendix A.

3.1 Leverage Economy with 3 States

As shown by Fostel and Geanakoplos (2012b), in equilibrium with debt two contracts are traded: a risk-free promise j_D promising D and a risky promise j_M promising M, with prices π^D and π_M. The interest rate on j_D is zero ($\pi^D = D$) because it is a risk-free promise. However, the delivery of j_M depends on the realization of the state at time 1 and j_M is therefore risky; j_M pays (M,M,D). This means that any agent making the promise j_M can only borrow $\pi_M < M$. Thus, the interest rate for j_M is strictly positive, defined by $i_M = \frac{M}{\pi_M} - 1$, and is endogenously determined in equilibrium. We refer to changes in the interest rate as changes in the *risk premium* for the debt contract.

In equilibrium there are three marginal investors h_M, h_D, h_J. Agents $h > h_M$ will sell their endowment of X, buy the asset Y, and promise M (issue j_M) for every unit of the asset bought.[2] These agents receive state-contingent payoffs $(1-M,0,0)$, equivalent to an Arrow U. Agents $h \in (h_D, h_M)$ will sell their endowment of X and buy the risky asset, promising D against every asset bought. These agents receive state-contingent payoffs $(1-D, M-D, 0)$, with payoffs in U and M. Agents $h \in (h_J, h_D)$ will sell their endowment of X and Y and buy j_M (effectively lending to agents $h > h_M$). Agents $h < h_J$ will sell their endowment of Y and buy both risk-free assets X and contracts j_D backed by the risky asset (these two are equivalent). Figure 2 illustrates the equilibrium regime. It is easy to see how the assumption on beliefs implies this ordering of investors.

Agents $h > h_M$ are "maximally leveraged" in the sense that making a larger promise would simply result in a transfer of resources to lenders in U, the state in which the asset has the maximum payoff. Agents can choose to promise more to attain additional leverage—they can make any promise j—but $j > M$ is unattractive to borrowers. Fundamentally, any contract $j > M$ has the same delivery as j_M in states M and D (because of default against the asset's payoff) and delivers

[2] Since the marginal agent has measure zero, to simplify notation we will use strict inequalities when referencing the marginal agent.

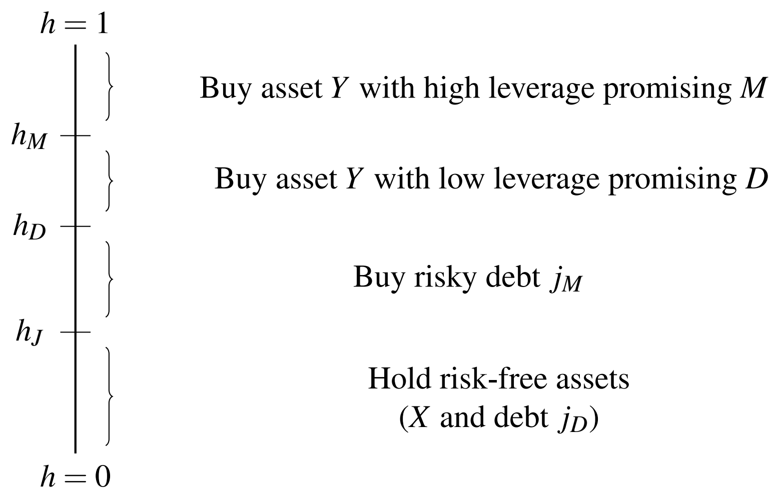

Figure 2: Equilibrium regime with leverage in static 3-state model.

more only in state U. While U is the state that investors $h > h_M$ believe to be comparatively the most likely to happen, the larger promise in U is priced by more pessimistic agents. Hence, a promise $j > M$ would result in raising less than the value of the promise. Agents $h \in (h_D, h_M)$, promising D against each unit of the asset, are not maximally leveraged because promising M changes the delivery to borrowers in both states U and M.

Equilibrium is described by the following set of equations. Agent h_M is indifferent between buying Y with high leverage promising M, and buying asset with low leverage promising D,

$$\frac{\gamma_U(h_M)(1-M)}{p-\pi_M} = \frac{\gamma_U(h_M)(1-D)+\gamma_M(h_M)(M-D)}{p-D}. \tag{5}$$

Agent h_D is indifferent between buying Y with leverage promising D, and holding risky debt j_M,

$$\frac{\gamma_U(h_D)(1-D)+\gamma_M(h_D)(M-D)}{p-D} = \frac{(1-\gamma_D(h_D))M+\gamma_D(h_D)D}{\pi_M}. \tag{6}$$

Agent h_J is indifferent between holding risky debt j_M and holding risk-free assets (X or risk-free debt),

$$\frac{\gamma_U(h_J)M+\gamma_M(h_J)M+\gamma_D(h_J)D}{\pi_M} = 1. \tag{7}$$

Market clearing for the risky asset Y requires

$$(1-h_M)\frac{1+p}{p-\pi_M}+(h_M-h_D)\frac{1+p}{p-D}=1, \tag{8}$$

and market clearing for the risky debt j_M requires

$$(1-h_M)\frac{1+p}{p-\pi_M}=(h_D-h_J)\frac{1+p}{\pi_M}. \tag{9}$$

Equation (8) states that the agents buying the risky asset, $h \in (h_D, 1)$, will spend all of their endowment, $(1+p)$, to purchase the risky asset, which costs price p, borrowing either π_M or D to leverage their purchases, and that the demand is equal to the supply of the risky asset, 1. Equation (9) states that the amount of risky debt demanded by agents $h \in (h_M, 1)$ is equal to the amount of risky debt supplied by agents $h \in (h_J, h_D)$.

3.2 Economy with Debt Collateralization

We now suppose agents can also trade contracts of the form $j^1_\ell = (\ell, j_M)$, i.e., $C^j = j_M$. This contract specifies a non-contingent promise (ℓ, ℓ, ℓ) backed by the risky debt j_M acting as collateral. The restriction to j_M is without loss of generality.[3] The payoff to j^1_ℓ is $\min\{\ell, d^{j_M}_s\}$, the minimum of the promise ℓ and the payoff of the debt contract j_M. The budget set now includes the constraint $\sum_{j\in J^1}\max(0,\varphi_j) \le \varphi_{j_M}$ in addition to the collateral constraint in (2). That is, they must hold sufficient positions in j_M to issue contracts backed by j_M. We denote equilibrium variables with debt collateralization by a 'hat' (ˆ) to distinguish them from their counterparts with leverage.

Consider how this expansion of the financial environment affects the ability to create state-contingent securities. For concreteness, let Y have payoffs $M = 0.3$ and let $D = 0.1$. Buying the risky asset with leverage and promising M splits the asset's cash flows into risky debt and an "Arrow U." Buying the risky asset and promising D splits the risky asset's cash flows into risk-free

[3]We could let any contract $j \in J^0$ serve as collateral; however, we show that in equilibrium only j_M will be traded and thus only j_M will serve as collateral. Making a non-contingent promised backed by j_D, which is non-contingent, is redundant, and using j_U is equivalent to using Y.

debt and payoffs in U and M.

$$d^Y = \begin{pmatrix} 1 \\ 0.3 \\ 0.1 \end{pmatrix} \rightarrow \begin{pmatrix} 0.3 \\ 0.3 \\ 0.1 \end{pmatrix} + \begin{pmatrix} 0.7 \\ 0 \\ 0 \end{pmatrix}, \quad d^Y = \begin{pmatrix} 1 \\ 0.3 \\ 0.1 \end{pmatrix} \rightarrow \begin{pmatrix} 0.1 \\ 0.1 \\ 0.1 \end{pmatrix} + \begin{pmatrix} 0.9 \\ 0.2 \\ 0 \end{pmatrix}.$$

With debt collateralization, the *risky debt* can also be split into risk-free debt and payoffs in U and M. Note that the act of holding j_M and selling the contract j_D^1 is equivalent to buying j_M with leverage promising D, yielding a payoff of $(M-D, M-D, 0)$, i.e., $(0.2, 0.2, 0)$ in our example. Our first result is that any investor buying risky debt will choose to use leverage in this way.

Lemma 1. *Suppose that in equilibrium agents are able to collateralize debt. Then every agent holding risky debt will maximally leverage their purchases of risky debt. That is, all agents holding j_M will sell the promise $j_D^1 = (D, j_M)$.*

The intuition is straightforward. In the leverage economy, only the marginal agent investing in risky debt thinks the debt is priced to exactly compensate for risk, while every other agent thinks the expected payoff is higher than implied by the price and thus would like to leverage their investment in the debt. Since agents investing in risky debt can leverage their purchases, all else equal the demand for risky debt increases, which decreases the risk premium on the risky debt. Promising D maximally leverages the investment in j_M; any agent that is not willing to maximally leverage their investment in j_M will be priced out by those who are.

When agents and leverage risky debt, demand for risky debt increases and increases the supply of safe assets. As a result, the marginal buyer of risky debt will be more optimistic, increasing the price of risky debt.

Proposition 1. *Suppose that in equilibrium agents are able to collateralize debt. Then, the price of risky debt increases.*

Critically, when risky debt can be used as collateral, *in equilibrium no agent chooses to leverage Y by promising risk-free debt*—no investor chooses the payoff $(0.9, 0.2, 0)$—which is stated in the following lemmas.

Lemma 2. *Let agents be allowed to collateralize debt. Then, every agent holding the risky asset will maximally leverage their purchases of the risky asset. In other words, every agent holding the risky asset will promise M.*

The intuition for Lemma 2 is that promising M creates a debt contract that can be used as collateral, while promising D does not. Additionally, debt collateralization decreases the risk premium of risky debt, increasing the amount of leverage agents get from risky debt. As a result, it becomes more attractive for investors to use Y to issue the risky debt (which has a higher price), rather than issuing risk-free debt, which can also be issued by owners of the risky debt. The general equilibrium consequences imply that any investor who is not willing to buy Y and promise M finds it more attractive to leverage the risky debt j_M rather than to buy Y and promise D. In other words, Y is priced so that the only efficient investment is to use a high level of leverage, and so investors who desire a low level of leverage will choose to buy a different asset. Thus, the set of state-contingent payoffs associated with buying Y with low leverage are priced so that no investor chooses those payoffs.

The key insight for our result is that the price of any asset is a sum of the payoff value and the collateral value. Allowing a debt contract to be used as collateral increases its price—it now has a collateral value—which increases the value to buying the risky asset and issuing that debt contract. Because only the risky asset will back risky debt in equilibrium (the risky debt will back risk-free debt in equilibrium), the collateral value of the risky debt, in effect, gets imparted to the risky asset. Using the risky asset to issue risk-free debt is "inefficient." Instead, by issuing risky debt against the asset, the risky asset can be used to back *both* risky debt and risk-free debt, where the risk-free debt has been issued against the risky debt. This process creates a new security with collateral value (risky debt), while using the asset to issue risk-free debt does not.

Proposition 2. *In equilibrium, there exist two marginal buyers $\hat{h}_M$ and $\hat{h}_J$ such that all $h \in (\hat{h}_M, \hat{h}_J)$ will hold risky debt with maximal leverage (promise D); all $h < \hat{h}_J$ will hold risk-free debt and X, and all $h > \hat{h}_M$ will hold the risky asset with maximal leverage (promise M).*

The proposition characterizes equilibrium in the 3-state model and follows directly from the previous two lemmas and the fact that marginal utilities/optimism is strictly and monotonically increasing in h. Figure 3 illustrates the equilibrium regimes with debt collateralization and with leverage. This result is analogous to Geerolf (2015), in which equilibrium with pyramiding produces the same ordering of lending in the economy with a continuum of states. Importantly, in our result the threshold promises are defined by the discrete payoffs of the states and the ordering of investors follows from valuations of payoffs in different states (either tolerance for risk or subjective probabilities of default), with debt prices compensating for risk. The qualitative break in the equilibrium regime in our model corresponds to changes in the sets of state contingent payoffs agents trade. Our result for maximal leverage would hold even if agents had some degree of risk-sharing needs so long as marginal utilities of agents are monotonic with dividends.[4]

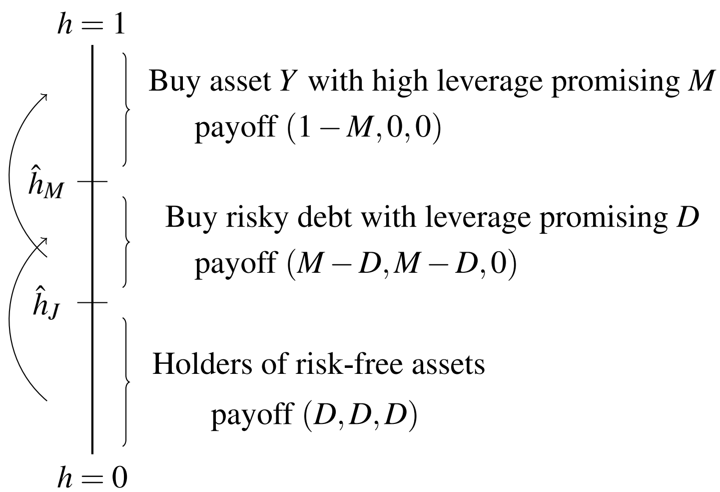

Figure 3: Equilibrium with debt collateralization in static 3-state model.

Thus, equilibrium is characterized by the following equations. Agent $\hat{h}_M$ is indifferent between holding the risky asset with high leverage promising M, and the risky debt with leverage,

$$\frac{\gamma_U(\hat{h}_M)(1-M)}{\hat{p}-\hat{\pi}_M} = \frac{\gamma_U(\hat{h}_M)(M-D)+\gamma_M(\hat{h}_M)(M-D)}{\hat{\pi}_M - D}. \tag{10}$$

In equilibrium both of these investment options are preferred over holding Y with low leverage (promising D). Agent $\hat{h}_J$ is indifferent between holding the risky debt with leverage and holding

[4] We could reproduce the distribution of marginal utilities we get from differences in prior probabilities by instead assuming common probabilities, strictly concave utilities, and by allocating endowments of consumption goods appropriately. An implication is that our results continue to hold (weakly) whether there are more agents than states or whether there are more states than agents. Our results continue to hold when marginal utilities are endogenous so long as there are appropriate bounds on risk aversion and endowments so that even with endogenous portfolio choices, optimists remain uniformly optimistic after accounting for changes in marginal utilities (see Phelan, 2015, for an analysis in a two-agent economy); see also the example in Appendix B.3.

risk-free assets,

$$\frac{\gamma_U(\hat{h}_J)(M-D)+\gamma_M(\hat{h}_J)(M-D)}{\hat{\pi}_M-D}=1. \tag{11}$$

Market clearing for the risky asset Y requires

$$\frac{(1-\hat{h}_M)(1+\hat{p})}{\hat{p}-\hat{\pi}_M}=1, \tag{12}$$

and market clearing for risk-free debt requires

$$\hat{h}_J(1+\hat{p})=1+D. \tag{13}$$

Collateralizing risky debt thus serves two purposes: it isolates upside payoffs to agents buying risky debt with leverage, and it creates risk-free debt for more pessimistic agents, increasing the supply of risk-free securities.

3.3 Asset Pricing

The effect of debt collateralization on the price of the risky asset is somewhat ambiguous because there are two forces affecting the price. There is a collateral effect, which raises the asset price, and a required return effect, which may decrease the asset price.

Let R and $\hat{R}$ denote the alternative return according to the most pessimistic investor who maximally leverages the asset in the leverage economy and the debt collateralization economy:

$$R=\frac{\gamma_U(h_M)(1-D)+\gamma_M(h_M)(M-D)}{p-D}, \quad \hat{R}=\frac{\gamma_U(\hat{h}_M)(M-D)+\gamma_M(\hat{h}_M)(M-D)}{\hat{\pi}_M-D},$$

which are taken from equations (5) and (10). Then we can write the asset prices as

$$p=\pi+\frac{\gamma_U(h_M)(1-M)}{R}, \quad \hat{p}=\hat{\pi}+\frac{\gamma_U(\hat{h}_M)(1-M)}{\hat{R}}.$$

The "collateral effect" implies that debt collateralization increases the collateral value of the risky asset because it can now be used to issue a contract (risky debt) that can serve as collateral ($\pi<\hat{\pi}$). This force increases the price of the risky asset and endogenously increases leverage in the economy. The "required return effect" implies that the required return for investing in the risky asset may increase because alternative investments have become more attractive, namely, investing in *risky debt* with leverage so that generally $R < \hat{R}$. In the leverage economy, the most optimistic agent h_M compares the return to Y with high leverage to the return to Y with low leverage. In the debt collateralization economy, the most optimistic agent $\hat{h}_M$ compares the return to Y with high leverage to the return to risky debt with leverage, and in the debt collateralization economy this investment is strictly preferred to buying Y with low leverage. The required return force tends to decrease the price of the risky asset.

Debt collateralization would decrease the asset price if (i) risky debt prices do not increase by much (i.e., $\hat{\pi}$ near π), (ii) the marginal investor becomes much less optimistic about U (i.e., $\gamma_U(h_M) >> \gamma_U(\hat{h}_M)$), and (iii) the perceived return on leveraged debt is more attractive than the return on Y with low leverage. For a wide range of parameters it appears that debt collateralization increases the asset price (Appendix B.1) because the primary effect of debt collateralization is to increase the price of risky debt. However, Appendix B.2 provides an example where the price p decreases with debt collateralization because the collateral effect is small. This result is in contrast to Geerolf (2015), where pyramiding strictly increases prices.

We can provide some restrictive sufficient conditions under which the collateral effect dominates the return effect so that debt collateralization will increase prices. We require three conditions. First, belief heterogeneity among "pessimists" is greater than among "optimists". Denote the hazard rates by $f_U(h) = \frac{\gamma_U(h)}{\gamma_U(h)+\gamma_M(h)}$ and $f_M(h) = \frac{\gamma_U(h)+\gamma_M(h)}{\gamma_U(h)+\gamma_M(h)+\gamma_D(h)}$. We require

$$f_U, f_M \text{ are concave.} \tag{A2}$$

Second, optimism about the down state *not* occurring must increase faster than the optimism about the conditional likelihood of the up state.

$$\text{For all } h \geq h', \quad f_U(h) - f_U(h') \leq f_M(h) - f_M(h') \tag{A3}$$

These two conditions combined ensure that collateralization has a relatively large effect on the price of the risky debt. As an example, constant hazard rates for each investor (i.e., $f_U(h) = f_M(h)$ for all h) satisfies this condition.

Third, the fraction of buyers using high leverage in the leverage economy must be sufficiently high, which implies that $\hat{h}_M$ does not differ too much from h_M and collateralization sufficiently expands the supply of safe debt. Let η denote the fraction of Y purchased by investors promising M (high leverage) in the leverage equilibrium. Then we can state the following proposition.

Proposition 3. *Suppose* $\eta > \frac{(1-M)^2}{(1-M)^2+(M-D)D}$ *and that beliefs satisfy A2, A3. Then* $\hat{p} > p$*.*

Additionally, we can isolate the collateral effect by considering an economy that simultaneously contains multiple assets, one that can be leveraged and one that can be used for debt collateralization. Then investors have access to all investment options and so the different leveraged investments will have common required returns. In this case, the collateral effect from debt collateralization will increase the asset price.

Proposition 4. *Consider an economy with risky assets Y and Z with identical dividends but debt backed by Z cannot be used as collateral (Z can be leveraged), while debt backed by Y can be used as collateral. Then in equilibrium the price of Y exceeds the price of Z.*

Because Y and Z are available to investors at the same time, the required return for any investor applies equally to both assets and so the required return force does not differentially affect Y over Z. But the risky promise backed by Y has collateral value, while the promise backed by Z does not, and thus the risky promise backed by Y has a higher price. As a result, Y must also have a higher price since it is used to issue a more valuable contract.

In reality not every financial contract can be used as collateral to issue further contracts. Perhaps debt collateralization is prevalent in one market, but not necessarily in others. (Consider how the mortgage market is often the vanguard of financial innovation.) To the extent that investors may have access to assets and financial contracts with differential degrees of collateralizability, investment opportunities will have common required returns but debt collateralization will isolate the collateral effect. We, therefore, suspect that the setting in Proposition 4 is an empirically realistic setting.

3.4 Numerical Example

A numerical example is helpful to suggest what happens to prices and economy-wide margins. We roughly "calibrate" the 3-state model so that the move from leverage-only to debt collateralization explains the following moments: we target economy-wide average margins with leverage to be 15% and with debt collateralization to be 5%, and we target risky debt spreads to be 3.9% with leverage and 1.6% with debt collateralization.[5] (Of course many other changes occurred pre-crisis, not just the innovation of debt collateralization.) We parametrize marginal utilities of the form $\gamma_U(h) = h^\zeta$ and $\gamma_M(h) = h^\zeta(1-h^\zeta)$. Thus we choose parameters M, D, and ζ to match the four moments. Our calibration yields $M = 0.93$, $D = 0.81$, $\zeta = 6.5$.[6] Appendix B.1 discusses parameter robustness.

Table 1 compares the equilibria with leverage and with debt collateralization ("DC"). With the introduction of debt collateralization, economy-wide average margins fall dramatically and the price of the asset rises. Economy-wide average margins decrease for two reasons: all agents who buy the risky asset use the low margin (high leverage) strategy, and the risky margin (buying the asset with j_M) decreases because the risky debt price increases by relatively more than the asset price p. In this example (and across a wide range of parameters), the first effect is much larger.

Table 1: Equilibrium with Leverage and with Debt Collateralization

Prices and Marginal Investors

	Leverage	DC ($\hat{\ }$)
p	0.9542	0.9608 ↑
π_M	0.9014	0.9103 ↑
h_M	0.9984	0.9742 ↓
h_D	0.9289	–
h_J	0.9021	0.9231 ↑

Interest Spread and Margins

	Leverage	DC ($\hat{\ }$)
i_M	3.17 %	2.17% ↓
Risky Margin	5.53%	5.26% ↓
Safe Margin	15.11%	–
Average Margin	14.78%	5.26% ↓

The asset price increases by a modest 0.7 percent. Across a range of parametrizations, the model typically delivers modest increases in p. In our numerical simulations, the price of risky debt always increases, and the price of the risky asset increases in most cases. This result is in line with evidence by Kaplan et al. (2015), who quantitatively assess the contributions of changes in mortgage margins, productivity, and expectations about future house prices to explain house prices

[5] Fostel and Geanakoplos (2012a) show that for subprime mortgages from 2000–2008, average margins decreased from 12% to 3% in 2006 and then increased to roughly 18% by end of 2007. Pre crisis 10 year Baa corporate bond credit spreads ranged from 3.9% to roughly 1.6% through 2007, which we use as a rough measure of financing spreads.

[6] An alternative, attractive parametrization is to set payoffs to $M = 0.9$ and $D = 0.65$: the middle payoff corresponds to a mild recession for firms or a bad-but-typical decrease in house prices; the down payoff is a deep recession or a dramatic (35%) decrease in house prices. We then choose beliefs so that risky spreads and margin changes correspond roughly to levels over the early 2000s, yielding $\zeta = 2$. In this case, introducing debt collateralization, average margins decrease from 30.27% to 8.7%, spreads decrease from 3.88% to 2.45%, and the price increases by 1.16%.

during the housing boom and bust, and find that house prices are explained primarily by changes in expectations about future appreciation, not by margins. Thus, our model is best understood as a model of margins and leverage.

Dynamic Extension The static model illustrates that debt collateralization leads to agents making larger promises, increasing the leverage in the economy. In Appendix C we consider a dynamic extension of the 3-state model in order to study the effect of debt collateralization on price crashes and volatility. The maximal leverage result has several important implications for economy-wide margins and asset price levels and volatility (crashes). First, debt collateralization exacerbates the Leverage Cycle (Geanakoplos, 2003, 2010), amplifying price fluctuations and creating more price volatility than occur with leverage alone. Higher leverage increases the risky asset's collateral value, which fluctuates in response to news about fundamentals. Second, higher leverage endogenously increases defaults after bad news. Accordingly, our analysis explains how financial innovations in CDO, LBO, and similar markets, can lead to credit expansions and potentially higher volatility.

4 Characterizing Equilibrium in the General Model

We characterize the set of contracts potentially traded in equilibrium in the general setting with N states and L levels of collateralization. The main result of this section is that the possible set of investment options chosen in equilibrium decreases with more levels of debt collateralization, with only higher-leverage strategies remaining.

When only debt contracts in J^0 can be traded, agents can buy the risky asset leveraged with any promise $s_1, \ldots, s_N$ by selling the promise $j_n = (s_n, Y)$. We let Y/j_n denote the act of holding Y and selling the debt contract j_n. Following Araujo et al. (2012) and Fostel and Geanakoplos (2012b), in the leverage economy agents will do one of the following in equilibrium:

1. hold Y/j_n, where $1 \leq n \leq N-1$,
2. hold risky debt j_n with $2 \leq n \leq N-1$,
3. hold risk-free debt j_1 or the risk-free asset X.

Debt collateralization will allow the contracts traded in the leverage economy to be used as collateral, and as a result the set of debt contracts traded will endogenously change.

Our main result is that every level of debt collateralization increases the minimum promise made by agents buying the asset, and with "complete collateralization"—when any existing risky debt contract can be used as collateral—agents make the maximum (natural) promise available for every investment, risky asset or risky debt. With more than 3 states, multiple *risky* contracts will typically be traded in equilibrium. When agents can use these initial debt contracts as collateral, in equilibrium some agents will invest in risky debt contracts and make *risky* promises. These second-level debt contracts (backed by debt backed by the asset) can potentially be used as collateral to make further promises. Equilibrium will thus depend on how many "levels of debt" can be used as collateral.

Proposition 5. *Consider an economy in which, when agents can leverage, $N-1$ contracts are traded in equilibrium. In any equilibrium, there exists an equivalent equilibrium such that at the L-th level of debt collateralization, at most the following leveraged positions exist in the economy*

1. Y/j_n, *where* $L < n < N$
2. j^l_m/j^{l+1}_k, *where* $0 \leq l < L$, $L-l < m < N-l$, $L-l \leq k < m$
3. j^L_ℓ, *where* $1 \leq \ell < N-L$.

Additionally, more optimistic investors invest in assets with larger face values, and within each asset-class investors are ordered by the amount of leverage they use.

This result is a generalization of the three-state environment and the intuition is similar. Each level of collateralization increases the collateral value of new promises and of every debt contract that could already be used as collateral. As collateralization increases, more debt contracts have collateral value, as do the "upstream" debt contracts that can back those promises. As a result, when a security can be used to back promises that serve as collateral L times, making a smaller promise than stipulated by the proposition would not maximize the collateral value of debt contracts. Thus, investors make the largest promise that maximizes the collateral value of "downstream" promises.

We state a few implications of the proposition to provide more meaning. Corollary 1 explicitly states that debt collateralization decreases the number of low-level leverage strategies, and Corollary 2 states that with maximal debt collateralization, only the highest leverage positions remain in equilibrium, which corresponds to the conjecture in Simsek (2013) that in multi-state models when debt contracts can be used as collateral, equilibrium will feature a pyramiding arrangement; in other words the conjecture in Simsek (2013) holds at the maximal level of collateralization. By simple accounting, there can be at most $N-2$ levels of debt collateralization.

Corollary 1. *With each additional level of debt collateralization, there is one fewer marginal buyer of the risky asset Y, and thus one fewer "low level" of leverage used to buy the risky asset.*

Corollary 2 (Pyramiding Arrangment)**.** *Consider the continuum of agents in the economy. At the maximum $N-2$ levels of debt collateralization, the interval $(0,1)$ is broken up into $N+1$ sub-intervals, denoted $(1,\hat{a}_1),(\hat{a}_1,\hat{a}_2),\ldots,(\hat{a}_N,0)$. The first interval, $(1,\hat{a}_1)$ consists entirely of agents holding Y/j_{N-1}. The second interval, $(\hat{a}_1,\hat{a}_2)$ consists only of agents holding j_{N-1}/j^1_{N-2}. In general, the k^{th} interval, where $k>1$, consists of agents holding $j^{k-2}_{N+1-k}/j^{k-1}_{N-k}$. In other words, every level of agents in the economy is lending directly to the level above and maximally leveraging the asset or contract in which they invest.*

The corollaries follow immediately from Proposition 5. In the pyramiding arrangement investors are maximally leveraged: every investor makes the largest promise (from among the discrete set of states), given the asset or contract in which they invest.

5 Debt Collateralization and Capital Structure

This section shows that senior-subordinated tranching schemes can exactly implement the competitive equilibrium with complete debt collateralization. Tranching refers to the process of using collateral to back promises of different types. Senior-subordinated capital structures define tranches with realized payoffs determined by the seniority of the tranche. We discuss the connection of collateralization to tranching and capital structures and then present the formal analysis relating maximal leverage and capital structure. The maximal leverage property shown in this paper might explain why in reality securities are often tranched according to seniority.

5.1 Tranching and Capital Structure

One of the key features of securitized mortgage markets is the *explicit* ability to use risky debt contracts as collateral for new financial contracts. CDOs and other structured capital structures explicitly use debt (ABS tranches, TruPS tranches, etc.) as collateral to support another senior-subordinated capital structure. CDOs do not create pass-through securities backed by subordinated ABS tranches (in which case the only purpose of a CDO would be diversification of idiosyncratic risk), but rather create leveraged investments in the ABS tranches—the underlying promises backed by the original collateral are used to make more promises. Thus, the equity tranche of a CDO creates a leveraged investment in ABS tranches, and the senior tranches of a CDO create investments in debt "issued" by the leveraged (equity) investors.[7] Hence, CDOs (and then CDO-squareds) increase the degree to which debt contracts can be used as collateral to make new promises.

Critically, subordinated tranches (and subordinated capital) are equivalent to leveraged positions in risky debt backed by equity tranches, giving investors the *implicit* ability to use debt as collateral. To see this, consider a typical ABS deal, which consists of a pool of mortgages (collateral) supporting senior, mezzanine, and equity/residual securities. The equity security behaves like a leveraged position in the collateral, with the payoff declining "linearly" with the value of the collateral and paying zero when the collateral falls below a certain level. The senior security behaves like debt, making a predetermined payoff unless the collateral value falls below a certain threshold, at which point the payoff declines linearly to zero only when the collateral is worth zero. The subordinated, or mezzanine, security, however, behaves like a leveraged debt position. For sufficient values of collateral the subordinated security gets the predetermined payoff (there is not additional upside as with a leveraged position in the collateral), but gets nothing if the value of the collateral is low (like a leveraged position). In fact, the subordinated tranches are leveraged positions in the debt implicitly "issued" by the equity tranche.

The process of implicitly and/or explicitly using debt as collateral is incredibly general and widespread, as it is common to fund assets with a capital structure and then to use the debt created from that capital structure as collateral for new structures; classically, banks take deposits and make loans, issuing debt to invest in debt. In other words, layered capital structures are essentially "CDOs" with different collateral. Examples go back to unit trusts in the 1920s, the "unit trust of unit trusts" created by Goldman Sachs in 1928, Trust Preferred ("TruPS") CDOs, and, more prevalent, structured leveraged buyouts ("LBOs"). Similarly, securitized second-lien mortgages

[7] Allen and Gale (1988) motivate mezzanine securities as arising from market incompleteness, which is a similar motivation to that in our paper. The corporate finance literature has extensively studied how informational problems affect capital structure (see Harris and Raviv (1991) for a review).

(see Bear Stearns Second Lien Trust 2007) create tranches in debt that is part of a complex capital structure financing housing (Chambers et al., 2011).

5.2 Theoretical Analysis of Tranching

Consider the N-state model. Suppose the asset Y can be split by a financial intermediary into the following tranches: $T_1, \ldots, T_{N-1}$ where T_1 pays s_1 in all states of the world, and for $k > 1$ T_k pays $s_k - s_{k-1}$ when $n \geq k$ and 0 otherwise. That is, one unit of the risky asset Y can be used to simultaneously back multiple promises, creating the following tranches:

$$\begin{aligned} T_N &: \ (s_N - s_{N-1},\ 0,\ 0,\ \ldots\ ,\ 0), \\ T_{N-1} &: \ (s_{N-1} - s_{N-2},\ s_{N-1} - s_{N-2},\ 0,\ \ldots\ ,\ 0), \\ &\vdots \qquad\qquad\qquad\qquad \vdots \\ T_2 &: \ (s_2 - s_1,\ s_2 - s_1,\ \ldots\ ,\ s_2 - s_1, 0), \\ T_1 &: \ (s_1,\ s_1,\ \ldots\ ,\ s_1). \end{aligned}$$

Note that $T_1 + T_2 + \cdots + T_N = Y$. We refer to this financial structure as *senior-subordinated tranching* to emphasize the state-contingency is defined according to a senior-subordinated capital structure (complete tranching would refer to the creation of Arrow securities, not just paying zero in down states). In this economy, investors buy and sell the tranches listed above rather than trading the risky asset Y (though they can exactly replicate Y by buying all the tranches). Each investor must hold a non-negative quantity of each tranche. We refer to equilibrium as the senior-subordinated tranching equilibrium. This yields the following result (with formal conditions in the appendix).

Proposition 6. *The senior-subordinated tranching equilibrium is equivalent to equilibrium with complete debt collateralization. That is, there exists a bijective mapping of assets and prices from the debt collateralization equilibrium to the senior-subordinated tranching equilibrium such that the buyers of assets remain the same.*

While the result follows essentially from accounting, the result is important: tranching and debt collateralization have an essential equivalence in terms of the state-contingent promises they create to maximize collateral values.

In reality financial innovation includes forms of both tranching and debt collateralization. Subprime mortgage pools have been used to create tranches of different seniority. Each tranche of the asset-backed security ("ABS") pays different amounts depending on the aggregate value of the mortgage pool (i.e., in different states of the world). A typical ABS deal tranches a pool of mortgages into 4 or 5 rated bonds and a residual, or equity, tranche. These tranches (typically the mezzanine bonds) are then be pooled together to serve as collateral for a CDO, which would issue another 4-5 bonds. And the process continues as the tranches from the CDO are collateralized into a CDO-squared. Each stage includes both tranching and collateralization of existing debt securities. Because mortgage pools do contain idiosyncratic risk, pooling tranches together to diversify this risk is an important step of the securitization process.

This discussion highlights precisely some of the key differences between Propositions 5 and 6. Up to $N-2$ levels of debt collateralization are possible when level-0 contracts are non-contingent, but *no* levels of debt collateralization are possible when level-0 contracts are complete senior-subordinated tranches. Just as Arrow-Debreu securities are not meaningful collateral (Lemma 3), senior-subordinated tranches are not meaningful collateral for other senior-subordinated tranches. If senior-subordinated tranches are available to start, collateralization is redundant. Informational frictions, cross-netting frictions, or agency frictions requiring risk retention may limit contract contingencies. Any of these limitations will have implications for the levels of collateralization that would occur in equilibrium. The degree of debt collateralization is clearly endogenous, depending on the financial sector's ability to track and clear payments backed to the L-th degree and the need for diversification (or retention) at every level of pooling.

6 Conclusion

When agents have the ability to use risky debt backed by a risky asset as collateral for other financial promises, agents use exclusively maximal leverage in equilibrium. Debt collateralization expands the set of possible contingent payoffs in the economy, and maximal leverage maximizes the ability of assets to serve as collateral, and thus providing a way of stretching scarce collateral. This shift in the set of state-contingent payoffs traded in the economy decreases margins on the risky asset (increases leverage), decreases the risk premia for risky debt, and generally increases the price of the risky asset. Our results offer important empirical implications for economy-wide margins, risk-premia, and asset prices.

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Appendices for Online Publication

The appendix is organized as follows. Section A contains proofs for results. Section B contains additional analyses in the static model. Section C presents a dynamic analysis in a 3-period model. Section D presents empirical and testable implications based on the results in our static and dynamic analyses.

A Proofs

Proof of Lemma 1. Suppose for contradiction that there exists an h_i who prefers to hold the risky debt with some amount of leverage L, $0 \leq L < D$, less than the maximum. Since $L < D$ it is risk-free and thus $\hat{\pi}^L = L$. The marginal utilities from investing in j_M against promise L, from investing in j_M against promise D, and from holding risk-free assets are:

$$\text{debt with leverage L:} \quad \frac{(\gamma_U(h_i)+\gamma_M(h_i))(M-L)+\gamma_D(h_i)(D-L)}{\hat{\pi}_M - L} \tag{14}$$

$$\text{debt with leverage D:} \quad \frac{\gamma_U(h_i)(M-D)+\gamma_M(h_i)(M-D)}{\hat{\pi}_M - D} \tag{15}$$

$$\text{risk-free asset:} \quad 1. \tag{16}$$

Since by assumption h_i strictly prefers the first option, it must be the case that (14) $>$ (15) and (14) $>$ (16). That is, the investor is optimistic enough to prefer the risky debt to risk-free debt but not so optimistic as to want zero payoff in D. Hence,

$$\frac{(\gamma_U(h_i)+\gamma_M(h_i))(M-L)+\gamma_D(h_i)(D-L)}{\hat{\pi}_M - L} > \frac{\gamma_U(h_i)(M-D)+\gamma_M(h_i)(M-D)}{\hat{\pi}_M - D}, \tag{17}$$

$$\frac{(\gamma_U(h_i)+\gamma_M(h_i))(M-L)+(1-\gamma_U(h_i)-\gamma_M(h_i))(D-L)}{\hat{\pi}_M - L} > 1 \tag{18}$$

Simplifying 17, we obtain

$$\hat{\pi}_M - (\gamma_U(h_i)+\gamma_M(h_i))M - \gamma_D(h_i)D > 0 \implies \hat{\pi}_M > (\gamma_U(h_i)+\gamma_M(h_i))M + \gamma_D(h_i)D$$

Simplifying 18, we obtain

$$\hat{\pi}_M - \gamma_D(h_i)D - (\gamma_U(h_i)+\gamma_M(h_i))M < 0 \implies \hat{\pi}_M < \gamma_D(h_i)D + (\gamma_U(h_i)+\gamma_M(h_i))M$$

Note that the above gives us $\hat{\pi}_M > \hat{\pi}_M$. This is a contradiction so long as any of the inequalities are strict. Given our strict monotonicity assumptions on beliefs/marginal utilities, if the above set of inequalities are weak for any agent (i.e., equalities), then they are strict inequalities for every other agent. Thus, in equilibrium, all agents holding risky debt (but potentially a measure zero) will do so with maximal leverage. □

Proof of Proposition 1. Market clearing for the risk-free asset in the leverage and DC economies are given by

$$h_J(1+p) = 1 + \left(\frac{(h_M - h_D)(1+p)}{p-D}\right) D < 1+D, \qquad \hat{h}_J(1+\hat{p}) = 1+D$$

This implies that $\hat{h}_J(1+\hat{p}) > h_J(1+p)$ so either $\hat{h}_J > h_J$ or $\hat{p} > p$.

Suppose $\hat{h}_J > h_J$. The marginal buyer pricing the risky debt in the leverage economy has

$$\pi = \gamma_U(h_J)M + \gamma_M(h_J)M + \gamma_D(h_J)D$$

The marginal buyer in the DC economy has

$$\hat{\pi} - D = \gamma_U(\hat{h}_J)(M-D) + \gamma_M(\hat{h}_J)(M-D) \implies \hat{\pi} = \gamma_U(\hat{h}_J)M + \gamma_M(\hat{h}_J)M + \gamma_D(\hat{h}_J)D$$

So $\hat{h}_J > h_J \iff \hat{\pi} > \pi$. Now suppose for contradition that $\hat{p} > p$ but $\hat{\pi} < \pi$. Then,

$$\frac{\hat{p}-D}{\hat{p}-\hat{\pi}} < \frac{p-D}{p-\pi}$$

Consider the marginal buyer h_M in the leverage economy who is indifferent between buying the asset with high or low leverage. This agent is defined by

$$\frac{\gamma_U(h_M)(1-M)}{p-\pi} = \frac{\gamma_U(h_M)(1-D) + \gamma_M(h_M)(M-D)}{p-D} \implies \frac{\hat{p}-D}{\hat{p}-\hat{\pi}} < \frac{\gamma_U(h_M)(1-D) + \gamma_M(h_M)(M-D)}{\gamma_U(h_M)(1-M)}$$

So, under the prices in the DC, this marginal buyer would strictly prefer to NOT buy the asset with high leverage, implying $\hat{h}_M > h_M$. Now, combining this with the market clearing for the risky asset in the leverage and DC economies imply

$$\frac{(1-h_M)(1+\hat{p})}{\hat{p}-\hat{\pi}} > \frac{(1-\hat{h}_M)(1+\hat{p})}{\hat{p}-\hat{\pi}} = 1 > \frac{(1-h_M)(1+p)}{p-\pi}$$

where the first inequality follows from $\hat{h}_M > h_M$, the equality follows from the market clearing in the DC economy, and the last inequality follows from market clearing in the leverage economy. Simplifying,

$$\frac{(1+\hat{p})}{\hat{p}-\hat{\pi}} > \frac{(1+p)}{p-\pi}$$

But this contradicts $\hat{p} > p$ and $\hat{\pi} < \pi$. $\square$

Proof of Lemma 2. In equilibrium, each unit of the leveraged risky asset must be backed by one unit of debt, either risk-free or risky and leveraged. By previous lemma, we have shown that all agents holding risky debt will be maximally leveraged. We therefore know that agents holding the risky asset must either be leveraged against state D or state M and not something in-between.

Suppose for contradiction that there is some agent h_i who prefers to hold the risky asset leveraged against state D and the price of debt is D. That is, the investor is optimistic enough to prefer the risky asset with low leverage to the leveraged risky debt, but not so optimistic as to want to maximally leverage the asset and get zero payoff in M. Note that returns from investment strategies are:

$$\text{Marginal utility from risky asset with debt } D: \quad \frac{\gamma_U(h_i)(1-D)+\gamma_M(h_i)(M-D)}{\hat{p}-D} \tag{19}$$

$$\text{Marginal utility from risky asset with debt } M: \quad \frac{\gamma_U(h_i)(1-M)}{\hat{p}-\hat{\pi}_M} \tag{20}$$

$$\text{Marginal utility from risky debt}: \quad \frac{(\gamma_U(h_i)+\gamma_M(h_i))(M-D)}{\hat{\pi}_M-D} \tag{21}$$

Since by assumption h_i strictly prefers the first option, it must be the case that (19) $>$ (20) and (19) $>$ (21). That is,

$$\frac{\gamma_U(h_i)(1-D)+\gamma_M(h_i)(M-D)}{\hat{p}-D} > \frac{\gamma_U(h_i)(1-M)}{\hat{p}-\hat{\pi}_M} \tag{22}$$

$$\frac{\gamma_U(h_i)(1-D)+\gamma_M(h_i)(M-D)}{\hat{p}-D} > \frac{(\gamma_U(h_i)+\gamma_M(h_i))(M-D)}{\hat{\pi}_M-D} \tag{23}$$

Simplifying 22, we obtain that

$$\gamma_M(h_i)(M\hat{p}+D\hat{\pi}_M-M\hat{\pi}_M-D\hat{p}) > \gamma_U(h_i)(\hat{\pi}_M+MD+D\hat{p}-D-D\hat{\pi}_M-M\hat{p})$$

Simplifying 23, we obtain

$$\gamma_U(h_i)(\hat{\pi}_M+MD+D\hat{p}-D-D\hat{\pi}_M-M\hat{p})+\gamma_M(h_i)(M\hat{\pi}_M+D\hat{p}-D\hat{\pi}_M-M\hat{p}) > 0$$

For convenience, let

$$\alpha := \gamma_M(h_i)(M\hat{p} + D\hat{\pi}_M - M\hat{\pi}_M - D\hat{p}), \quad \beta := \gamma_U(h_i)(\hat{\pi}_M + MD + D\hat{p} - D - d\hat{\pi}_M - M\hat{p})$$

Notice that the above two equations simplify to $\alpha > \beta$ and $\beta - \alpha > 0$. Again, given our monotonicity assumptions on beliefs/utilities, these inequalities must be strict for all except a measure zero of investors. Hence, this is clearly a contradiction since we cannot have both $\alpha > \beta$ and $\beta > \alpha$. Thus, all investors holding the risky asset will be maximally leveraged against state M. □

Proof of Proposition 3. First, suppose for contradiction that $\hat{p} \leq p$. From market clearing for safe assets, we have

$$h_J(1+p) \leq 1 + D = \hat{h}_J(1+\hat{p}),$$

and so $\hat{p} \leq p$ implies $\hat{h}_J > h_J$, which also means that $\hat{\pi} > \pi$. If $\hat{p} \leq p$ then we must also have $\hat{h}_M < h_M$, so that altogether $h_J < \hat{h}_J < \hat{h}_M < h_M$.

Second, using equations (10) and (11) for the marginal buyers, we can write the asset price with debt collateralization as

$$\hat{p} = \hat{\pi} + (1-M)\frac{\gamma_U(\hat{h}_M)}{\gamma_U(\hat{h}_M) + \gamma_M(\hat{h}_M)}\left(\gamma_U(\hat{h}_J) + \gamma_M(\hat{h}_J)\right). \tag{24}$$

Rearranging equation (5) for the marginal investor h_M in the leverage economy, we have

$$\begin{aligned}
\frac{\gamma_U(1-M)}{p-\pi} &= \frac{\gamma_U(1-M) + (\gamma_U + \gamma_D)(M-D)}{p-D}, \\
(p-D)\gamma_U(1-M) &= (p-\pi)\gamma_U(1-M) + (p-\pi)(\gamma_U + \gamma_D)(M-D), \\
p - \pi &= \frac{\gamma_U(h_M)}{\gamma_U(h_M) + \gamma_M(h_M)}\left(\frac{1-M}{M-D}\right)(\pi - D),
\end{aligned}$$

which together with equation (7) gives

$$p = \pi + (1-M)\frac{\gamma_U(h_M)}{\gamma_U(h_M) + \gamma_M(h_M)}(\gamma_U(h_J) + \gamma_M(h_J)). \tag{25}$$

Let $f_U(h) = \frac{\gamma_U(h)}{\gamma_U(h) + \gamma_M(h)}$ and $f_M(h) = \gamma_U(h) + \gamma_M(h)$ denote the hazard ratios for states U and

M. Then we can combine equations (24) and (25) to

$$\hat{p} - p = (M - D)(f_M(\hat{h}_J) - f_M(h_J)) + (1 - M)\left(f_U(\hat{h}_M)f_M(\hat{h}_J) - f_U(h_M)f_M(h_J)\right). \tag{26}$$

By assumption A1, $f_U(h)$ and $f_M(h)$ are increasing so that $f_M(\hat{h}_J) > f_M(h_J)$ and $f_U(h_M) > f_U(\hat{h}_M)$. To show that $\hat{p} > p$, therefore, requires showing that

$$(M - D)(f_M(\hat{h}_J) - f_M(h_J)) > (1 - M)\left(f_U(h_M) - f_U(\hat{h}_M)\right).$$

By our assumptions on the concavity of the hazard rates, and the ordering of the marginal buyers, it is sufficient to show that

$$(M - D)(\hat{h}_J - h_J) > (1 - M)\left(h_M - \hat{h}_M\right)$$

We proceed by providing bounds on $\hat{h}_J - h_J$ and $h_M - \hat{h}_M$. From market clearing for risk-free assets, we have

$$\hat{h}_J - h_J > \frac{\eta D}{1 + \hat{p}},$$

and from market clearing for risky assets, we have

$$h_M - \hat{h}_M < \frac{p - \eta(p - \pi) - \hat{\pi}}{1 + \hat{p}} < \frac{(p - \pi)(1 - \eta)}{1 + \hat{p}}.$$

By assumption A2 and A3, we therefore have:

$$\begin{aligned} \frac{(h_M - \hat{h}_M)}{(p - \pi)(1 - \eta)} &< \frac{(\hat{h}_J - h_J)}{\eta D} \\ \implies \frac{(h_M - \hat{h}_M)}{(\hat{h}_J - h_J)} &< \frac{(p - \pi)(1 - \eta)}{\eta D} \end{aligned}$$

Note that we have $p - \pi < 1 - M$. Combining this with our assumption on η, we get

$$\frac{(p - \pi)(1 - \eta)}{\eta D} < \frac{(1 - M)(1 - \eta)}{\eta D} < \frac{M - D}{1 - M}$$

which proves that $\hat{p} > p$ from our earlier inequality.

□

Proof of Proposition 4. Denote the prices of Y and Z by $\hat{p}$ and p. Denote risky debt promising M backed by Y and Z by $\hat{j}_M$ and j_M, and denote their prices by $\hat{\pi}_M$ and π_M respectively. Since $\hat{j}_M$ can be used as collateral while j_M cannot, it must be that $\hat{\pi}_M > \pi_M$ in equilibrium. Note that buying Y and promising M yields the same payoffs as buying Z and promising M—namely, $1-M$ in state U and zero otherwise. In there is a marginal buyer $\hat{h}_M$ willing to hold an Arrow U. Since both investment strategies yield identical payoffs, they must have the same prices, i.e., $\hat{p}-\hat{\pi}_M = p-\pi_M$. Using $\hat{\pi}_M > \pi_M$ we have that $\hat{p} > p$. □

Lemma 3. *Arrow-Debreu securities cannot serve as meaningful collateral for promises: making a promise backed by an Arrow-Debreu security is equivalent to merely selling a portion of the Arrow-Debreu security.*

This lemma states that using Arrow-Debreu securities as collateral does nothing to change the set of contracts or promises available to agents.

Proof of Lemma 3. First, consider a contract with promise $(j_1, j_2, \ldots, j_N)$, backed by a single state-n Arrow-Debreu security (paying 1 in state n) as collateral. Such a contract would deliver zero in every state $s \neq n$ since the collateral, an Arrow-Debreu security, pays zero in those states. The contract would deliver $\min(1, j_n)$ in state n, thus leaving the issuer of the contract with $\max(0, 1-j_n)$ in state n. Thus, issuing this contract is equivalent to selling $\min(1, j_n)$ units of the original Arrow-Debreu security. Second, generalizing this argument, allowing multiple Arrow-Debreu securities to act as collateral to back a promise is equivalent to selling $\min(1, j_n)$ units of security n, for each security n used as collateral. □

A.1 Proof of Proposition 5

We proceed with the proof by induction, and break the proof into the following two parts. (1) The equilibrium at the first level of collateralization. (2) Equilibrium at the L^{th} level of collateralization.

A.1.1 Equilibrium at the first level of collateralization

To prove the base case, we will show that agents will hold one of the following assets in equilibrium: (i) Y/j_i, where $2 \leq i \leq N-1$; (ii) j_i/j_j^1, where $2 \leq i \leq N-1$, $1 \leq j < i$; (iii) j_j^1, where $1 \leq j \leq N-2$. That is, agents will hold the risky asset, Y, leveraged against states $S_2, \ldots, S_{N-1}$; the risky debt contract, j_n (backed by the risky asset), leveraged against some state S_j with $j < n$; or a debt security j_k^1 ($1 \leq k \leq N-2$), which is backed by risky debt.

Note that the j_k^1 contracts are just securities created in the first round of debt collateralization. Apart from this, the only difference from equilibrium with leverage is that all risky debt contracts backed directly by Y are now bought with leverage, and no agent holds Y, leveraged against S_1. Thus, to prove the base case, it suffices to prove the following two lemmas:

Lemma 4. *In the first level of collateralization, no agent will hold Y/j_1.*

Proof of Lemma 4. The intuition for this lemma is nearly identical to the intuition for lemmas 1 and 2. Suppose for contradiction that some agent, h prefers to hold Y/j_1. Then, it must be the case that the expected return of holding Y/j_1 is greater than holding Y/j_{N-1}. This implies that we must have

$$\frac{\sum_{i=1}^{N} \gamma_i(h)(s_i - s_1)}{\hat{p} - \hat{\pi}_1} > \frac{\gamma_N(h)(s_N - s_{N-1})}{\hat{p} - \hat{\pi}_{N-1}} \tag{27}$$

Rearranging and simplifying, we obtain

$$\sum_{i=1}^{N-1} \gamma_i(h)(s_i - s_1)(\hat{p} - \hat{\pi}_{N-1}) + \gamma_N(h)[(s_N - s_1)(\hat{p} - \hat{\pi}_{N-1}) - (s_1 - s_{N-1})(\hat{p} - \hat{\pi}_{N-1})] > 0. \tag{28}$$

Furthermore, we know that the expected return of holding Y/j_1 is greater than holding j_{N-1}/j_1^1, which gives

$$\frac{\sum_{i=1}^{N} \gamma_i(h)(s_i - s_1)}{\hat{p} - \hat{\pi}_1} > \frac{\sum_{i=1}^{N-1} \gamma_i(h)(s_i - s_1) + \gamma_N(h)(s_{N-1} - s_1)}{\hat{\pi}_{N-1} - \hat{\pi}_1^1} \tag{29}$$

Rearranging and simplifying, we obtain

$$\sum_{i=1}^{N-1} \gamma_i(h)(s_i - s_1)(\hat{\pi}_{N-1} - \hat{p}) + \gamma_N(h)[(s_N - s_1)(\hat{\pi}_{N-1} - \hat{\pi}_1^1) - (s_{N-1} - s_1)(\hat{p} - \hat{\pi}_1^1)] > 0 \tag{30}$$

A quick check will assure readers that equations (28) and (30) provide a contradiction because the expressions to the left of the $>$ sign are additive inverses and therefore cannot be both strictly greater than 0.

$\square$

Lemma 5. *In the first level of collateralization, any agent buying the promise j_j with $j > 1$ will also sell a promise j_k^1 with $1 \leq k < j$.*

Proof of Lemma 5. Now suppose that some agent h prefers to hold a promise j_ℓ with $j > 1$, but not sell a debt security. Then, it must be the case that the expected return of holding j_ℓ^0 is greater than the expected return of holding j_ℓ^0/j_1^1. That is,

$$\frac{\sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) + \sum_{i=\ell}^{N} \gamma_i(h)(s_\ell)}{\hat{\pi}_\ell^0} > \frac{\sum_{i=1}^{\ell-1} \gamma_i(h)(s_i - s_1) + \sum_{i=\ell}^{N} \gamma_i(h)(s_\ell - s_1)}{\hat{\pi}_\ell^0 - \hat{\pi}_1^1} \tag{31}$$

Note that $\hat{\pi}_1^1 = s_1$, since j_1^1 promises s_1 in all states and is therefore risk-free debt. Thus, rearranging and simplifying 31, we obtain

$$s_1 \left(\hat{\pi}_\ell^0 - \sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) - \sum_{i=\ell}^{N} \gamma_i(h)(s_\ell) \right) > 0 \implies \hat{\pi}_\ell^0 - \sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) - \sum_{i=\ell}^{N} \gamma_i(h)(s_\ell) > 0$$

We also know that the expected return of holding j_ℓ^0 must be greater than holding the risk-free asset. Consequently,

$$\frac{\sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) + \sum_{i=\ell}^{N} \gamma_i(h)(s_\ell)}{\hat{\pi}_\ell^0} > 1 \tag{32}$$

Rearranging and simplifying 32, we obtain

$$\sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) + \sum_{i=\ell}^{N} \gamma_i(h)(s_\ell) > \hat{\pi}_\ell^0 \implies \sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) + \sum_{i=\ell}^{N} \gamma_i(h)(s_\ell) - \hat{\pi}_\ell^0 > 0$$

The above clearly cannot happen because we have that the two equations are additive inverses of each other and therefore cannot both be strictly greater than 0. Thus, no agent holding a risky debt contract will prefer to hold the contract unleveraged.

□

A.1.2 Induction Hypothesis

We now assume that the proposition holds for all levels of collateralization T with $T < L$. Specifically, this means that the proposition holds with $L-1$ levels of collateralization. Looking at this level, we have that agents will hold one of the following assets in equilibrium: (i) Y/j_i, with $L-1 < i \leq N-1$, (ii) j_j^l/j_k^{l+1}, with $0 \leq l < L-1$, $L-1-l < j < N-l$, and $L-1-l \leq k < j$, (iii) j_ℓ^{L-1}, with $1 \leq \ell < N-L+1$.

A.1.3 Equilibrium at the L^{th} level of collateralization

At the L^{th} level of collateralization, we allow all agents holding j_ℓ^{L-1} (with $1 \leq \ell < N-L+1$) to sell the promise $j_n^L(j_\ell^{L-1}) = (s_n, j_\ell^{L-1})$ where $1 \leq p < \ell$.

We will prove the following: (i) No agent holds Y/j_M. This implies that the asset j_M/j_{M-1}^1 no longer exists; (ii) No agent holding the debt security j_ℓ^{L-1} with $1 < \ell < N-L+1$ will do so without leveraged; (iii) No agent will hold j_j^l/j_{L-1-l}^{l+1}, for $0 \leq l < L-1$ and $L-1-l < j < N-l$. This implies that all $j_{L-1-l}^{l+1}/A_{L-2-l}^{l+2}$ no longer exist in equilibrium.

Note that the above are the changes between the $L-1$ and L^{th} levels of collateralization given by the proposition. We break up the proof into three lemmas, corresponding to the three claims listed above.

Lemma 6. *At the L^{th} level of collateralization, no agent will hold Y/j_L.*

Proof of Lemma 6. Suppose for contradiction that some investor h wants to hold Y/j_L. Then the leveraged expected return to this asset must be strictly greater than the expected return to holding Y/j_{N-1}. This means that

$$\frac{\sum_{i=L}^{N} \gamma_i(h)(s_i - s_L)}{\hat{p} - \hat{\pi}_L} > \frac{\gamma_N(h)(s_N - s_{N-1})}{\hat{p} - \hat{\pi}_{N-1}}. \tag{33}$$

Rearranging and simplifying the above, we obtain

$$\sum_{i=L}^{N-1} \gamma_i(h)(s_i - s_L)(\hat{p} - \hat{\pi}_{N-1}) + [\gamma_N(h)(s_N - s_L)(\hat{p} - \hat{\pi}_{N-1}) - (s_N - s_{N-1})(\hat{p} - \hat{\pi}_L)] > 0 \tag{34}$$

Additionally, holding Y/j_L must have a higher expected return than holding j_{N-1}/j_L^1. Note that at the L^{th} level of collateralization, both j_L and j_L^1 are fully securitized so they have the same price. That is $\hat{\pi}_L = \hat{\pi}_L^1$.

$$\frac{\sum_{i=L}^{N} \gamma_i(h)(s_i - s_L)}{\hat{p} - \hat{\pi}_L} > \frac{\sum_{i=L}^{N-1} \gamma_i(h)(s_i - s_L) + \gamma_N(h)(s_{N-1} - s_L)}{\hat{\pi}_{N-1} - \hat{\pi}_L} \tag{35}$$

Rearranging and simplifying, we have

$$\sum_{i=L}^{N-1} \gamma_i(h)(s_i - s_L)(\hat{\pi}_{N-1} - \hat{p}) + \gamma_N(h)[(s_N - s_L)(\hat{\pi}_{N-1} - \hat{\pi}_L) - (s_{N-1} - s_L)(\hat{p} - \hat{\pi}_L)] > 0 \tag{36}$$

The expressions on the left side of the $>$ sign in equations (34) and (36) are additive inverses, and therefore cannot both be strictly greater than 0. Thus, we have a contradiction and no agent will hold Y/j_L.

□

Lemma 7. *At the L^{th} level of collateralization, every agent holding j_i^{L-1} with $1 < i < N-L+1$ will sell a promise j_m^L, where $1 \leq m < i$.*

Proof of Lemma 7. Suppose that there exists an agent, h, holding j_i^{L-1} with $1 < \ell < N-L+1$ and prefers not to sell any promises. Then, it must be the case that the expected return of holding j_ℓ^{L-1} is greater than the expected return of holding j_ℓ^{L-1}/j_1^L. That is,

$$\frac{\sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) + \sum_{i=\ell}^{N} \gamma_i(h)(s_\ell)}{\hat{\pi}_\ell^{L-1}} > \frac{\sum_{i=1}^{\ell-1} \gamma_i(h)(s_i - s_1) + \sum_{i=\ell}^{N} \gamma_i(h)(s_\ell - s_1)}{\hat{\pi}_\ell^{L-1} - \hat{\pi}_1^L} \tag{37}$$

Note that $\hat{\pi}_1^L = s_1$, since j_1^L promises s_1 in all states and is therefore risk-free debt. Thus, rearranging and simplifying 37, we obtain

$$s_1 \left(\hat{\pi}_\ell^{L-1} - \sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) - \sum_{i=\ell}^{N} \gamma_i(h)(s_\ell) \right) > 0 \implies \hat{\pi}_\ell^{L-1} - \sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) - \sum_{i=\ell}^{N} \gamma_i(h)(s_\ell) > 0$$

We also know that the expected return of holding j_ℓ^{L-1} must be greater than holding the risk-free asset. Consequently,

$$\frac{\sum_{i=1}^{\ell-1}\gamma_i(h)(s_i)+\sum_{i=\ell}^{N}\gamma_i(h)(s_\ell)}{\hat{\pi}_\ell^{L-1}}>1 \tag{38}$$

Rearranging and simplifying 38, we obtain

$$\sum_{i=1}^{\ell-1}\gamma_i(h)(s_i)+\sum_{i=\ell}^{N}\gamma_i(h)(s_\ell)>\hat{\pi}_\ell^{L-1} \implies \sum_{i=1}^{\ell-1}\gamma_i(h)(s_i)+\sum_{i=\ell}^{N}\gamma_i(h)(s_\ell)-\hat{\pi}_\ell^{L-1}>0$$

The above clearly cannot happen because we have that the two equations are additive inverses of each other and therefore cannot both be strictly greater than 0. Thus, no agent holding a risky debt contract will prefer to hold the contract unleveraged at the L^{th} level of collateralization.

□

Lemma 8. *At the L^{th} level of collateralization, for all $0 \leq l < L-1$, no agent will hold j_k^l/j_{L-1-l}^{l+1}, where $L-l \leq k < N-l$.*

Proof of Lemma 8. Suppose for contradiction that there exist some agent h who prefers to be in the position stated above. Then, it must be the case that the is greater than the expected return of holding j_k^l/j_{L-l}^{l+1}. Thus,

$$\frac{\sum_{i=L-l}^{k-1}\gamma_i(h)(s_i-s_{L-1-l})+\sum_{i=k}^{N}\gamma_i(h)(s_k-s_{L-1-l})}{\hat{\pi}_k^l-\hat{\pi}_{L-1-l}^{l+1}}>\frac{\sum_{i=L-l}^{k-1}\gamma_i(h)(s_i-s_{L-l})+\sum_{i=k}^{N}\gamma_i(h)(s_k-s_{L-l})}{\hat{\pi}_k^l-\hat{\pi}_{L-l}^{l+1}}. \tag{39}$$

We rearrange and simplify the above to obtain

$$\sum_{i=L-l}^{k-1}\gamma_i(h)\Omega+\sum_{i=k}^{N}\gamma_i(h)\Psi>0 \tag{40}$$

where

$$\Omega := (s_i-s_{L-1-l})(\hat{\pi}_k^l-\hat{\pi}_{L-l}^{l+1})-(s_i-s_{L-l})(\hat{\pi}_k^l-\hat{\pi}_{L-1-l}^{l+1}) \tag{41}$$

and

$$\Psi := (s_k - s_{L-1-l})(\hat{\pi}_k^l - \hat{\pi}_{L-l}^{l+1}) - (s_k - s_{L-l})(\hat{\pi}_k^l - \hat{\pi}_{L-1-l}^{l+1}) \tag{42}$$

Furthermore, it must also be the case that the expected return of holding j_k^l/j_{L-1-l}^{l+1} is greater than the expected return from holding $j_{L-l}^{l+1}/j_{L-1-l}^{l+2}$. It is important to note here that the price of j_{L-1-l}^{l+2} is the same as the price of j_{L-1-l}^{l+1} because at the L^{th} level of collateralization, both have been securitized to the exact same degree, so the two have the same value. Thus, abusing notation, we can write $\hat{\pi}_{L-1-l}^{l+2} = \hat{\pi}_{L-1-l}^{l+1}$. This give us

$$\frac{\sum_{i=L-l}^{k-1} \gamma_i(h)(s_i - s_{L-1-l}) + \sum_{i=k}^{N} \gamma_i(h)(s_k - s_{L-1-l})}{\hat{\pi}_k^l - \hat{\pi}_{L-1-l}^{l+1}} > \frac{\sum_{i=L-l}^{N} \gamma_i(h)(s_{L-l} - s_{L-1-l})}{\hat{\pi}_{L-l}^{l+1} - \hat{\pi}_{L-1-l}^{l+1}} \tag{43}$$

Rearranging and simplifying the above inequality, we obtain

$$\sum_{L-l}^{k-1} \gamma_i(h)\Upsilon + \sum_{i=k}^{N} \gamma_i(h)\Phi > 0, \tag{44}$$

where

$$\Upsilon := (s_i - s_{L-1-l})(\hat{\pi}_{L-l}^{l+1} - \hat{\pi}_{L-1-l}^{l+1}) - (s_{L-l} - s_{L-1-l})(\hat{\pi}_k^l - \hat{\pi}_{L-1-l}^{l+1}), \tag{45}$$

and

$$\Phi := (s_k - s_{L-1-l})(\hat{\pi}_{L-l}^{l+1} - \hat{\pi}_{L-1-l}^{l+1}) - (s_{L-l} - s_{L-1-l})(\hat{\pi}_k^l - \hat{\pi}_{L-1-l}^{l+1}). \tag{46}$$

A quick check will assure the readers that $\Upsilon = -\Omega, \Phi = -\Psi$, a contradiction, meaning equations (40) and (44) cannot both be true. Thus, no agent will hold j_k^l/j_{L-1-l}^{l+1}, where $0 \leq l < L-1$ and $L-l \leq k < N-l$.

□

A.2 Proof of Proposition 6

The full statement of Proposition 6: *The senior-subordinated tranching equilibrium is equivalent to equilibrium with complete debt collateralization. That is, there exists a bijective mapping of assets and prices from the debt collateralization equilibrium to the senior-subordinated tranching equilibrium such that the buyers of assets remain the same. Specifically,*

1. *Any agent buying Y/j_{N-1} (collateralization) will buy T_N (senior-subordinated tranching).*
2. *Any agent holding j_n^l/j_{n-1}^{l+1} with $N > n > 1$ (collateralization) will buy T_n (senior-subordinated tranching).*
3. *Any agent holding j_1^L (collateralization) will buy T_1 (down-traching).*
4. *Letting q_N denote the price of T_N, senior-subordinated tranching equilibrium will have (i) $q_N = \hat{p} - \hat{\pi}_{N-1}$, (ii) $q_n = \hat{\pi}_n^{N-n-1} - \hat{\pi}_{n-1}^{N-n}$, (iii) $q_1 = \hat{\pi}_1^{N-2} = s_1$, where $\hat{p}$ and $\hat{\pi}_j^k$, are the equilibrium prices for the asset and debt securities in the complete collateralization equilibrium, respectively.*

Proof. This follows because the expected return of holding T_n in the senior-subordinated tranching equilibrium is the same as holding j_n^l/j_{n-1}^{l+1} in the collateralization equilibrium, when $N > n > 1$. Similarly, the expected return of Y/j_{N-1} is identical to that of T_N; the expected return to holding q_1 is exactly the return of j_1^{N-2} $\square$

B Extensions in the Static Model

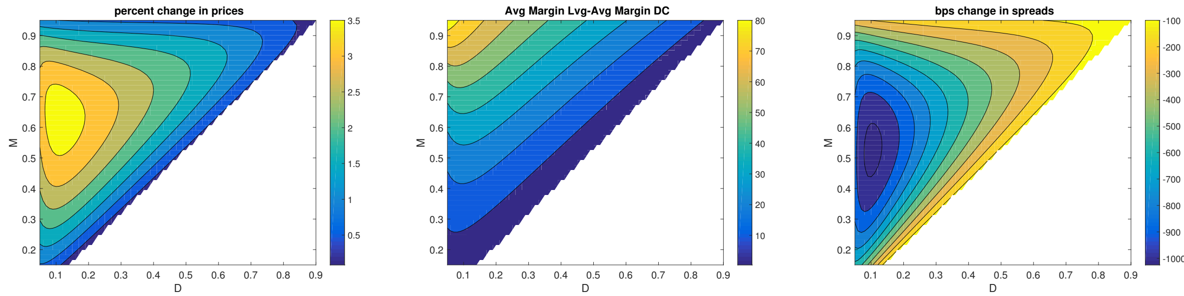
B.1 Parameter Robustness and Comparative Statics

Robustly, debt collateralization decreases average margins by shifting agents to high-leverage contracts, decreases margins on high-leverage contracts, decreases spreads, and increases the asset price. We consider a wide range of payoffs pairs, with $M \in (0.2, 0.95)$ and $D \in (0.05, 0.95 * M)$, and a broad range of belief parameterizations for $\zeta = 1/3, 1, 3$. Our results—that debt collateralization decreases margins, increases prices, and decreases spreads—hold for every combination. (Our results extend for ζ outside the range, but we omit the figures.)

Figures 4-6 display how moving from leverage to debt collateralization affects prices, the economy-wide average margin, and the interest rate i_M on the risky debt. The figures plot these changes for pairs of payoffs (M,D). The figures respectively use a belief parameterization with $\zeta = 3, 1, \frac{1}{3}$. For each of these cases, for every pair (M,D), debt collateralization increases asset prices, decreases economy-wide average margins, and decreases the interest rate on risky debt. The effects on prices and spreads are larger when ζ is larger.

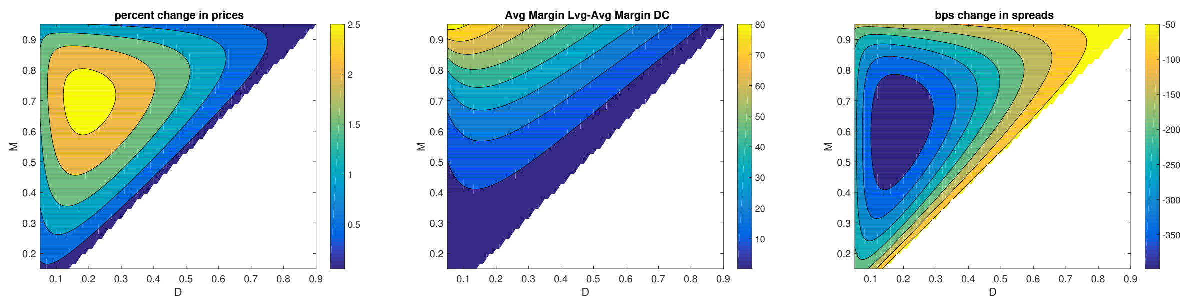
Marginal Utilities/Beliefs Because it is the least intuitive of the parameters, we discuss the role of the marginal utilities/belief parameterization in greater detail. The parameter ζ determines the relative frequency of optimists and pessimists in the economy; equivalently, the frequency of pessimists can be interpreted as the relative demand for assets that pay in bad states (negative-beta assets), perhaps from hedging needs or from risk aversion. High ζ corresponds to relatively more pessimists and low ζ to more optimists (with $\zeta > 1$, γ's are convex; $\zeta < 1$, concave). Increasing ζ, has the following consequences: (i) in both financial environments risky spreads increase and risky and safe margins decrease; (ii) moving from leverage to debt collateralization results in a bigger change in spreads; (iii) average margins under leverage converge to the safe margin (for low ζ (many optimists), more agents use risky margins); (iv) moving from leverage to debt collateralization, margins decrease by a greater amount; (v) moving from leverage to debt collateralization, the percent change in p increases (nonlinearly) with ζ so that p increases by more when ζ is high.

Comparative Statics Additionally, the model provides several interesting comparative statics for each parameter. Figure 7 plots comparative statics for prices, spreads, and margins. Figures 7a and 7b show how varying the down-payoff D affects the change in prices and spreads. We fix $M = 0.9$ (results are robust to varying M) and plot results for three values of ζ. Debt collateralization always increases asset prices and decreases debt spreads, and the effects are greatest when D is small and when ζ is large. The effect from D is intuitive: as D increases, the risk-free promise becomes closer to the risky promise, and thus the value from using the risky promise as collateral diminishes (the risk-free debt is already good for leverage). Figure 7c shows how varying the middle payoff M affects changes in economy-wide average margins, setting $D = 0.8$ (again, results are robust to varying D). As M increases the decrease in average margins when moving to debt collateralization becomes much greater, precisely because the margin on the risky debt decreases as M increases, and average margins decrease because investors shift toward using the risky debt instead of the risk-free debt.



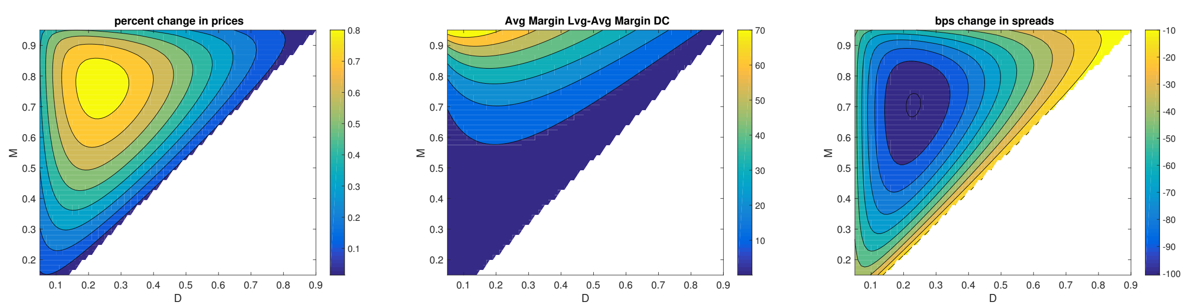
(a) Percent change in price. (b) Change in average margin. (c) Change in debt spread i_M.

Figure 4: Equilibrium changes moving from leverage to debt collateralization. Beliefs exponent $\zeta = 3$.



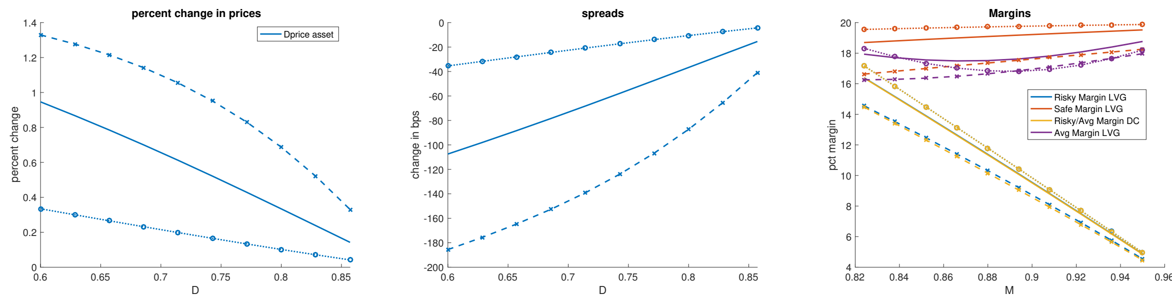
(a) Percent change in price. (b) Change in average margin. (c) Change in debt spread i_M.

Figure 5: Equilibrium changes moving from leverage to debt collateralization. Beliefs exponent $\zeta = 1$.



(a) Percent change in price. (b) Change in average margin. (c) Change in debt spread i_M.

Figure 6: Equilibrium changes moving from leverage to debt collateralization. Beliefs exponent $\zeta = 1/3$.



(a) Percent change in price, varying D; $M = 0.9$.

(b) Change in debt spread i_M, varying D; $M = 0.9$.

(c) Change in average margin, varying M; $D = 0.8$.

Figure 7: Comparative statics. Plain line is $\zeta = 1$; '- -x' is $\zeta = 3$; ':x' is $\zeta = 1/3$.

B.2 Debt Collateralization Does Not Necessarily Increase Prices

We demonstrate that prices do not necessarily increase with the introduction of securitization when beliefs are weakly monotonic. In this example, the beliefs of the marginal agents under each regime are very different—there is a discontinuous jump in beliefs. In contrast, in the examples in the robustness exercises with beliefs defined by $\gamma = h^{\zeta}$, beliefs vary smoothly (and not very much) over the relevant range where marginal buyers may fall.

As before, we let $M = 0.3$ and $D = 0.1$, be the payouts of asset Y in states M and D. We also define the following marginal agents: $h_1 = 0.6$, $h_2 = 0.65$, $h_3 = 0.69$. Beliefs are given as follows:

For $h \leq h_1$: $\gamma_U(h) = 1-(1-h_1)^2$, $\gamma_M(h) = h_1(1-h_1)^2$, $\gamma_D(h) = (1-h_1)^3$.

For $h \in (h_1, h_2)$: $\gamma_U(h) = 1-(1-h)^2$, $\gamma_M(h) = h(1-h)^2$, $\gamma_D(h) = (1-h)^3$.

For $h \in [h_2, h_3)$: $\gamma_U(h) = 1-(1-h_2)^2$, $\gamma_M(h) = h_2(1-h_2)^2$, $\gamma_D(h) = (1-h_2)^3$.

Finally, for $h > h_3$: $\gamma_U(h) = 1-(1-(h-(h_3-h_2)))^2$, $\gamma_M(h) = (h-(h_3-h_2))(1-(h-(h_3-h_2)))^2$, $\gamma_D(h) = (1-(h-(h_3-h_2)))^3$.

Equilibrium in the economy with leverage has price $p = 0.894$ while equilibrium in the debt collateralization economy has price $\hat{p} = 0.888$.[8] Introducing debt collateralization in this case causes the price to *decrease*. The reason is that the leveraged return on debt has increased sufficiently, which increases the required return for investing in the risky asset, decreasing its price.

[8] Furthermore, $h_M = 0.715, h_D = 0.669, h_J = 0.534, \pi_M = 0.287$ and $\hat{h_M} = 0.682, \hat{h_J} = 0.583, \hat{\pi_M} = 0.287$.

B.3 A Numerical Example With 3 Agents

We provide an example with 3 agents (not the only possible one) that replicates the numerical example with a continuum of agents. There are 3 agents with log-utility over consumption in period-1, $u_i(c) = \log(c)$, and the three states be equiprobable. Agents are endowed with units of the safe asset X and the risky asset Y, and endowments are as follows: $x^1 = 0.212$, $x^2 = 0.185$, and $x^3 = 0.603$ units of the safe asset X, and the risky asset Y is endowed entirely to agent 1, $y_0^1 = 1$. Agents have future endowments given by: $e^1 = (14.41, 38.18, 100)$, $e^2 = (100, 1, 100)$, and $e^3 = (3.25, 66.67, 100)$. Assets are priced according to the standard marginal analysis using the marginal utilities of the buyers. Two contracts, j_M and j_D, continue to be traded in equilibrium.

Given these parameters, in equilibrium with leverage, agent 3 buys all of the risky asset and issues contracts promising M and D; agents 2 buys risky contract M; and agent 1 buys contracts M and D and all of the safe asset X. In equilibrium with debt collateralization, agent 3 buys the asset leveraged with M; agent 2 buys the risky debt leveraged with D; agent 1 buys safe assets. Coincidentally, the asset prices p and $\hat{p}$ and debt prices π_M and $\hat{\pi}_M$ are exactly the same as in the economy in Section 3.

C Price Volatility in a Dynamic Model

In this section we examine how debt collateralization affects volatility and default. Geanakoplos (2003, 2010) demonstrate that using an asset as collateral creates a "Leverage Cycle" in which asset prices become more volatile because of fluctuations in the asset's collateral value and the distribution of investors' wealth.

C.1 Setup in the Dynamic Model

We consider a dynamic variation of the model in Section 3 with three periods, $t = 0, 1, 2$ following Geanakoplos (2003, 2010). Uncertainty in the payoffs of Y is represented by a tree $S = \{0, U, M, D, UU, MU, MD, DU, DD\}$, illustrated in Figure 8. The asset pays only at $t = 2$ with payoffs d_s^Y. To simplify, we will normalize the asset payoffs so that $d_{UU}^Y = d_{MU}^Y = d_{DU}^Y = 1$. Thus the possible "down payoffs" of the asset are d_{MD}^Y and $d_{DD}^Y < d_{MD}^Y$. In other words, the payoff tree is binary at $t = 1$ with a worse possible realization at state M than at D, and at $t = 0$ there is uncertainty about what the minimum possible asset payoff will be.

The risky asset Y has price p_0 at $t = 0$ and prices p_M and p_D in states M and D in $t = 1$. (In state U the price is trivially 1.) Just as before, we first look at an economy where leverage is the only financial innovation and then move on to explore the consequences of debt collateralization. Endogenously, all financial contracts traded in equilibrium are one-period contracts.

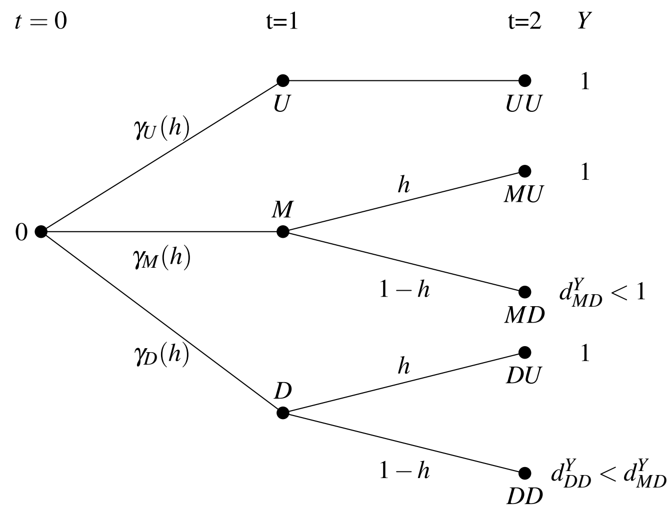


Figure 8: Payoff tree for risky asset in dynamic three-state model.

C.2 The Dynamic Economy with Leverage

With leverage, the dynamic equilibrium is essentially different from the static equilibrium because of the interaction between prices and leverage across time. However, the equilibrium regimes in each state resemble the equilibrium regime in the static economy of Section 3. The dynamic equilibrium with leverage is as follows.

In equilibrium, at time 0 there are three marginal agents, h_{M0}, h_{D0}, and h_{J0}. Agents $h > h_{M0}$ buy the risky asset and promise p_M (i.e., they sell the contract j_{p_M}), which is a risky promise (the contract j_{p_M} delivers $p_D < p_M$ in state D); agents $h \in (h_{D0}, h_{M0})$ buy the risky asset and promise p_D (i.e., they sell the contract j_{p_D}), which is a risk-free promise; agents $h \in (h_{J0}, h_{D0})$ buy the risky debt j_{p_M}; and agents $h < h_{J0}$ buy risk-free asset X and risk-free debt j_{p_D}. Unlike in a binomial economy, there is a possibility of default in the down state D because agents $h \in (h_{M0}, 1)$ cannot pay off the entirety of their debt, having promised p_M when the asset is only worth $p_D < p_M$. We denote the price of the risky debt j_{p_M} by π_0, which has interest rate $i_0 = \frac{p_M}{\pi_0} - 1$.

At time 1, agents receive news about the economy, borrowers repay their debts (margin calls occur) and the remaining agents trade assets and issue new promises. Because the economy is binomial at time 1, in equilibrium agents trade only risk-free contracts. In equilibrium there is one marginal investor in each state, with the remaining optimistic investors buying the risky asset against the maximal risk-free promise possible given the state. Thus, in state M there is a marginal investor h_{MM}. Investors $h > h_{M0}$ have zero wealth after repaying their promise. Investors $h \in (h_{MM}, h_{M0})$ buy the risky asset and promise M, which is the minimum payoff at $t = 2$. Investors $h < h_{MM}$ buy risk-free assets. In state D there is one marginal investor h_{DD}. Investors $h > h_{D0}$ have zero wealth after repaying their promise. Investors $h \in (h_{DD}, h_{D0})$ buy the risky asset and promise D, which is the minimum payoff at $t = 2$. Investors $h < h_{DD}$ buy risk-free assets.[9]

It is instructive to compare the 3-state model with leverage to the standard binomial model. Compared to the binomial model, crashes in this economy are larger. One reason the crash in D is so large is that investors who bought the risky debt are receiving less than the face value, and less than they invested. This "default mechanism" depresses p_D because remaining investors have less wealth. Larger crashes occur precisely because the three-state model with collateralization has more bankrupt agents at time 1 in the down state when compared to the two-state models.[10]

C.3 The Dynamic Economy with Debt Collateralization

Given our results in the previous section, we can easily characterize equilibrium in the dynamic model with debt collateralization. In equilibrium there are two marginal agents at time 0, $\hat{h}_{M0}$, and $\hat{h}_{J0}$. Agents $h > \hat{h}_{M0}$ buy the risky asset and promise p_M; agents $h \in (\hat{h}_{J0}, \hat{h}_{M0})$ buy the risky debt with promise p_M and use it as collateral to promise p_D; and agents $h < \hat{h}_{J0}$ buy the risk-free asset X and the risk-free debt (with promise p_D). In equilibrium, at time 1 there is one marginal investor in each state as discussed previously. Notice that if the economy is in state M at time 1, then agents

[9] Since we do not know the positions of h_{MM} and h_{DD} relative to the marginal investors at time 0, there are several possible equilibrium cases. These cases, as well as the equations defining equilibrium, are listed in Appendix C.5.

[10] We isolate the impact of the default mechanism in Appendix C.4 by considering a surprise bailout in $s = D$ to replace the wealth lost to default. Appendix C.4 also compares the 3-state dynamic model to corresponding binomial models and shows that in each case the price crash in the down state of the 3-state world is greater than the price crash in any of the two-state models and that the biggest price crash is obtained in the case of debt collateralization.

$h \in (\hat{h}_{M0}, 1)$ will be bankrupt; if the economy is in state D at time 1, agents $h \in (\hat{h}_{J0}, 1)$ will be bankrupt.[11]

We solve the system numerically with $\zeta = 2$, and payoffs $d^Y_{MD} = 0.3$ and $d^Y_{DD} = 0.1$. Table 2 gives the equilibrium with debt collateralization and compares it to the equilibrium with leverage. The price crash in states M and D are larger with debt collateralization. Note that the "default mechanism" is effectively much greater with debt collateralization since all debt in the economy is fully collateralized and leveraged. Rather than being poorer in the down state, agents holding risky debt will be completely out of the market. In this case, debt collateralization leads to even more volatility since the agents buying the asset in the down state will be more pessimistic.

Table 2: Dynamic Equilibrium with Debt Collateralization and with Leverage

	Leverage	Collateralization ($\hat{}$)
p_0	0.805	0.814 $\uparrow$
p_M	0.663	0.659 $\downarrow$
p_D	0.434	0.431 $\downarrow$
π_0	0.595	0.611 $\uparrow$
h_{M0}	0.963	0.888 $\downarrow$
h_{D0}	0.823	–
h_{J0}	0.719	0.789 $\uparrow$
h_{MM}	0.720	0.717 $\downarrow$
h_{DD}	0.610	0.606 $\downarrow$
M Crash	17.58 %	19.02% $\uparrow$
D Crash	46.00 %	47.09% $\uparrow$

Crashes in states M and D increase by 1.44% and 1.09%. Our result that debt collateralization increases volatility is closely related to previous work studying collateral values and volatility.[12] We have shown that debt collateralization increases the collateral value of debt contracts and of the risky asset, and as a result, asset price volatility increases because debt collateralization increases fluctuations in both collateral values and the distribution of wealth.[13]

[11]Note that we do not know the position of $\hat{h}_{MM}$ relative to the positions of the other marginal investors at time 0, but we do know the relative position of $\hat{h}_{DD}$. The equations defining equilibrium are in Appendix C.5.

[12]Fostel and Geanakoplos (2012a) show that asset price volatility increases when agents can tranche assets. Tranching increases the collateral value of the risky asset, and in a dynamic setting the "Tranching Cycle" exhibits larger fluctuations in collateral values and in the distribution of wealth.

[13]In general, debt collateralization improves welfare for pessimistic agents, whose wealth increases while the price of risk-free assets remains the same, but decreases welfare for optimistic agents because the risky asset and risky debt

The model is highly nonlinear and so it is difficult to characterize in general how debt collateralization affects price volatility. Thus, for comparative dynamics we focus on how the model parameters affect how volatility changes (how the M and D crashes change) when we introduce debt collateralization. Robustly, the D crash increases with debt collateralization. However, when beliefs at $t = 0$ are very convex, the M crash may decrease with debt collateralization. Hence, for a wide range of parameters (which almost surely includes the empirically relevant cases), debt collateralization increases volatility and fat tails.

The complete results are in the following sections, but we briefly summarize the results here. When beliefs are concave ($\zeta < 1$) indicating a larger proportion of optimism, the changes in both crashes are larger for larger D. Furthermore, the change in both crashes are smaller for smaller ζ; with more optimism, price crashes are already large and debt collateralization does little to amplify fluctuations. When beliefs are convex ($\zeta > 1$) indicating a larger proportion of pessimism, debt collateralization can *decrease* the crash in the M state for moderate levels of M but *increase* the crash for high M. For high M the price increases ($p_M > p_0$), so the price increase in state M would be muted. Changes in crashes can also be much larger (over 3%).

C.4 Default in the Dynamic 3-state Model

This section isolates the role of default in the dynamic model in two ways. First, it considers a surprise injection of wealth to bailout agents who lost money to default. Second, it maps the 3-state model onto binomial models, in which there is no default, and compares equilibrium in each case.

We solve our model numerically with $\gamma_U(h) = h^2$, $\gamma_M(h) = h^2(1-h^2)$, and payoffs $d^Y_{MD} = 0.3$ and $d^Y_{DD} = 0.1$. For these parameters, h_J holds the risk-free asset in state M and the risky asset in state D. Notice that the probability of down states (M or D) is $1-h$, similar to parametrization in

are more expensive. It is not necessarily the case in this model that increased volatility is bad for welfare. Given the stylized nature of this model, it is worth considering the the effects of volatility in a richer model. Our model leaves out many important factors, such as production, investment, cash flow problems, agency issues, and bankruptcy costs, just to name a few. Geanakoplos (2010) discusses a number of these issues in greater detail. Phelan (2016) uses a richer model to show that financial leverage creates a pecuniary externality through excess volatility, resulting in an economy with higher likelihood of crises and recessions and lower welfare. One can suspect that incorporating the mechanisms of our paper into a richer framework with any of the aforementioned issues would result in welfare losses from increased volatility.

binomial models. The marginal investors and prices in equilibrium are:

$$h_{M0} = 0.963, \quad h_{D0} = 0.823, \quad h_{J0} = 0.719, \quad p_0 = 0.805, \quad \pi_0 = 0.595,$$
$$h_{MM} = 0.720, \quad h_{DD} = 0.610, \quad p_M = 0.663, \quad p_D = 0.434, \quad i_0 = 2.25\%.$$

The percent price drop, given by $1 - \frac{p_s}{p}$, is 17.6% in state M, and 46% in state D.

C.4.1 The Default Mechanism

We demonstrate the impact that default has on asset prices at time 1 by considering the counterfactual scenario. We now suppose that in the down state, holders of risky debt receive an unexpected, exogenous wealth increase at time 1: suppose that the holders of risky debt are compensated the difference between p_M and p_D. This wealth shock is unexpected; it does not change the equilibrium at time 0 and only at time 1. In the down-state, we now have $h \in (h_{J0}, h_{D0})$ holding p_M units of wealth and $h \in (0, h_{J0})$ holding $1 + p_0$ units of wealth. The marginal investor and market clearing equations defining equilibrium are:

$$\frac{\gamma_U(h_{DD})(1 - d^Y_{DD})}{p_D - d^Y_{DD}} = 1, \text{ and } \frac{(h_{D0} - h_{J0})\left(\frac{1+p}{\pi_0}\right) p_M}{p_D - d^Y_{DD}} + \frac{(h_{J0} - h_{DD})(1 + p_0)}{p_D - d^Y_{DD}} = 1.$$

Using the same specifications as before, as well as the results for $h_{D0}, h_{J0}, \pi_0, p_0$ and p_M from the previous subsection, we find that $h_{DD} = 0.598, p_D = 0.638, \text{down crash} = 33.08\%$.

Without this injection of cash, $p_D = 0.607$ and the crash was 36.33%. The wealth increase, offsetting the default mechanism, increases the asset price in D and lowers the volatility. However, it is important to note that this result only occurs if agents do not expect the wealth shock. If agents anticipated the wealth increase at time $t = 0$, then the increase in the p_D price will lead to higher margins at the initial time period and the expectation that all debt is actually risk-free.

C.4.2 Comparison With Dynamic Two-State Model

We can compare the volatility in the dynamic three-period model to the dynamic 2-period model in several ways. We can normalize the expected payoff of the asset, normalize beliefs in the up state, as well as normalize the beliefs in the downstate.

We first consider normalization by the expected payoff of the asset. We use $\gamma_U(h) = h$ and $\gamma_M(h) = h(1-h)$ to for the probabilities in the three-state model. We want to set the belief of the upstate, $\xi(h)$, in the two-state model so that for every agent, the ultimate expected payout of the risky asset is the same in the two models. That is,

$$h+h^2(1-h)+h(1-h)^2+h(1-h)^2\alpha d_{DD}^Y+(1-h)^3 d_{DD}^Y = \xi(h)+(1-\xi(h))\xi(h)+(1-\xi(h))^2 d_{DD}^Y,$$

where $d_{MD}^Y = \alpha d_{DD}^Y$. Solving the above, we find that

$$\varphi_i = 1+\frac{\sqrt{(d_{DD}^Y-1)(1-h)^2(d_{DD}^Y-1+(\alpha-1)d_{DD}^Y h)}}{d_{DD}^Y-1}.$$

Letting $\alpha = 3$, we can solve for equilibrium in the two-state dynamic model.

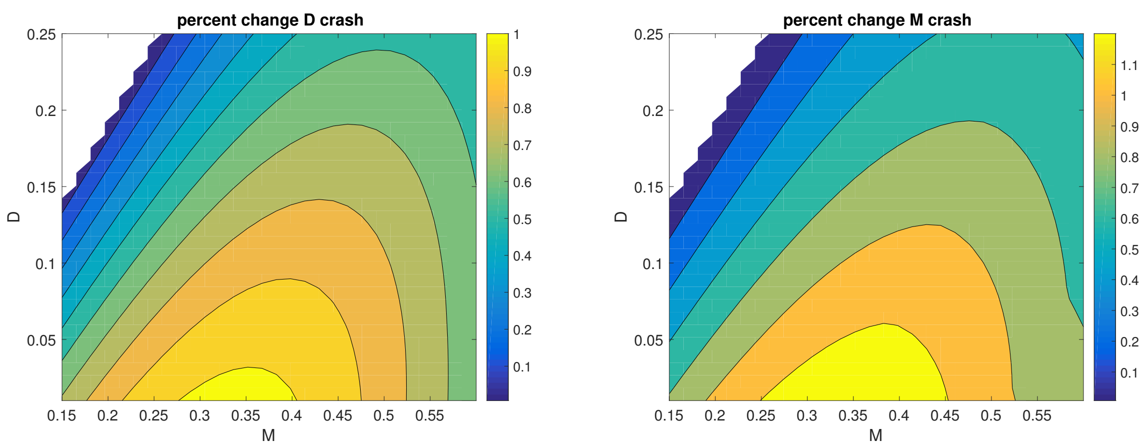
When we normalize the belief about the up state, we have $\xi(h) = \gamma_U(h) = h$. Alternatively, if we normalize the belief about the down state, we need $1-\xi(h) = \gamma_D(h)$. In summary, we obtain the following differences by normalization.

	Dynamic 3-State		Dynamic 2-State		
	Collateralized debt	No Collateralization	Expected Payout	Belief Up	Belief Down
p_0	.970	0.9531	0.9470	0.9351	0.9876
p_D	.602	0.6069	0.6339	0.6116	0.7314
crash	37.91%	36.33%	33.06%	34.593 %	25.94%

Note that for every normalization, the price crash in the down state of the three state world is greater than the price crash in any of the two-state models and that the biggest price crash is obtained in the case of debt collateralization. Furthermore, the price of the asset in the down state is lowest in the three-state world with debt collateralization. This phenomenon occurs precisely because the three-state model with collateralization has more bankrupt agents at time 1 in the down state when compared to the two-state models. Thus, the agents who are left to buy the asset are more pessimistic and do not value the asset as highly.

C.4.3 Comparative Dynamics

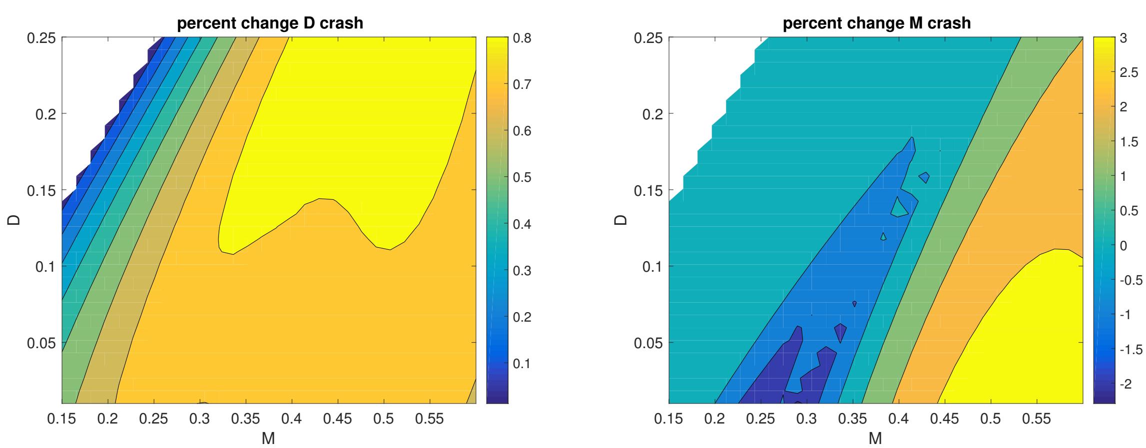
Figures 9-10 show the percent change in the crashes in the M and D states for (M,D) pairs. When ζ is not too high, both crashes are larger with debt collateralization. However, for high ζ ($\zeta \geq 3.2$ in this case), the crash in the M-state could decrease with debt collateralization; the D-crash increases. While debt collateralization often increases both crashes, but not always, we have not been able to solve for a set of parameters where the D crash decreases.



(a) Crash in D-state.

(b) Crash in M-state.

Figure 9: Percent Changes in crashes moving from leverage to debt collateralization. Beliefs exponent $\zeta = 1$.



(a) Crash in D-state.

(b) Crash in M-state.

Figure 10: Percent changes in crashes moving from leverage to debt collateralization. Beliefs exponent $\zeta = 3.2$.

C.5 Equilibrium Conditions in the Dynamic Model

C.5.1 Equilibrium Conditions with Leverage: Section C.2

For a few marginal investors, we have no doubt about their course of action. The equations defining them are as follows.

Marginal Investors, known

h_{M0}: indifferent between leveraging against p_M and p_D at time $t = 0$. If at time $t = 1$ we are in state D, then h_{M0} is no longer in the market. If at $t = 1$ we are in state M, h_{M0} will choose to hold the risky asset because he is the most optimistic investor in the market. Thus, we have

$$\frac{\gamma_U(h_{M0})(1-p_M)}{p_0-\pi_0} = \frac{\gamma_U(h_{M0})(1-p_D)}{p_0-p_D} + \left(\frac{\gamma_M(h_{M0})(p_M-p_D)}{p_0-p_D}\right)\left(\frac{\gamma_U(h_{M0})(1-d^Y_{MD})}{p_M-d^Y_{MD}}\right).$$

The above equates the marginal utility divided by payoff of holding the risky asset leveraged against p_M and the marginal utility divided by the payoff of holding the asset leveraged against p_D.

h_{MM} : Indifferent between risky asset and riskless asset given the realization of state M at $t = 1$. Since this marginal investor only exists at time $t = 1$ in state M. There is no ambiguity.

$$\frac{\gamma_U(h_{MM})(1-d^Y_{MD})}{p_M-d^Y_{MD}} = 1$$

h_{DD} : Indifferent between risky asset and riskless asset given the realization of state D at $t = 1$. h_{DD} also only exists at time $t = 1$ in state D.

$$\frac{\gamma_U(h_{DD})(1-d^Y_{DD})}{p_D-d^Y_{DD}} = 1$$

Marginal Investors, unknown

h_{D0}: indifferent between leveraging against p_D and holding risky debt at time $t = 0$. If at time $t = 1$ the world is in state D, h_{D0} will choose to hold the risky asset. But, at time $t = 1$ in state M, h_{D0} can either choose to hold the risky asset or the risk-free asset. Let h_{D0} hold the risky asset. To simplify notation, let $\gamma^{D0}_U = \gamma_U(h_{D0})$, $\gamma^{D0}_M = \gamma_M(h_{D0})$ and $\gamma^{D0}_D = \gamma_D(h_{D0})$.

Then, equating payoffs (multiplied by continuation values, we have:

$$\frac{\gamma_U^{D0}(1-p_D)}{p_0-p_D}+\left(\frac{\gamma_M^{D0}(p_M-p_D)}{p_0-p_D}\right)\left(\frac{\gamma_U^{D0}(1-d_{MD}^Y)}{p_M-d_{MD}^Y}\right)$$
$$=\frac{\gamma_U^{D0}p_M}{\pi_0}+\left(\frac{\gamma_M^{D0}(p_M)}{\pi_0}\right)\left(\frac{\gamma_U^{D0}(1-d_{MD}^Y)}{p_M-d_{MD}^Y}\right)+\left(\frac{\gamma_D^{D0}p_D}{\pi_0}\right)\left(\frac{\gamma_U^{D0}(1-d_{DD}^Y)}{p_D-d_{DD}^Y}\right)$$

When h_{D0} holds the risk-free asset, we have:

$$\frac{\gamma_U^{D0}(1-p_D)+\gamma_M^{D0}(p_M-p_D)}{p-p_D}=\frac{(\gamma_U^{D0}+\gamma_M^{D0})p_M}{\pi_0}+\left(\frac{\gamma_D^{D0}p_D}{\pi_0}\right)\left(\frac{\gamma_U^{D0}(1-d_{DD}^Y)}{p_D-d_{DD}^Y}\right)$$

h_{J0}: indifferent between holding risky debt and holding riskless asset at time $t=0$. There are several possibilities for this agent. At time $t=1$ in state M, the agent can either hold the risk-free or risky asset. At time $t=1$ in state D, the agent can either hold the risk-free or risky asset. To simplify notation, let $\gamma_U^{\pi 0}=\gamma_U(h_{J0})$, $\gamma_M^{\pi 0}=\gamma_M(h_{J0})$ and $\gamma_D^{\pi 0}=\gamma_D(h_{J0})$. Thus, we have four possible equations defining this agent:

M safe, D safe.

$$\frac{(\gamma_U^{\pi 0}+\gamma_M^{\pi 0})p_M+(\gamma_D^{\pi 0})p_D}{\pi_0}=1$$

M risky, D risky.

$$\frac{\gamma_U^{\pi 0}p_M}{\pi_0}+\left(\frac{\gamma_M^{\pi 0}p_M}{\pi_0}\right)\left(\frac{\gamma_U^{\pi 0}(1-d_{MD}^Y)}{p_M-d_{MD}^Y}\right)+\left(\frac{\gamma_D^{\pi 0}p_D}{\pi_0}\right)\left(\frac{\gamma_U^{\pi 0}(1-d_{DD}^Y)}{p_D-d_{DD}^Y}\right)$$
$$=\gamma_U^{\pi 0}+\gamma_M^{\pi 0}\left(\frac{\gamma_U^{\pi 0}(1-d_{MD}^Y)}{p_M-d_{MD}^Y}\right)+\gamma_D^{\pi 0}\left(\frac{\gamma_U^{\pi 0}(1-d_{DD}^Y)}{p_D-d_{DD}^Y}\right)$$

M safe, D risky.

$$\frac{(\gamma_U^{\pi 0}+\gamma_M^{\pi 0})p_M}{\pi_0}+\left(\frac{\gamma_D^{\pi 0}p_D}{\pi_0}\right)\left(\frac{\gamma_U^{\pi 0}(1-d_{DD}^Y)}{p_D-d_{DD}^Y}\right)=\gamma_U^{\pi 0}+\gamma_M^{\pi 0}+\gamma_D^{\pi 0}\left(\frac{\gamma_U^{\pi 0}(1-d_{DD}^Y)}{p_D-d_{DD}^Y}\right)$$

M risky, D safe.

$$\frac{\gamma_U^{\pi 0}p_M}{\pi_0}+\left(\frac{\gamma_M^{\pi 0}p_M}{\pi_0}\right)\left(\frac{\gamma_U^{\pi 0}(1-d_{MD}^Y)}{p_M-d_{MD}^Y}\right)+\frac{\gamma_D^{\pi 0}p_D}{\pi_0}=\gamma_U^{\pi 0}+\gamma_M^{\pi 0}\left(\frac{\gamma_U^{\pi 0}(1-d_{MD}^Y)}{p_M-d_{MD}^Y}\right)+\gamma_D^{\pi 0}$$

The known market clearing conditions are for the asset and debt at time 0:

Time $t = 0$, Asset:

$$(1-h_{M0})\frac{(1+p_0)}{(p_0-\pi_0)}+(h_{M0}-h_{D0})\frac{(1+p_0)}{(p_0-p_D)}=1$$

Time $t = 0$, Risky Debt:

$$(1-h_{M0})\frac{(1+p_0)}{(p_0-\pi_0)}=(h_{D0}-h_{J0})\frac{(1+p_0)}{\pi_0}$$

Market Clearing, unknown

Time $t = 1$, State M Asset: We are unsure whether $h_{MM} \in (0, h_{J0})$, $h_{MM} \in (h_{J0}, h_{D0})$, or $h_{MM} \in (h_{D0}, h_{M0})$. This issue can be resolved by considering all cases, solving for equilibrium, and checking that h_{MM} is indeed in the specified interval. $h_{MM} \in (0, h_{J0})$:

$$\frac{(h_{M0}-h_{D0})\left(\frac{1+p_0}{p_0-p_D}\right)(p_M-p_D)}{p_M-d^Y_{MD}}+\frac{(h_{D0}-h_{J0})\left(\frac{1+p_0}{\pi_0}\right)(p_M)}{p_M-d^Y_{MD}}+\frac{(h_{J0}-h_{MM})(1+p_0)}{p_M-d^Y_{MD}}=1$$

$h_{MM} \in (h_{J0}, h_{D0})$:

$$\frac{(h_{M0}-h_{D0})\left(\frac{1+p_0}{p_0-p_D}\right)(p_M-p_D)}{p_M-d^Y_{MD}}+\frac{(h_{D0}-h_{MM})\left(\frac{1+p_0}{\pi_0}\right)(p_M)}{p_M-d^Y_{MD}}=1$$

$h_{MM} \in (h_{D0}, h_{M0})$:

$$\frac{(h_{M0}-h_{MM})\left(\frac{1+p_0}{p_0-p_D}\right)(p_M-p_D)}{p_M-d^Y_{MD}}=1$$

Time $t = 1$, State D Asset: We do not know whether $h_{DD} \in (0, h_{J0})$ or $h_{DD} \in (h_{J0}, h_{D0})$. In the first case, we have:

$$\frac{(h_{D0}-h_{J0})\left(\frac{1+p_0}{\pi_0}\right)p_D}{p_D-d^Y_{DD}}+\frac{(h_{J0}-h_{DD})(1+p_0)}{p_D-d^Y_{DD}}=1.$$

In the second case, we have

$$\frac{(h_{D0}-h_{DD})\left(\frac{1+p_0}{\pi_0}\right)p_D}{p_D-d^Y_{DD}}=1.$$

Thus, we obtain the following possible cases in equilibrium:

1. h_{D0} holds risky asset at time 1 in state M. (a) h_{J0} holds risky asset in state M and risky asset

in state D. This implies that $h_{MM}, h_{DD} \in (0, h_{J0})$. (b) h_{J0} holds safe asset in state M and safe asset in state D. This implies that $h_{MM}, h_{DD} \in (h_{J0}, h_{D0})$. (c) h_{J0} holds risky asset in state M and safe asset in state D. This implies that $h_{MM} \in (0, h_{J0})$ and $h_{DD} \in (h_{J0}, h_{D0})$. (d) h_{J0} holds safe asset in state M and risky asset in state D. This implies that $h_{MM} \in (h_{J0}, h_{D0})$ and $h_{DD} \in (0, h_{J0})$.

2. h_{D0} holds safe asset at time 1 in state M. This implies that $h_{MM} \in (h_{D0}, h_{M0})$ and h_{J0} holds safe asset in state M. (a) h_{J0} holds safe asset in state D. This implies that $h_{DD} \in (h_{J0}, h_{D0})$. (b) h_{J0} holds risky asset in state D. This implies that $h_{DD} \in (0, h_{J0})$.

C.5.2 Equilibrium Conditions with Collateralization: Section C.3

The equations defining equilibrium are as follows

Marginal Investors, known

$\hat{h}_{M0}$: indifferent between holding risky asset, leveraged against state M and risky debt leveraged against state D at time 0

$$\frac{\gamma_U(\hat{h}_{M0})(1-\hat{p}_M)}{\hat{p}_0-\hat{\pi}_0} = \frac{\gamma_U(\hat{h}_{M0})(\hat{p}_M-\hat{p}_D)}{\hat{\pi}_0-\hat{p}_D} + \frac{\gamma_M(\hat{h}_i)(\hat{p}_M-\hat{p}_D)}{\hat{\pi}_0-\hat{p}_D}\left(\frac{\gamma_U(1-d^Y_{MD})}{\hat{p}_M-d^Y_{MD}}\right)$$

$\hat{h}_{MM}$: Indifferent between holding risky asset and safe asset at time 1, state M.

$$\frac{\gamma_U(\hat{h}_{MM})(1-d^Y_{MD})}{\hat{p}_M-d^Y_{MD}} = 1$$

$\hat{h}_{DD}$: Indifferent between holding risky asset and safe asset at time 1, state D.

$$\frac{\gamma_U(\hat{h}_{DD})(1-d^Y_{DD})}{\hat{p}_D-d^Y_{DD}} = 1$$

Marginal Investors, unknown

$\hat{h}_{J0}$: Indifferent between holding risky debt with leverage and holding the safe asset. There are two possibilities for this agent: at time $t=1$ in state M, the agent can either hold the safe or risky asset; at time $t=1$ in state D, $\hat{h}_{J0}$ will be the most optimistic agent still in the market, forcing him to hold the risky asset. Thus, we have the following two possibilities

M safe.

$$\frac{(\gamma_U(\hat{h}_{J0})+\gamma_M(\hat{h}_{J0}))(\hat{p}_M-\hat{p}_D)}{\hat{\pi}_0-\hat{p}_D}=\gamma_U(\hat{h}_{J0})+\gamma_M(\hat{h}_{J0})+\gamma_D(\hat{h}_{J0})\left(\frac{\gamma_U(\hat{h}_{J0})(1-d^Y_{DD})}{p_D-d^Y_{DD}}\right)$$

M risky.[14]

$$\frac{\gamma_U(\hat{h}_{J0})(\hat{p}_M-\hat{p}_D)}{\hat{\pi}_0-\hat{p}_D}+\left(\frac{\gamma_M(\hat{h}_{J0})(\hat{p}_M-\hat{p}_D)}{\hat{\pi}_0-\hat{p}_D}\right)\left(\frac{\gamma_U(\hat{h}_{J0})(1-d^Y_{MD})}{\hat{p}_M-d^Y_{MD}}\right)$$
$$=\gamma_U(\hat{h}_{J0})+\gamma_M(\hat{h}_{J0})\left(\frac{\gamma_U(\hat{h}_{J0})(1-d^Y_{MD})}{\hat{p}_M-d^Y_{MD}}\right)+\gamma_D(\hat{h}_{J0})\left(\frac{\gamma_U(\hat{h}_{J0})(1-d^Y_{DD})}{p_D-d^Y_{DD}}\right)$$

Market Clearing, known

Time $t=0$, Risky Asset:

$$(1-\hat{h}_{M0})\left(\frac{1+\hat{p}_0}{\hat{p}_0-\hat{\pi}_0}\right)=1$$

Time $t=1$, state D, Risky Asset:

$$(\hat{h}_{J0}-\hat{h}_{DD})\left(\frac{1+\hat{p}_0}{\hat{p}_D-d^Y_{DD}}\right)=1$$

Time $t=0$, Risky Debt:

$$(1-\hat{h}_{M0})\left(\frac{1+\hat{p}_0}{\hat{p}_0-\hat{\pi}_0}\right)=(\hat{h}_{M0}-\hat{h}_{J0})\left(\frac{1+\hat{p}_0}{\hat{\pi}_0-\hat{p}_D}\right)$$

Market Clearing, unknown

Time $t=1$, State M Asset: We are unsure whether $\hat{h}_{MM}\in(0,\hat{h}_{J0})$ of $\hat{h}_{MM}\in(\hat{h}_{J0},\hat{h}_{M0})$. This issue can be resolved by considering both cases, solving for equilibrium, and checking that $\hat{h}_{MM}$ is indeed in the specified interval.

$\hat{h}_{MM}\in(0,\hat{h}_{J0})$:

$$\frac{(\hat{h}_{M0}-\hat{h}_{J0})\left(\frac{1+\hat{p}_0}{\pi-\hat{p}_D}\right)(\hat{p}_M-\hat{p}_D)}{\hat{p}_M-d^Y_{MD}}+\frac{(\hat{h}_{J0}-\hat{h}_{MM})(1+\hat{p}_0)}{\hat{p}_M-d^Y_{MD}}=1$$

[14]the payout of the asset in state M at time 1 is multiplied by the continuation value of the asset in time 2 since we have specified that $\hat{h}_{J0}$ will hold the risky asset.

$\hat{h}_{MM} \in (\hat{h}_{J0}, \hat{h}_{M0})$:

$$\frac{(\hat{h}_{M0} - \hat{h}_{MM}) \left(\frac{1+\hat{p}_0}{\pi - \hat{p}_D}\right) (\hat{p}_M - \hat{p}_D)}{\hat{p}_M - d^Y_{MD}} = 1$$

D Empirical Implications

Our model offers several empirical implications regarding how explicit and implicit uses of debt as collateral affect returns and leverage: (i) capital structures are affected by funding markets and ease of financing, designed in part to stretch collateral; (ii) debt collateralization decreases economy-wide margins as borrowers shift to using high-leverage, risky contracts; (iii) debt collateralization decreases risk premia, increases defaults, and tends to increase asset prices and volatility; (iv) these effects are strongest when the economy has larger demand for negative-beta assets.

D.1 Static Implications

Capital Structures The main result of our analysis is that debt collateralization (implicit or explicit) leads investors to take maximal leverage, or more broadly fewer investors use low leverage. While the corporate finance literature has emphasized the role of capital structure in mitigating infomational frictions, our results imply that capital structures are defined in part to stretch scarce collateral: when "collateral is tight" capital structures should be designed to further stretch collateral.[15] An empirical test of this prediction could be to use measures of ease of financing (e.g. loan margins or haircuts in funding markets, or measures that typically correlate with risk and other determinants of credit conditions) to see how capital structures (for LBOs, syndicated loans, mortgages, etc.) respond to changes in funding markets, controlling for incentive conflicts.[16]

Empirically, growth in the origination of CDOs, CDO-squareds, etc., should all else equal translate into changes in capital structures in the ABS deals underlying CDO structures. Implicit or

[15]Critically, our analysis assumes that default is "costless," which is obviously problematic in many situations. Asquith et al. (1994) find that debt structure affects restructuring decisions, and Alderson and Betker (1995) find that when liquidation costs are high, firms choose capital structure to minimize firm distress.

[16]For example, Axelson et al. (2013) find that public firms have high leverage when credit spreads are high (when credit is expensive), whereas LBO deals have high leverage when credit spreads are low; they interpret LBO's "buy expensive when credit is cheap" to reflect agency problems between private equity sponsors and their investors. Rauh and Sufi (2010) show that low-credit-quality firms more likely to have multi-tiered capital structure, with secured bank debt (tight covenants) and subordinated non-bank debt (loose covenants), in order to reduce incentive conflicts.

explicit debt collateralization would imply that subordinated tranches would be larger and riskier, with the largest effect on the most subordinated tranches. However, rather than affecting capital structure, the effect of greater collateralization could manifest itself by changing the underlying composition of ABS collateral. If higher prices would change incentives for mortgage lenders, then debt collateralization could contribute to those incentives. While Benmelech et al. (2012) find no evidence that corporate loans securitized in CLOs were riskier, there is evidence that securitization affected underlying loans in the subprime mortgage market.[17] Our analysis takes as given the collateral quality and ignores any informational asymmetries.

Risk Premia Our results imply that risky debt that can be used as collateral should experience decreased risk premia. Empirically, Nadauld and Weisbach (2012) find evidence that securitization of corporate loans was associated with lower spreads by 17 basis points, consistent with a reduction in cost of capital. Lemmon et al. (2014) find that securitization of receivables by nonfinancial firms decreases financing costs by providing access to segmented markets, and innovations in capital structure increase firm value even for large, mid-tier credit firms. Measures of how funding markets treat derivative debt contracts should correspond to increases in debt riskiness together with lower risk premia. In our model the face values of promises are fixed, but one might expect more generally to see debt collateralization leading to larger face values as well, and thus riskier debt.

We show that under sufficient conditions, debt collateralization increases asset prices.Thus, increased ability to use debt as collateral should increase prices of underlying collateral (or push down risk premia). However, as noted earlier, to the extent that collateral quality can change or is subject to informational frictions, prices may stay the same and instead composition changes.

Origination and Securitization Volume Debt collateralization may provide incentives to produce collateral.[18] Shivdasani and Wang (2011) find evidence that the leveraged buyout (LBO) boom of 2004 to 2007 was fueled by growth in CDOs and other forms of securitization, which facilitated

[17] Keys et al. (2010) show that securitization does not change the interest rate or the LTV ratio for mortgages, but nonetheless affects the subsequent performance of mortgages through reduced screening by lenders. Nadauld and Weisbach (2012) find that securitization led to less screening for subprime borrowers. Additionally, Wang and Xia (2014) find that banks active in CLO securitization exert less effort on ex-post monitoring.

[18] see Fostel and Geanakoplos 2016.

much larger LBOs than historically possible, potentially because it helped relax balance sheet constraints that banks faced in financing large LBOs.

Securitization decreased spectacularly after the crisis (see Chernenko et al. 2014). While there are many explanations for why this occurred, one possibility is that investors realized many ABS tranche payoffs were better described by "pay full or zero"—in this case debt collateralization is not meaningful. It is well understood that correlations in ABS tranches were the primary determinant of CDO quality, and as the mortgage bubble burst it became clear that CDOs would be essentially worthless or of full-value, with very little likelihood of intermediate values. As the comparative statics demonstrate, when uncertainty is less dispersed (M closer to D), there is less debt collateralization in equilibrium. The incentive for debt collateralization would decrease if uncertainty more closely followed a binomial model (in a binomial model all equilibrium debt is risk free). An implication is that the extent of debt collateralization (whether measured by origination volume or margins on securitized debt) should vary with perceptions of dispersion of payoffs conditional on default. Greater risk dispersion should lead to less debt collateralization.

Negative-Beta Assets/Pessimism Finally, an economy with relatively more pessimists, or more demand for negative-beta assets, exhibits more sensitivity to debt collateralization: the increase in prices is greater; the change in risk premia is greater; the composition of margins is more affected. (An empirical proxy for these measures could be risk-premia variation as a function of an asset's beta.) Critically, with relatively more pessimists, agents with leverage primarily use low leverage by making safe promises, but with debt collateralization agents switch to using high leverage by making risky promises.

D.2 Dynamic Implications

Our analysis suggests that debt collateralization increases volatility, particularly in response to very bad news, which has the following implications:

(i) Assets with easily financed derivative debt should have prices that fluctuate more in response to news or other changes in fundamentals (given the volatility of fundamentals). However, one might expect that less volatile assets receive better treatment as collateral (endogeneity is a concern). Additionally, debt collateralization should be associated with increased default rates (how funding markets treat collateral is clearly endogenous, reflecting in part expected default rates).

(ii) Since debt collateralization can also be a metaphor for capital structure, the price of firms whose capital structure implicitly provides debt collateralization (through greater use of subordinated tranches, for example) should be more volatile. Since capital structure is endogenous, testing this prediction would require using some exogenous variation in capital structure (perhaps driven by corporate governance or regulation).

(iii) The model with only leverage suggests that when there is greater dispersion in payoffs conditional on default (M and D are more different) then price crashes after really bad news becomes worse because agents make more risky promises. Measures of uncertainty should correlate with larger price crashes, holding fixed expectations about worst-case scenarios.