

Debt Collateralization, Capital Structure, and Maximal Leverage

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This version: July 12, 2016

Abstract

Many capital structures implicitly or explicitly give agents the ability to use debt contracts as collateral for other financial promises. We study the effects of allowing debt to be used as collateral in a general equilibrium model with heterogeneous agents, collateralized financial contracts, and multiple states of uncertainty. When agents cannot use debt contracts as collateral, some agents borrow using risky debt and others borrow with risk-free debt. With debt collateralization, agents switch to high-leverage contracts, margin requirements decrease, and risk premia decrease. We provide testable implications regarding how funding markets affect capital structure, economy-wide margins, and price volatility.

Keywords: Leverage, margins, asset prices, default, securitized markets, asset-backed securities, collateralized debt obligations

JEL classification: D52, D53, G11, G12

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1 Introduction

Despite the Modigliani and Miller (1958) irrelevance result, complex and innovative capital structures prevail, and senior-subordinated debt structures are incredibly common. While the corporate finance literature has extensively studied how informational problems affect capital structure,¹ less is known about how capital structures affect the equilibrium prices of assets. Asset-backed securities (ABS) and collateralized debt obligations (CDOs) significantly contributed to the growth of the market for subprime mortgages and leveraged buyouts,² yet their effects on the equilibrium prices of underlying assets and the amount of leverage used to purchase these assets are not fully understood.

Innovations in capital structures are particularly important when one considers that tranching a pool of loans is similar to financing an asset with a capital structure. Critically, subordinated tranches (and subordinated capital) are essentially leveraged positions in debt backed by equity tranches, giving investors the *implicit* ability to use debt as collateral.³ Furthermore, one of the key features of securitized mortgage markets is the *explicit* ability to use debt contracts as collateral for new financial contracts.⁴ The process of implicitly and/or explicitly using debt as collateral is incredibly general and widespread, as it is common to fund assets with a capital structure and then to use the debt created from that capital structure as collateral for new structures; classically, banks take deposits and make loans, issuing debt to invest in debt.⁵ This paper, thus, asks two equivalent questions: (i) how do equilibrium leverage and asset prices change when debt contracts can be used as collateral for new financial contracts, and (ii) how do innovations in capital structures determine

¹See Harris and Raviv (1991) for a review. Mello and Quintin (2015) motivate mezzanine finance as arising from moral hazard for the manager, where they explicitly consider the back-up management rights of mezzanine investors. In contrast, Allen and Gale (1988) motivate mezzanine securities as arising from market incompleteness, which is a similar motivation to that in our paper.

²see Shivdasani and Wang (2011).

³We discuss this in greater detail below; see Section 3.4 for a formalization.

⁴Securitization has many other important features that we abstract away from in our analysis in order to isolate the effect of debt collateralization. Securitization pools loans with similar characteristics to diversify away idiosyncratic risks. Diversification removes less easily quantifiable risks, resulting in securities that are relatively more standardized and thus more liquid. Tranching pools into bonds with different characteristic creates securities with different state-contingencies, improving investors' abilities to hedge and share risks.

⁵In other words, layered capital structures are essentially "CDOs" with different collateral. Examples go back to unit trusts in the 1920s and the "unit trust of unit trusts" created by Goldman Sachs in 1928. Consider also Trust Preferred ("TruPS") CDOs, and, more prevalent, structured leveraged buyouts ("LBOs"). Similarly, securitized second-lien mortgages (see Bear Stearns Second Lien Trust 2007) create tranches in debt that is part of a complex capital structure financing a house. See Chambers et al. (2011).

investors' leverage and affect the risk premia for underlying assets?⁶

To study this issue, we use a general equilibrium model featuring heterogeneous agents and collateralized financial contracts following Geanakoplos (1997, 2003). The central element of our analysis is repayment enforceability problems: we suppose that collateral is the only means of enforcing promises, with lenders seizing collateral that has been agreed upon in advance by contract. We consider a model with multiple states of uncertainty so that in an economy with debt contracts, agents trade multiple contracts in equilibrium.⁷ We then allow agents to use debt contracts as collateral to back new financial contracts, a process we call “debt collateralization,” and we characterize the equilibrium. Equilibrium with debt collateralization also corresponds to equilibrium with contingent claims defined by senior-subordinated capital structures (tranching). The key insight of our results is that debt collateralization is a way of stretching scarce collateral. Our results suggest that one motivating factor for senior-subordinated capital structures is to provide a way to stretch scarce collateral.

We show that in equilibrium, debt collateralization increases the total amount of leverage in the economy so that investors use the maximum amount of leverage for whatever investment they choose. All agents use maximal leverage because doing so “creates” new securities that can be further collateralized/leveraged (doing otherwise creates securities with fewer collateralization opportunities). Debt prices increase (risk premia decrease) because debt contracts now have “collateral value.” Decreased risk premia provide greater incentives for leveraged investors to make large promises, which in turn increases the price of the risky assets. In fact, allowing debt to back debt (to back debt, etc.) further increases collateral values, increasing leverage in each contract. In other words, each “level of debt collateralization” reinforces these effects. In our dynamic analysis, debt collateralization also increases volatility in asset prices since leverage increases, which endogenously increases defaults after bad news.

Our modeling environment is a natural way to capture the complexities and innovations in securitized mortgage markets and in many structured finance products. First, the collateral

⁶The corporate finance literature typically takes as given the value of assets and considers how informational frictions and asymmetries determine capital structure.

⁷Much of the recent literature on collateral in equilibrium, with a few important exceptions, has used binomial models. With binomial models, in equilibrium all debt in the economy is risk-free. Studying debt collateralization in general equilibrium requires the consideration of a model with risky debt. Multi-state models also allow the study of interactions between different financial innovations such as debt collateralization, tranching and CDS. Gong and Phelan (2016) study the equilibrium consequences of CDS and the use of different assets as collateral.

underlying ABS and other structured products, whether mortgages or other loans, are themselves debt backed by assets (houses in the case of mortgages).⁸ Second, the senior-subordinated structure implicitly creates securities in which debt contracts serve as collateral for additional debt. To see this, consider a typical ABS deal, which consists of a pool of mortgages (collateral) supporting senior, mezzanine, and equity/residual securities. The equity security behaves like a leveraged position in the collateral, with the payoff declining “linearly” with the value of the collateral and paying zero when the collateral falls below a certain level. The senior security behaves like debt, making a predetermined payoff unless the collateral value falls below a certain threshold, at which point the payoff declines linearly to zero only when the collateral is worth zero. The subordinated, or mezzanine, security, however, behaves like a leveraged debt position. For sufficient values of collateral the subordinated security gets the predetermined payoff (there is not additional upside as with a leveraged position in the collateral), but gets nothing if the value of the collateral is low (like a leveraged position). In fact, the subordinated tranches are leveraged positions in the debt implicitly “issued” by the equity tranche.⁹

Finally, CDOs and structured capital structures explicitly use debt (ABS tranches, TruPS, etc.) as collateral to support another senior-subordinated capital structure. CDOs do not create pass-through securities backed by subordinated ABS tranches (in which case the only purpose of a CDO would be diversification of idiosyncratic risk), but rather create leveraged investments in the ABS tranches—the underlying promises backed by the original collateral are used to make more promises. Thus, the equity tranche of a CDO creates a leveraged investment in ABS tranches, and the senior tranches of a CDO create investments in debt “issued” by the leveraged (equity) investors. Hence, CDOs (and then CDO-squareds) increase the degree to which debt contracts can

⁸There may still be good reasons to consider these loans directly as “assets” rather than as loans backed by another asset. This is typically the way mortgage assets are treated in related papers.

⁹Consider a simple, stylized version of a deal with senior, mezzanine, and equity tranches all with face-values of 1, and suppose the value of the collateral could take values of 1, 2, or 3. The senior bonds would get paid 1 for sure; the mezzanine bond would get paid 1 in only when the collateral is worth 2 or 3, and zero otherwise; and the equity would get paid 1 only in the best state, and zero otherwise. This structure can be equivalently implemented with leveraged investments in the collateral, in the debt backed by the collateral, *and in debt backed by the debt backed by collateral*. The equity investor is effectively buying the collateral with leverage, promising to repay 2 units and defaulting whenever the collateral is worth 2 or less. The mezzanine investor is effectively buying *the promise from the equity investor* and using *this promise as collateral* to borrow 1 from the senior investor. The senior investor buys this promise from the mezzanine investor. This investment scheme exactly replicates the payoffs to the tranches, giving (the mezzanine) investors the ability to use debt as collateral to make new promises. In practice the payoffs to ABS tranches are complicated by timing of prepayments and how principal payments are allocated to the different tranches.

be used as collateral to make new promises.¹⁰

Related Literature

Our paper follows the model of collateral equilibrium developed in Geanakoplos (1997, 2003); Geanakoplos and Zame (2014), and is closely related to the literature on collateral and financial innovation (Fostel and Geanakoplos, 2008, 2012a,b, 2015a,b; Fostel et al., 2015). This literature uses binomial models to explain asset prices and investment, and defines the financial environment as the permissible set of assets that serve as collateral and the set of promises that can be made with existing collateral. Debt collateralization, which expands the set of assets that can be used as collateral, fits directly into this definition of financial environment.

Several papers study collateral equilibrium with multiple states.¹¹ Simsek (2013) uses a model with a continuum of states to show how belief disagreements about the future state endogenously create constraints on the amount of leverage that can be used to buy the risky asset. Simsek (2013) conjectures that in multi-state models, when debt contracts can be used as collateral, equilibrium will feature a pyramiding arrangement. We prove that his conjecture holds only when the *maximum* level of securitization has been reached. Araujo et al. (2012) examine the effects of default and collateral on risk sharing and prove that with N states, at most $N - 1$ contracts are traded in a collateral equilibrium. Phelan (2015) studies how changing asset risk and endowment covariances affects asset prices and leverage in equilibrium.

Few papers study debt collateralization, or “pyramiding,” in equilibrium. Gottardi and Kubler (2015) implicitly assume that all financial securities serve as collateral, and show that any Arrow–Debreu equilibrium allocation with limited pledgeability can also be attained at a collateral-constrained financial market equilibrium, provided the financial markets are sufficiently rich in terms of the specification of pay-offs and also of collateral requirements. Geerolf (2015) studies an economy with a continuum of states and a continuum of agents with differing point-beliefs about the asset payoff. A continuum of contracts are traded in equilibrium, and with pyramiding (i) the asset price increases with each layer of pyramiding, (ii) the measure of contracts traded decreases, and (iii)

¹⁰In this process, the supply of safe assets (i.e., perceived to be close to risk-free) increases.

¹¹The shortcoming of binomial models is that all equilibrium debt in the economy is risk-free. Studying debt collateralization in general equilibrium requires the consideration of a model with risky debt. Our work involves general equilibrium models with more than two states, so that in equilibrium risky debt contracts are traded.

the distribution of leverage changes.¹² While these results are closely related to ours, there are important distinctions. Most importantly, in our setting we prove that with debt collateralization agents use maximal leverage on the assets in which they invest—agents switch to using risky contracts rather than safe contracts—where leverage and investment decisions are intuitively related to the set of states. In our model, economy-wide margins decrease because the composition of leverage changes—agents switch from low-leverage to high-leverage contracts—not primarily because margins decrease on each contract.¹³

Our results relate to the literature on how securitized markets create safe and liquid assets. It is well-understood that one of the motivations for the structure of ABS and CDOs was the creation of “safe” (close to risk-free) assets (see for example Gorton and Metrick 2009). We show that this process increases volatility.¹⁴ Shen et al. (2014) propose a collateral view of financial innovation driven by the cross-netting friction. In our model, debt collateralization and innovative capital structures are ways of stretching collateral, which is similar to their insight that financial innovation is a response to scarce collateral. Dang et al. (2011) study how debt collateralization can alleviate asymmetric information problems by creating information-insensitive securities, and they show that the optimal financial instrument is debt backed by debt. Other papers have studied the problems of valuing or rating CDOs (e.g. Bolton et al. 2012; Benmelech and Dlugosz 2010) or the information problems that arise (e.g. DeMarzo and Duffie 1999; DeMarzo 2005). In our model there are no informational problems and agents rationally value collateralized debt contracts.

¹²Additionally, Geerolf (2015) characterizes the asymptotic distribution of leverage levels in the economy and also shows that interest rates are disconnected from default probabilities, thus explaining the credit-spread puzzle.

¹³Additionally, his approach differs from our model because agents’ disagreements are of the form of point-expectations about the asset’s value; one can view our analysis as generalizing his results to more flexible beliefs/hedging needs. Our model is a reduced-form version of a model with (potentially) finitely many heterogeneous risk-averse agents with *common* beliefs and is not fundamentally about belief disagreements; see Appendix D.

¹⁴Others have shown that the additional leverage from securitization is not without risk. Gorton and Metrick (2012) argue that a combination of securitization and repo finance was at the heart of the 2007–2008 financial crisis. Krishnamurthy (2009) discusses how feedback effects in risk capital and risk aversion, repo financing, and counterparty risk decrease liquidity and increase financing cost, causing debt markets to break down as fundamental and market values diverge. Longstaff (2010) shows that malfunctions in debt markets likely occurs through liquidity and risk premium channels, creating contagion for other markets. Also within the literature on securitization, Geanakoplos and Zame (2011) study how “security pools” affect efficiency of equilibrium, and Toda (2014) presents a model in which securitized ABS markets are a way of sharing idiosyncratic risk, and entrepreneurs endogenously borrow from loans with the lowest collateral requirement. ABS reallocate capital to high-risk, high-return technologies, and enhance welfare through improved risk sharing.

2 General Equilibrium Model with Collateral

This section presents the basic general equilibrium model with collateralized borrowing. To simplify the analysis and the exposition, we consider a multi-state extension of Geanakoplos (2003), which is a reduced-form version of a model with multiple (potentially finite) risk-averse agents with heterogeneity in endowments and preferences. This section introduces a general version of the model with collateral and financial contracts, and later sections will exogenously specify what types of financial contracts are allowed.

Time and Assets

We begin by considering a two-period, three-state general equilibrium model with time $t = 0, 1$. Uncertainty is represented by a tree $S = \{0, U, M, D\}$ with a root $s = 0$ at time $t = 0$ and three states of nature $s = U, M, D$ at time 1.

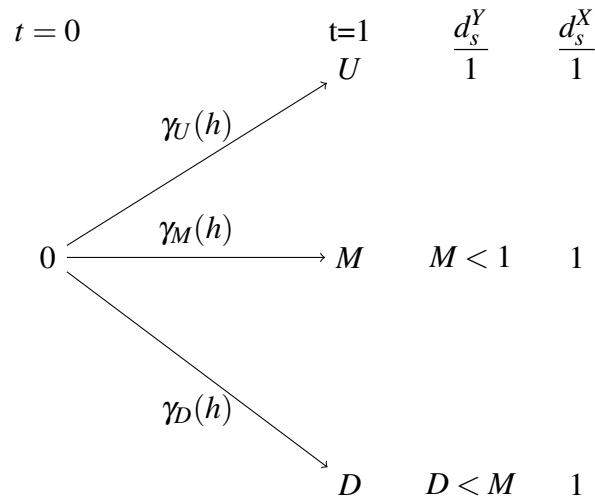


Figure 2.1: Payoff tree of assets X and Y in three-state world

There are two assets, X and Y , which produce dividends of the consumption good at time 1. Asset X is risk-free, producing (as a normalization) 1 unit of the consumption good in every state of the world. Asset Y is risky, producing $d_U^Y = 1$ unit in state U , $d_M^Y < 1$ units in state M , and $d_D^Y < d_M^Y$ units of the consumption good. To simplify notation, we let $d_M^Y = M$ and $d_D^Y = D$, denoting the state and the payoff by the same variable. Asset payoffs are shown in Figure 2.

We suppose that agents are uniformly distributed in $(0, 1)$, that is they are described by Lebesgue measure.¹⁵ Agents are risk-neutral and have linear utility for the consumption good c at time 1. Each agent $h \in (0, 1)$ assigns probability $\gamma_U(h)$ to the state U , $\gamma_M(h)$ to the state M , and $\gamma_D(h) = 1 - \gamma_U(h) - \gamma_M(h)$ to the state D . The probabilities $\gamma_U(h)$ and $\gamma_M(h)$ are continuous in h . We further specify that $\gamma_U(h)$ and the ratio $\gamma_M(h)/(\gamma_M(h) + \gamma_D(h))$ are monotonically increasing in h . The second condition implies that the subjective conditional probability of state M , given that U does not occur, is increasing in h . Hence, both conditions imply that a higher h indicates more optimism “uniformly.”

The expected utility of agent h is

$$U^h(c_U, c_M, c_D) = \gamma_U(h)c_U + \gamma_M(h)c_M + \gamma_D(h)c_D,$$

where c_s is consumption in state s . At time 0, each investor is endowed with one unit of each asset.

This setup is equivalent to a model with finitely many heterogeneous risk-averse agents, with restrictions on endowments and preferences such that marginal utilities or “hedging needs” are monotonic and uniformly increasing by state (see Appendix D). We stress that our paper is *not* fundamentally about belief disagreements and heterogeneous priors; none of the results of our analysis depend on risk-neutrality or heterogeneous priors. Our view is that the slightly unconventional modeling is a small price to pay for the simple tractability of the analysis.¹⁶

Financial Contracts and Collateral

The heart of our analysis involves contracts and collateral. We explicitly incorporate repayment enforceability problems, but exclude cash flow problems;¹⁷ collateral acts as the only enforcement mechanism. At time 0, agents trade financial contracts. A financial contract $j = (A^j, C^j)$, consists of a promise $A^j = (A_U^j, A_M^j, A_D^j)$ of payment in terms of the consumption good, and an asset

¹⁵We will use the terms “agents” and “investors” interchangeably.

¹⁶We could reproduce the distribution of marginal utilities we get from differences in prior probabilities by instead assuming common probabilities, strictly concave utilities, and by allocating endowments of consumption goods appropriately. An implication is that our results continue to hold (weakly) whether there are more agents than states or whether there are more states than agents.

¹⁷For an extensive analysis on the of the implications on asset prices, leverage and production arising from the distinction see Fostel and Geanakoplos (2015a,b). Crucially, in our model all safe assets truly are safe; assets perceived to be risk-free do not suddenly become risky. See Gennaioli et al. (2013) for a model in which a crisis occurs when “safe assets” are not truly safe.

C^j serving as collateral backing the promise. The lender has the right to seize as much of the collateral as was promised, but no more. Therefore, upon maturity, the financial contract yields $(\min\{A_U^j, d_U^{C^j}\}, \min\{A_M^j, d_M^{C^j}\}, \min\{A_D^j, d_D^{C^j}\})$ in states U , M , and D respectively. Agents must own collateral before making promises.

Let J be the set of all possible financial contracts. We first suppose that every contract is collateralized by Y , and without loss of generality we normalize the collateral to one unit of Y .¹⁸ We let J^Y be the set of promises j backed by one unit of Y . In Section 3.2 we introduce debt collateralization so that agents are also allowed to borrow against the debt contracts they hold (i.e., the amount owed to them by other agents). Note that even in this scenario, all financial contracts are ultimately collateralized by the risky asset.

Throughout our analysis we primarily consider non-contingent debt contracts so that $A^j = (j, j, j)$ for all $j \in J$, whether contracts are collateralized by Y or by debt contracts. However, in Section 3.4 we consider an environment with senior-subordinated state-contingent contracts.

We denote the sale of $|\varphi_j|$ units of a promise $j \in J$ when $\varphi_j > 0$ and the purchase of $|\varphi_j|$ units of the contract when $\varphi_j < 0$. The sale of a contract corresponds to borrowing the sale price and the purchase of a promise is equivalent to lending the price in return for the promise. The sale of $\varphi_j > 0$ units of a contract requires ownership of φ_j units of that asset, whereas the purchase of such contracts does not require ownership.

Budget Set

Each contract $j \in J$ trades for a price π^j . An investor can borrow π^j by selling contract j in exchange for a promise to pay A^j tomorrow, provided that he owns C^j . We normalize by the price of asset X to be 1 in all states of the world, making X the numeraire good. (Holding X is analogous to holding cash without inflation.) We let p denote the price of the risky asset Y . Given asset and contract prices at time 0, each agent decides how much X and Y he holds and trades contracts φ_j

¹⁸Notice that a contract promising .4 collateralized by 2 units of Y is exactly the same as 2 units of a contract promising .2 collateralized by 1 unit of Y .

to maximize utility, subject to the budget set

$$B^h(p, \pi) = \{(x, y, \varphi, c_U, c_M, c_D) \in R_+ \times R_+ \times R^J \times R_+ \times R_+ \times R_+ : \\ (x - 1) + p(y - 1) \leq \sum_{j \in J} \varphi_j \pi^j \quad (1)$$

$$\sum_{j \in J^Y} \max(0, \varphi_j) \leq y \quad (2)$$

$$c_s = x + y d_s^Y - \sum_{j \in J} \varphi_j \min(A_s^j, d_s^{C^j}) \}. \quad (3)$$

Equation (1) states that expenditures on assets (purchased or sold) cannot be greater than the resources borrowed by selling contracts. Equation (2) is the collateral constraint, requiring that agents must hold sufficient number of assets to collateralize the contracts they sell.¹⁹ Equation (3) states that in the final states, consumption must equal dividends of the assets held minus debt repayment. Recall that a positive φ_j denotes that the agent is selling a contract or borrowing π^j , while a negative φ_j denotes that the agent is buying the contract or lending π^j . Thus there is no sign constraint on φ_j . Additionally, we assume that short selling of assets is not possible ($y \geq 0$ and $x \geq 0$).

Collateral Equilibrium

A collateral equilibrium in this economy is a price of asset Y , contract prices, asset purchases, contract trade, and consumption decisions all by agents $((p, \pi), (x^h, y^h, \varphi^h, c_U^h, c_M^h, c_D^h)_{h \in (0,1)}) \in (R_+ \times R_+^J) \times (R_+ \times R_+ \times R^J \times R_+ \times R_+ \times R_+)^H$ such that

1. $\int_0^1 x^h dh = 1$
2. $\int_0^1 y^h dh = 1$
3. $\int_0^1 \varphi_j^h dh = 0 \forall j \in J$
4. $(x^h, y^h, \varphi^h, c_U^h, c_M^h, c_D^h) \in B^h(p, \pi), \forall h$
5. $(x, y, \varphi, c_U, c_M, c_D) \in B^h(p, \pi) \Rightarrow U^h(c) \leq U^h(c^h), \forall h.$

Conditions 1 and 2 are the asset market clearing conditions for X and Y at time 0 and condition 3 is the market clearing condition for financial contracts. Condition 4 requires that all portfolio and

¹⁹When we expand the assets that can serve as collateral for contracts we will have an additional collateral constraint for each asset serving as collateral.

consumption bundles satisfy agents' budget sets, and condition 5 requires that agents maximize their expected utility given their budget sets. Geanakoplos and Zame (2014) show that equilibrium in this model exists under the assumptions made thus far.

Discussion of the Financial Environment

The financial environment in our model is the set of assets used as collateral or the permissible promises that can be backed by the same collateral (the set of contracts J). Debt collateralization expands the set of contracts in J . We take the financial environment as exogenous.²⁰

Our analysis primarily considers debt contracts (non-contingent promises) and debt contracts collateralized by debt. Since the environment with debt collateralization is equivalent to allowing agents to issue state-contingent claims defined by senior-subordinated capital structure, there are at least three interpretations of our restriction to non-contingent contracts.

First, as we've discussed, debt contracts are the building blocks for senior-subordinated state-contingent payoffs. Thus, one can interpret our results with non-contingent contracts and collateralization as "a metaphor" for environments when agents can trade state-contingent contracts derived from senior-subordinated capital structures.

Second, agents may be restricted to non-contingent promises because of some un-modeled informational friction, or because markets are segmented and some investors are restricted to buying "tier-1" securities.²¹ Leverage and debt collateralization are mechanisms that create new state-contingent payoffs—and increase the supply of risk-free assets—from underlying non-contingent contracts *without* violating the informational friction (they depend on collateral seizure and limited repayment enforceability). Thus, our results provide an explanation for how financial markets create state-contingent contracts in the presence of these informational frictions.

Third, our results could extend to environments with cross-netting frictions. State-contingent contracts may be available, but agents may not be able to use an asset as collateral to back multiple

²⁰For motivations for why financial markets may *decrease* the available set of assets serving as collateral, see the informational explanations in Dang et al. (2009); Gennaioli et al. (2013); Gorton and Ordoñez (2014). One could note in addition that in our model after bad news, the natural buyers of assets used for securitized markets are wiped out.

²¹For examples relating to securitization see DeMarzo (2005); Pagano and Volpin (2012); Friewald et al. (2015). Mada and Soubra (1991) show that nonextremal securities (debt and equity rather than "Arrow Securities") may be optimal when securities must be marketed at a cost. Lemmon et al. (2014) provide evidence that one value of securitization (for nonfinancial firms) is providing access to segmented markets.

promises, even when doing so would still guarantee repayment. As shown by Geanakoplos and Zame (2014), equilibrium can be *endogenously incomplete* when collateral is scarce (agents may trade debt contracts even when Arrow securities are available because debt contracts economize on collateral). Shen et al. (2014) show that financial innovations are likely to occur in such a setting. Senior-subordinated capital structures allow an asset to simultaneously collateralize multiple state-contingent contracts.²² Thus, our restrictions reflect some combination of informational frictions limiting state-contingencies together with some degree of cross-netting frictions.

Financial systems differ in a myriad of both subtle and complex ways—for example, the level of insurance and risk sharing—but the financial structures we assume allow us to focus on the abilities to leverage and securitize assets in the most straightforward setting. Additionally, while investors in our model borrow or issue contracts directly against assets, this is without loss of generality. This process could also correspond to financial intermediaries producing the financial assets that correspond to these cash flows, or to firms issuing these securities as part of their capital structure.²³

3 Static Model

In this section we characterize equilibrium with debt contracts (“leverage economy”) and with debt collateralization. In the leverage economy, agents can issue non-contingent promises using the asset Y as collateral: $J = J^Y$, and each $A^j = (j, j, j)$ for all $j \in J^Y$. With debt collateralization contracts $j \in J^Y$ can also serve as collateral. We begin with the 3-state model and then generalize to N states. Equilibrium conditions are in Appendix A and all proofs are in Appendix B.

3.1 Leverage Economy

As shown by Fostel and Geanakoplos (2012b), in equilibrium two contracts are traded: $j_D = D$ and $j_M = M$, with prices π^D and π^M . The interest rate on j_D is zero because it is a safe promise,

²²Suppose that agents can trade Arrow contracts that pay in exactly one state. A cross-netting friction might require that one asset can only back a single Arrow security at a time, even though the asset could feasibly back a security for each state. Section 3.4 considers an economy when the asset can simultaneously back multiple state-contingent claims.

²³Similarly, investors could attain higher leverage through an intermediary when collateral is rehypothecated, as is common with Prime brokerage.

and thus $\pi^D = D$. However, the delivery of j_M depends on the realization of the state at time 1 and j_M is therefore risky (j_M pays (M, M, D)). This means that any agent making the promise j_M can only borrow $\pi^M < M$. Thus, the interest rate for j_M is strictly positive, defined by $i_M = \frac{M}{\pi^M} - 1$, and is endogenously determined in equilibrium.

In equilibrium there are three marginal investors h_M, h_D, h_π . Agents $h > h_M$ will sell their endowment of X , buy the asset Y , and promise M (issue j_M) for every unit of the asset bought.²⁴ Agents $h \in (h_D, h_M)$ will sell their endowment of X and buy the risky asset, promising D against every asset bought. Agents $h \in (h_\pi, h_D)$ will sell their endowment of X and Y and buy j_M (effectively lending to agents $h > h_M$). Notice that these agents do not hold any assets, only promises. Agents $h < h_\pi$ will sell their endowment of Y and buy both risk-free assets X and contracts j_D backed by the asset (these two are equivalent). Figure 3.1 illustrates the equilibrium regime.

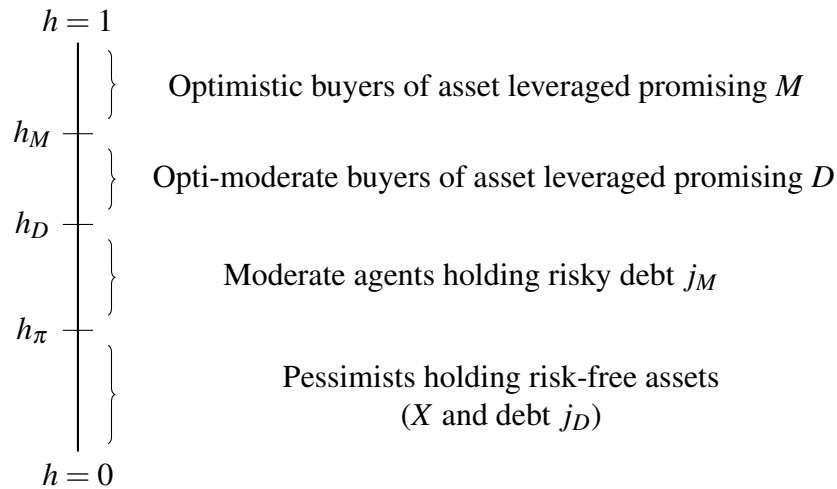


Figure 3.1: Equilibrium regime with leverage in static model.

We say that all agents $h > h_M$ are “maximally leveraged” in the sense that making a larger promise would simply result in a transfer of resources to lenders in the state(s) in which the asset has the maximum payoff. Agents can choose to promise more to attain additional leverage—they can make any promise j —but any promise $j > M$ is unattractive to borrowers. To see that this is the case, notice any contract $j > M$ has the same delivery as j_M in states M and D (because of

²⁴To simplify notation we will use strict inequalities when referencing the marginal agent, leaving the marginal agent’s decision ambiguous. This is without loss of generality since the marginal agent has measure-zero.

default against the asset's payoff) and delivers more only in state U . However, while U is that state that investors $h > h_M$ believe to be comparatively the most likely to happen, the larger promise in U is priced by more pessimistic agents. Hence, a promise $j > M$ would result in raising less than the value of the promise. Agents $h \in (h_D, h_M)$, promising D against each unit of the asset, are not maximally leveraged because promising M changes the delivery to borrowers in both states U and M .

3.2 Economy with Debt Collateralization

Let agents also be allowed to trade contracts of the form $j_\ell^1 = (\ell, j_M)$. This contract specifies a non-contingent promise (ℓ, ℓ, ℓ) backed by the risky debt j_M acting as collateral (the restriction to j_M is without loss of generality). In other words, we allow contracts with $C^j = j_M$ and denote such a contract by a superscript-1.²⁵ The payoff to j_ℓ^1 in each state is the minimum of the promise ℓ and the payoff of the debt contract j_M (i.e., $\min\{\ell, d_s^{j_M}\}$). Note that the act of holding j_M and selling the contract j_D^1 is equivalent to buying j_M with leverage promising D , yielding a payoff of $(M - D, M - D, 0)$.

Denote the set of contracts backed by j_M by J^1 . Now the set of contracts available for trade is $J = J^Y \cup J^1$. We denote equilibrium variables with debt collateralization by a 'hat' ($\hat{\cdot}$) to distinguish them from their counterparts with leverage. This expansion of the financial environment leads to the following results.

Lemma 1. *Suppose that in equilibrium agents are able to collateralize debt. Then every agent holding risky debt will maximally leverage their purchases of risky debt. That is, all agents holding j_M will sell the promise $j_D^1 = (D, j_M)$.*

The full proof is in the appendix, but the intuition is straightforward. Only the marginal agent investing in risky debt thinks the debt is priced to exactly compensate for risk, while every other agent thinks the expected payoff is higher than implied by the price and thus would like to leverage their investment in the debt. Since j_M pays (M, M, D) , promising D maximally leverages the investment in j_M .

²⁵We could let any contract $j \in J^Y$ serve as collateral; however, we show that in equilibrium only j_M will be traded and thus only j_M will serve as collateral.

This result has four important implications for equilibrium: (i) Agents investing in j_M use borrowed money to invest. (ii) Issuing risk-free debt against j_M increases the supply of safe assets. (iii) Because the risk premium on risky debt decreases, agents buying the risky asset with risky debt can borrow more for the same promise, making this investment strategy more attractive. (This implies that the marginal high-leveraged investor need not be as optimistic, i.e., $\hat{h}_M < h_M$.) (iv) Because investing in risky debt is more attractive (owing to the ability to leverage the investment) the marginal investor willing to buy Y is more optimistic because the least optimistic agents holding the risky asset in the leverage economy will now prefer to hold the risky debt with leverage because of its higher return. Critically, *in equilibrium no agent chooses to leverage Y by promising safe debt*, which is stated in the following lemma and proposition.

Lemma 2. *Let agents be allowed to collateralize debt. Then, every agent holding the risky asset will maximally leverage their purchases of the risky asset. In other words, every agent holding the risky asset will promise M .*

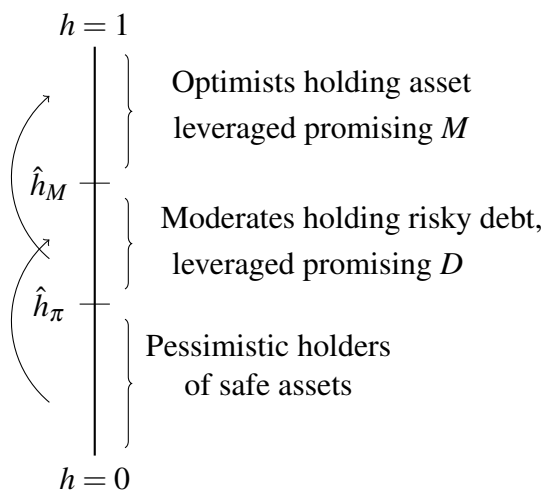
Proposition 1. *In equilibrium, there exist two marginal buyers \hat{h}_M and \hat{h}_π such that all $h \in (\hat{h}_M, \hat{h}_\pi)$ will hold risky debt with maximal leverage (promise D); all $h < \hat{h}_\pi$ will hold safe debt and X , and all $h > \hat{h}_M$ will hold the risky asset with maximal leverage (promise M).*

This result follows directly from the previous two lemmas and the fact that optimism is strictly and monotonically increasing in h . Figure 3.2 illustrates the equilibrium regimes with debt collateralization and with leverage. Theorem A.1 in Appendix A.3 generalizes the result to characterize equilibrium with N states and when debt can back debt, etc., L times, which we call “ L levels of debt collateralization.”

Notice that the marginal safe-debt investor increases with debt collateralization, reflecting a greater supply of safe debt. Collateralizing risky debt has thus served two purposes: it isolates upside payoffs to agents buying risky debt with leverage, and it creates safe debt for more pessimistic agents, increasing the supply of risk-free securities.

The key insight for this result is that the price of any asset is a sum of the payoff value and the collateral value. Allowing a debt contract to be used as collateral increases its price—it now has a collateral value—which increases the value to buying the risky asset and issuing that debt contract. Furthermore, because only the risky asset will back risky debt in equilibrium (the risky debt will

Equilibrium with Debt Collateralization



Equilibrium with Leverage

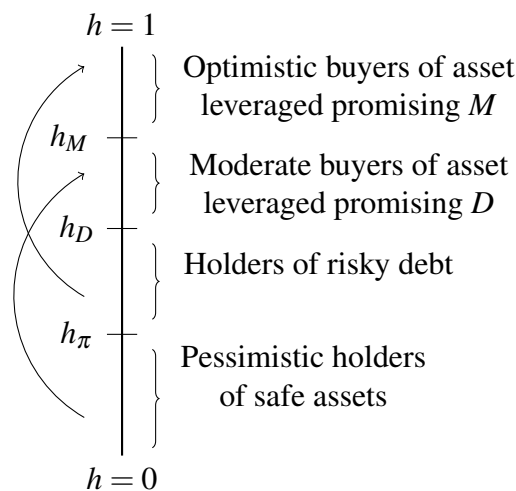


Figure 3.2: Equilibrium with leverage versus equilibrium with debt collateralization.

back safe debt in equilibrium), the collateral value of the risky debt, in effect, gets imparted to the risky asset. Using the risky asset to issue safe debt is “inefficient.” Instead, the risky asset can be used to back *both* risky debt and safe debt, by issuing risky debt against the asset, and then issuing safe debt against the risky debt. This process creates a new security with collateral value, while using the asset to issue safe debt does not.²⁶

3.3 Numerical Example

While our results hold across parameters and are not quantitative, a numerical example is helpful to suggest what happens to prices and economy-wide margins. We roughly “calibrate” in the following way. While surely many changes occurred pre-crisis, we choose to calibrate the model so that the move from leverage to debt collateralization explains the following moments. We target economy-wide average margins with leverage to be 15% and with debt collateralization to be 5%. We target risky debt spreads to be 3.9% with leverage and 1.6% with debt collateralization.²⁷ We

²⁶In terms of economic efficiency and welfare, the allocation with debt collateralization does *not* Pareto dominate the allocation with leverage. Pessimists benefit from debt collateralization because their wealth increases (the asset price is higher) but the price of risk-free assets is the same. Optimists and moderates are slightly hurt because, while they can more easily leverage their purchases, the assets they buy are more expensive.

²⁷Fostel and Geanakoplos (2012a) show that for subprime mortgages from 2000–2008, average margins decreased from 12% to 3% in 2006 and then increased to roughly 18% by end of 2007. Pre crisis 10 year Baa corporate bond credit spreads ranged from 3.9% to roughly 1.6% through 2007, which we use as a rough measure of financing spreads.

parametrize beliefs of the form $\gamma_U(h) = h^\zeta$ and $\gamma_M(h) = h^\zeta(1 - h^\zeta)$. Thus we choose parameters M , D , and ζ to match the moments. Our calibration yields $M = .93$, $D = .81$, $\zeta = 6.5$.²⁸ We discuss comparative statics below.

3.3.1 Baseline Results

Table 1 compares the equilibria with Leverage and with debt collateralization (“DC”). Beyond what we learn from the propositions, with the introduction of debt collateralization, economy-wide average margins fall dramatically and the price of the asset rises.

Prices and Marginal Investors			Interest Spread and Margins		
	Leverage	DC (^)		Leverage	DC (^)
p	0.9542	0.9608 ↑			
π^M	0.9014	0.9103 ↑	i_M	3.17 %	2.17% ↓
h_M	0.9984	0.9742 ↓	Risky Margin	5.53%	5.26% ↓
h_D	0.9289	–	Safe Margin	15.11%	–
h_π	0.9021	0.9231 ↑	Average Margin	14.78%	5.26% ↓

Margins With debt collateralization, economy-wide average margins decrease for two reasons: first, all agents who buy the risky asset use the low margin (high leverage) strategy; second, the risky margin (buying the asset with j_M) decreases because the risky debt price increases by relatively more than the asset price p . As is clear from this example (and across a wide range of parameters), the first effect is much larger. In this case average margins fall by 9.5%, close to our targeted calibration.²⁹

²⁸An alternative, attractive parametrization is to set payoffs to $M = .9$ and $D = .65$: the middle payoff corresponds to a mild recession for firms or a bad-but-typical decrease in house prices; the down payoff is a deep recession or a dramatic (35%) decrease in house prices. We then choose beliefs so that risky spreads and margin changes correspond roughly to levels over the early 2000s, yielding $\zeta = 2$. In this case, introducing debt collateralization, average margins decrease from 30.27% to 8.7%, spreads decrease from 3.88% to 2.45%, and the price increases by 1.16%.

²⁹The margin with debt collateralization is essentially determined by the payoff value M , with higher M leading to lower margins. The change in average margins from leverage to debt collateralization is primarily, though nonlinearly, driven by the difference between D and M . By choosing lower D the model can yield substantial decreases in average margins. For example, setting $D = .1$ yields average margins of 75% and 5.4% with leverage and with debt collateralization.

Prices The asset price increases by a modest .7 percent. Across a range of parametrizations, the model typically delivers modest increases in p .³⁰ This result is in line with recent evidence by Kaplan et al. (2015), who quantitatively assess the contributions of changes in mortgage margins, productivity, and expectations about future house prices to explain house prices during the housing boom and bust. They find that changing margins alone explain almost none of the increases in house prices, but that house prices are explained primarily by changes in expectations about future appreciation (margins are important for explaining other measures). Thus, our model is best understood as a model of margins and leverage. Accordingly, our model shows that debt collateralization can have very large effects on average margins.

The model highlights two competing forces affecting the asset price. First, there is a “leverage effect”: debt collateralization endogenously increases leverage in the economy and increases the collateral value of the risky asset, which increases the price.³¹ Second, there is a “required return effect”: the required return for investing in the risky asset increases because alternative investments—namely, investing in risky debt—have become more attractive owing to the ability to use leverage. This force tends to decrease the asset price.³²

3.3.2 Comparative Statics

Robustly, debt collateralization decreases average margins by shifting agents to high-leverage contracts, decreases margins on high-leverage contracts, decreases spreads, and increases the asset price. We discuss comparative statics for each calibrated parameter.

³⁰Prices change by more for lower D so that margins change by more as we change financial environments. However, we have not found examples with increases exceeding 4%.

³¹Fostel and Geanakoplos (2012a) show that, with sufficient heterogeneity regarding how states are valued, increasing the contingency of available promises increases the collateral value, and thus the price, of an asset. Using collateral to back contingent promises can increase the collateral value of the asset when agents differentially value state-contingent payoffs. Debt collateralization has a similar effect. Each level of debt collateralization increases the collateral value of a larger set of “upstream” debt contracts, which adds to the collateral value of the asset. In the limit, agents isolate payoffs to be above a certain threshold, receiving zero in default states. Equivalently, contracts have a higher collateral value because buying a contract with a larger promise creates a greater degree of state-contingency in the payoff. Debt collateralization is a way of adding a greater degree of state-contingency to non-contingent debt contracts.

³²For every set of beliefs we have tried defined by continuous and parametric functions, the first effect always dominates so that the asset price increases. Appendix A.4 provides an example where the price p decreases with debt collateralization.

Beliefs The parameter ζ determines the relative frequency of optimists and pessimists in the economy; equivalently, the frequency of pessimists can be interpreted as the relative demand for assets that pay in bad states (negative-beta assets), perhaps from hedging needs or risk aversion. High ζ corresponds to relatively more pessimists and low ζ to more optimists (with $\zeta > 1$, γ 's are convex; $\zeta < 1$, concave). Increasing ζ , has the following consequences: (i) in both financial environments risky spreads increase and risky and safe margins decrease; (ii) moving from leverage to debt collateralization results in a bigger change in spreads; (iii) average margins under leverage converge to the safe margin (for low ζ (many optimists), more agents use risky margins); (iv) moving from leverage to debt collateralization, margins decrease by a greater amount; (v) moving from leverage to debt collateralization, the percent change in p increases (nonlinearly) with ζ so that p increases by more when ζ is high.

Moderate Risk The parameter M determines the middle payoff. Increasing M has the following consequences: (i) risky margins drop and spreads increase (the risky promise is higher); (ii) safe margins increase for high ζ but are essentially unchanged for low ζ ; (iii) with leverage, average margins increase for high ζ , but for low ζ average margins are U-shaped (with lowest average margins occurring at roughly $M = \frac{D+1}{2}$); increasing M initially increases the fraction of borrowers using risky debt, but beyond a certain point fewer investors use risky debt; (iv) the percent change in p is hump shaped; (v) with debt collateralization, h_M and h_π diverge; in other words, as M and D differ, there is greater degree of debt collateralization (or securitization) in the economy.

Downside Risk The parameter D determines the worst payoff. Increasing D has the following consequences: (i) spreads decrease (default is not so costly), and the change in spreads decreases; (ii) all margins decrease, with the greatest change coming (directly) from safe margins; (iii) the percent change in p decreases.

3.4 Debt Collateralization and Senior-Subordinated Tranching

Tranching refers to the process of using collateral to back promises of different types. Senior-subordinated capital structures define tranches whose payoffs are state-contingent, with realized payoffs determined by the seniority of the tranche. By simple accounting, these capital structures

are equivalent to an economy with debt collateralization.

Let the states be given by $S = \{S_1, \dots, S_N\}$ with the payout of the asset Y being s_n in the state S_n . For convenience, we well-order the states so that $s_1 < s_2 < \dots < s_N$ and normalize so that $s_N = 1$. Consider an economy with no borrowing, but the asset Y can be split into the following tranches by a financial intermediary: T_1, \dots, T_{N-1} where T_1 pays s_1 in all states of the world, and T_k , where $k > 1$, pays $s_k - s_{k-1}$ for all states of the world S_n where $n \geq k$ and 0 otherwise. That is, one unit of the risky asset Y can be used to simultaneously back multiple promises, creating the following tranches:

$$\begin{aligned}
T_N &: (s_N - s_{N-1}, 0, 0, \dots, 0), \\
T_{N-1} &: (s_{N-1} - s_{N-2}, s_{N-1} - s_{N-2}, 0, \dots, 0), \\
&\vdots \\
T_2 &: (s_2 - s_1, s_2 - s_1, \dots, s_2 - s_1, 0), \\
T_1 &: (s_1, s_1, \dots, s_1).
\end{aligned}$$

Note that $T_1 + T_2 + \dots + T_N = Y$. We refer to this financial structure as *senior-subordinated tranching* to emphasize the state-contingency is defined according to a senior-subordinated capital structure.³³

In this economy investors buy and sell the tranches listed above rather than trading the risky asset Y (though they can exactly replicate Y by buying all the tranches). We specify that each investor must hold a non-negative quantity of each tranche and refer to equilibrium as the senior-subordinated tranching equilibrium. In Appendix A.3 we formally define the concept of “complete collateralization,” which refers to when all debt can be used to back financial promises, and we characterize equilibrium. This yields the following result (with formal conditions in the Appendix).

Proposition 2. *The senior-subordinated tranching equilibrium is equivalent to equilibrium with complete debt collateralization. That is, there exists a bijective mapping of assets and prices from the debt collateralization equilibrium to the senior-subordinated tranching equilibrium such that the buyers of assets remain the same.*

³³In contrast, complete tranching would refer to the creation of Arrow securities for each state so that tranche T_s would be pay 1 in state- s and zero otherwise, not just paying zero in down states.

While the result follows essentially from accounting, the result is important: tranching and debt collateralization have an essential equivalence. Additionally, the senior-subordinated tranching equilibrium does not require the existence of a separate financial intermediary. To see this, consider an economy where anyone holding the asset can use it to back the promises stated above. Then, every agent will keep/buy the tranche that provides the highest expected return and sell the rest. This provides the equilibrium that we have already stated. One natural candidate for the “intermediary” would be the most optimistic buyers who buy the asset, keep the tranche T_N , and sell off the remaining tranches to the other investors.

In reality financial innovation included forms of both tranching and debt collateralization. Subprime mortgage pools have been used to create tranches of different seniority. Each tranche of the asset-backed security (“ABS”) pays different amounts depending on the aggregate value of the mortgage pool (i.e., in different states of the world). A typical ABS deal tranches a pool of mortgages into 4 or 5 rated bonds and a residual, or equity, tranche. These tranches (typically the mezzanine bonds) are then be pooled together to serve as collateral for a CDO, which would issue another 4-5 bonds. And the process continues as the tranches from the CDO are collateralized into a CDO-squared. Each stage includes both tranching and collateralization of existing debt securities. Because mortgage pools do contain idiosyncratic risk, pooling tranches together to diversify this risk is an important step of the securitization process.

4 Price Volatility in a Dynamic Model

The static models illustrates that debt collateralization leads to agents making larger promises, increasing the leverage in the economy. In this section we examine how debt collateralization affects volatility and default. The main result of this section is that in equilibrium debt collateralization exacerbates and amplifies price fluctuations, creating more price volatility and more defaults than occur with leverage alone.³⁴

We consider a dynamic variation of the model in Section 3 with three periods, $t = 0, 1, 2$ following Geanakoplos (2003, 2010). Uncertainty in the payoffs of Y is represented by a tree

³⁴Geanakoplos (2003, 2010) demonstrate that using an asset as collateral creates a “Leverage Cycle” in which asset prices become more volatile because of fluctuations in the asset’s collateral value and the distribution of investors’ wealth.

$S = \{0, U, M, D, UU, MU, MD, DU, DD\}$, illustrated in Figure 4.1. The asset pays only at $t = 2$ with payoffs d_s^Y . To simplify, we will normalize the asset payoffs so that $d_{UU}^Y = d_{MU}^Y = d_{DU}^Y = 1$. Thus the possible “down payoffs” of the asset are d_{MD}^Y and $d_{DD}^Y < d_{MD}^Y$. In other words, the payoff tree is binary at $t = 1$ with a worse possible realization at state M than at D , and at $t = 0$ there is uncertainty about what the minimum possible asset payoff will be.

The risky asset Y has price p_0 at $t = 0$ and prices p_M and p_D in states M and D in $t = 1$. (In state U the price is trivially 1.) Just as before, we first look at an economy where leverage is the only financial innovation and then move on to explore the consequences of debt collateralization.

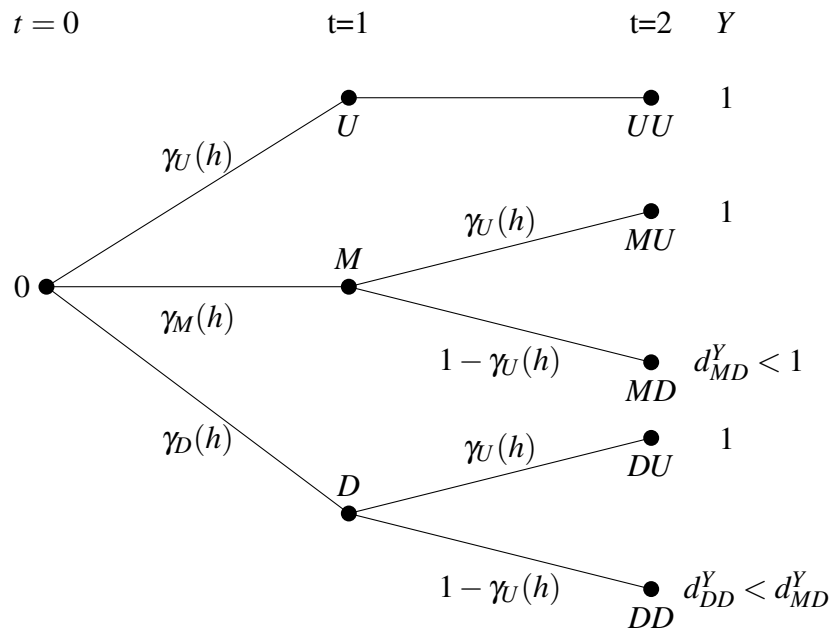


Figure 4.1: Payoff tree for risky asset in dynamic three-state model.

4.1 The Dynamic Economy with Leverage

With leverage, the dynamic equilibrium is essentially different from the static equilibrium because of the interaction between prices and leverage across time. However, the equilibrium regimes in each state resemble the equilibrium regime in the static economy of Section 3. The dynamic equilibrium with leverage is as follows.

In equilibrium, at time 0 there are three marginal agents, h_{M0} , h_{D0} , and $h_{\pi 0}$. Agents $h > h_{M0}$ buy the risky asset and promise p_M (i.e., they sell the contract j_{p_M}), which is a risky promise (the

contract j_{p_M} delivers $p_D < p_M$ in state D); agents $h \in (h_{D0}, h_{M0})$ buy the risky asset and promise p_D (i.e., they sell the contract j_{p_D}), which is a risk-free promise; agents $h \in (h_{\pi_0}, h_{D0})$ buy the risky debt j_{p_M} ; and agents $h < h_{\pi_0}$ buy risk-free asset X and risk-free debt j_{p_D} . Unlike in a binomial economy, there is a possibility of default in the down state D because agents $h \in (h_{M0}, 1)$ cannot pay off the entirety of their debt, having promised p_M when the asset is only worth $p_D < p_M$. We denote the price of the risky debt j_{p_M} by π_0 , which has interest rate $i_0 = \frac{p_M}{\pi_0} - 1$.

At time 1, agents receive news about the economy, borrowers repay their debts (margin calls occur) and the remaining agents trade assets and issue new promises. Because the economy is binomial at time 1, in equilibrium agents trade only risk-free contracts. In equilibrium there is one marginal investor in each state, with the remaining optimistic investors buying the risky asset against the maximal risk-free promise possible given the state. Thus, in state M there is a marginal investor h_{MM} . Investors $h > h_{M0}$ have zero wealth after repaying their promise. Investors $h \in (h_{MM}, h_M)$ buy the risky asset and promise M , which is the minimum payoff at $t = 2$. Investors $h < h_{MM}$ buy risk-free assets. In state D there is one marginal investor h_{DD} . Investors $h > h_{D0}$ have zero wealth after repaying their promise. Investors $h \in (h_{DD}, h_{D0})$ buy the risky asset and promise D , which is the minimum payoff at $t = 2$. Investors $h < h_{DD}$ buy risk-free assets.³⁵ Table 2 gives the wealth of agents at time 1 based on their previous investment choices.

Table 2: Wealth of agents at time 1

	State M	State D
$h \in (h_{M0}, 1)$	0	0
$h \in (h_{D0}, h_{M0})$	$\left(\frac{1+p_0}{p-p_D}\right)(p_M - p_D)$	0
$h \in (h_{\pi_0}, h_{D0})$	$\left(\frac{1+p_0}{\pi_0}\right)p_M$	$\left(\frac{1+p_0}{\pi_0}\right)p_D$
$h \in (0, h_{\pi_0})$	$1 + p_0$	$1 + p_0$

It is instructive to compare the 3-state model with leverage to the standard binomial model. Compared to the binomial model, crashes in this economy are larger. One reason the crash in D is so large is that investors who bought the risky debt are receiving less than the face value, and less than they invested. This “default mechanism” depresses p_D because remaining investors have

³⁵Since we do not know the positions of h_{MM} and h_{DD} relative to the marginal investors at time 0, there are several possible equilibrium cases. These cases, as well as the equations defining equilibrium, are listed in Appendix C.

less wealth. Larger crashes occur precisely because the three-state model with collateralization has more bankrupt agents at time 1 in the down state when compared to the two-state models.³⁶

4.2 The Dynamic Economy with Debt Collateralization

Given our results in the previous section, we can easily characterize equilibrium in the dynamic model with debt collateralization. In equilibrium there are two marginal agents at time 0, \hat{h}_{M0} , and $\hat{h}_{\pi 0}$. Agents $h > \hat{h}_{M0}$ buy the risky asset and promise p_M ; agents $h \in (\hat{h}_{\pi 0}, \hat{h}_{M0})$ buy the risky debt with promise p_M and use it as collateral to promise p_D ; and agents $h < \hat{h}_{\pi 0}$ buy the risk-free asset X and the risk-free debt (with promise p_D). In equilibrium, at time 1 there is one marginal investor in each state as discussed previously. Notice that if the economy is in state M at time 1, then agents $h \in (\hat{h}_{M0}, 1)$ will be bankrupt; if the economy is in state D at time 1, agents $h \in (\hat{h}_{\pi 0}, 1)$ will be bankrupt.³⁷

We solve the system numerically with $\zeta = 2$, and payoffs $d_{MD}^Y = .3$ and $d_{DD}^Y = .1$. Table 3 gives the equilibrium with debt collateralization and compares it to the equilibrium with leverage. The price crash in states M and D are larger with debt collateralization. Note that the “default mechanism” is effectively much greater with debt collateralization since all debt in the economy is fully collateralized and leveraged. Rather than being poorer in the down state, agents holding risky debt will be completely out of the market. In this case, debt collateralization leads to even more volatility since the agents buying the asset in the down state will be more pessimistic.

Crashes in states M and D increase by 1.44% and 1.09%. Our result that debt collateralization increases volatility is closely related to previous work studying collateral values and volatility (Fostel and Geanakoplos, 2012a; Fostel et al., 2015). We have shown that debt collateralization increases the collateral value of debt contracts and of the risky asset, and as a result, asset price volatility increases because debt collateralization increases fluctuations in both collateral values

³⁶We isolate the impact of the default mechanism in Appendix C.3 by considering a surprise bailout in $s = D$ to replace the wealth lost to default. Appendix C.3 also compares the 3-state dynamic model to corresponding binomial models and shows that in each case the price crash in the down state of the 3-state world is greater than the price crash in any of the two-state models and that the biggest price crash is obtained in the case of debt collateralization. Appendix C.4 explores comparative statics for how prices and marginal investors change depending on the payout in d_{MD}^Y .

³⁷Note that we do not know the position of \hat{h}_{MM} relative to the positions of the other marginal investors at time 0, but we do know the relative position of \hat{h}_{DD} . The equations defining equilibrium are in Appendix C.

Table 3: Dynamic Equilibrium with Debt Collateralization and with Leverage

	Leverage	Collateralization ($\hat{\cdot}$)
p_0	0.805	0.814 \uparrow
p_M	0.663	0.659 \downarrow
p_D	0.434	0.431 \downarrow
π_0	0.595	0.611 \uparrow
h_{M0}	0.963	0.888 \downarrow
h_{D0}	0.823	–
h_{π_0}	0.719	0.789 \uparrow
h_{MM}	0.720	0.717 \downarrow
h_{DD}	0.610	0.606 \downarrow
M Crash	17.58 %	19.02% \uparrow
D Crash	46.00 %	47.09% \uparrow

and the distribution of wealth.^{38,39}

Comparative Dynamics

The model is highly nonlinear. For comparative dynamics we focus on how the model parameters affect how volatility changes (how the M and D crashes change) when we introduce debt collateralization.

When beliefs are concave ($\zeta < 1$) indicating a larger proportion of optimism, the changes in both crashes are larger for larger D . Furthermore, the change in both crashes are smaller for smaller ζ ; with more optimism, price crashes are already large and debt collateralization does little to amplify fluctuations. As a function of M , changes in crashes are hump-shaped.

When beliefs are convex ($\zeta > 1$) indicating a larger proportion of pessimism, changes in M

³⁸Fostel and Geanakoplos (2012a) show that asset price volatility increases when agents can tranche assets. Tranching increases the collateral value of the risky asset, and in a dynamic setting the “Tranching Cycle” exhibits larger fluctuations in collateral values and in the distribution of wealth. In a two country model, Fostel et al. (2015) show that this result is amplified by international financial flows, which can further increase collateral value because of international demand for collateral-backed financial promises.

³⁹In general, debt collateralization improves welfare for pessimistic agents, whose wealth increases while the price of risk-free assets remains the same, but decreases welfare for optimistic agents because the risky asset and risky debt are more expensive. It is not necessarily the case in this model that increased volatility is bad for welfare. Given the stylized nature of this model, it is worth considering the effects of volatility in a richer model. Our model leaves out many important factors, such as production, investment, cash flow problems, agency issues, and bankruptcy costs, just to name a few. Geanakoplos (2010) discusses a number of these issues in greater detail. Phelan (2016) uses a richer model to show that financial leverage creates a pecuniary externality through excess volatility, resulting in an economy with higher likelihood of crises and recessions and lower welfare. One can suspect that incorporating the mechanisms of our paper into a richer framework with any of the aforementioned issues would result in welfare losses from increased volatility.

crashes are non-monotonic in M (changes in the D crash is still hump shaped). Over these regions the “required return effect” exceeds the “leverage effect” leading to more moderate fluctuations. For high convexity, debt collateralization can *decrease* the crash in the M state for moderate levels of M but *increase* the crash for high M .⁴⁰ Changes in crashes can also be much larger (over 3%).

5 Testable Implications

Our model offers several testable implications regarding how explicit and implicit uses of debt as collateral affect returns and leverage: (i) capital structures are affected by funding markets and ease of financing, designed in part to stretch collateral; (ii) debt collateralization decreases economy-wide margins as borrowers shift to using high-leverage, risky contracts; (iii) debt collateralization decreases risk premia, increases defaults, and tends to increase asset prices and volatility; (iv) these effects are strongest when the economy has larger demand for negative-beta assets.

5.1 Static Implications

Capital Structures The main result of our analysis is that debt collateralization (implicit or explicit) leads investors to take maximal leverage, or more broadly fewer investors use low leverage. While the corporate finance literature has emphasized the role of capital structure in mitigating informational frictions, our results imply that capital structures are defined in part to stretch scarce collateral: when “collateral is tight” capital structures should be designed to further stretch collateral.⁴¹ An empirical test of this prediction could be to use measures of ease of financing (e.g. loan margins or haircuts in funding markets, or measures that typically correlate with risk and other determinants of credit conditions) to see how capital structures (for LBOs, syndicated loans, mortgages, etc.) respond to changes in funding markets, controlling for incentive conflicts.⁴²

⁴⁰For high M the price increases ($p_M > p_0$), so the price increase in state M would be muted.

⁴¹Critically, our analysis assumes that default is “costless,” which is obviously problematic in many situations. Asquith et al. (1994) find that debt structure affects restructuring decisions, and Alderson and Betker (1995) find that when liquidation costs are high, firms choose capital structure to minimize firm distress.

⁴²For example, Axelson et al. (2013) find that public firms have high leverage when credit spreads are high (when credit is expensive), whereas LBO deals have high leverage when credit spreads are low; they interpret LBO’s “buy expensive when credit is cheap” to reflect agency problems between private equity sponsors and their investors. Rauh and Sufi (2010) show that low-credit-quality firms more likely to have multi-tiered capital structure, with secured bank debt (tight covenants) and subordinated non-bank debt (loose covenants), in order to reduce incentive conflicts.

Empirically, growth in the origination of CDOs, CDO-squareds, etc., should all else equal translate into changes in capital structures in the ABS deals underlying CDO structures. Implicit or explicit debt collateralization would imply that subordinated tranches would be larger and riskier, with the largest effect on the most subordinated tranches. However, rather than affecting capital structure, the effect of greater collateralization could manifest itself by changing the underlying composition of ABS collateral. If higher prices would change incentives for mortgage lenders, then debt collateralization could contribute to those incentives. While Benmelech et al. (2012) find no evidence that corporate loans securitized in CLOs were riskier, there is evidence that securitization affected underlying loans in the subprime mortgage market.⁴³ Our analysis takes as given the collateral quality and ignores any informational asymmetries.

Risk Premia Our results imply that risky debt that can be used as collateral should experience decreased risk premia. Empirically, Nadauld and Weisbach (2012) find evidence that securitization of corporate loans was associated with lower spreads by 17 basis points, consistent with a reduction in cost of capital. Lemmon et al. (2014) find that securitization of receivables by nonfinancial firms decreases financing costs by providing access to segmented markets, and innovations in capital structure increase firm value even for large, mid-tier credit firms. Measures of how funding markets treat derivative debt contracts should correspond to increases in debt riskiness together with lower risk premia. In our model the face values of promises are fixed, but one might expect more generally to see debt collateralization leading to larger face values as well, and thus riskier debt.

While debt collateralization does not always increase the price of the asset, we conjecture that the “leverage effect” pushing up prices should in general outweigh the “required return effect” pushing down prices. Thus, increased ability to use debt as collateral should increase prices of underlying collateral (or push down risk premia). However, as noted earlier, to the extent that collateral quality can change or is subject to informational frictions, prices may stay the same and instead composition changes.

⁴³Keys et al. (2010) show that securitization does not change the interest rate or the LTV ratio for mortgages, but nonetheless affects the subsequent performance of mortgages through reduced screening by lenders. Nadauld and Weisbach (2012) find that securitization led to less screening for subprime borrowers. Additionally, Wang and Xia (2014) find that banks active in CLO securitization exert less effort on ex-post monitoring.

Origination and Securitization Volume Debt collateralization may provide incentives to produce collateral.⁴⁴ Shivdasani and Wang (2011) find evidence that the leveraged buyout (LBO) boom of 2004 to 2007 was fueled by growth in CDOs and other forms of securitization, which facilitated much larger LBOs than historically possible, potentially because it helped relax balance sheet constraints that banks faced in financing large LBOs.

Securitization decreased spectacularly after the crisis (see Chernenko et al. 2014). While there are many explanations for why this occurred, one possibility is that investors realized many ABS tranche payoffs were better described by “pay full or zero”—in this case debt collateralization is not meaningful. It is well understood that correlations in ABS tranches were the primary determinant of CDO quality, and as the mortgage bubble burst it became clear that CDOs would be essentially worthless or of full-value, with very little likelihood of intermediate values. As the comparative statics demonstrate, when uncertainty is less dispersed (M closer to D), there is less debt collateralization in equilibrium. The incentive for debt collateralization would decrease if uncertainty more closely followed a binomial model (in a binomial model all equilibrium debt is risk free). An implication is that the extent of debt collateralization (whether measured by origination volume or margins on securitized debt) should vary with perceptions of dispersion of payoffs conditional on default. Greater risk dispersion should lead to less debt collateralization.

Pessimism Finally, an economy with relatively more pessimists, or more demand for negative-beta assets, exhibits more sensitivity to debt collateralization: the increase in prices is greater; the change in risk premia is greater; the composition of margins is more affected. Critically, with relatively more pessimists, agents with leverage primarily use low leverage by making safe promises, but with debt collateralization agents switch to using high leverage by making risky promises.

5.2 Dynamic Implications

Our analysis suggests that debt collateralization increases volatility, which has the following implications:

(i) Assets with easily financed derivative debt should have prices that fluctuate more in response to news or other changes in fundamentals (given the volatility of fundamentals). However, one

⁴⁴see Fostel and Geanakoplos 2015a.

might expect that less volatile assets receive better treatment as collateral (endogeneity is a concern). Additionally, debt collateralization should be associated with increased default rates (how funding markets treat collateral is clearly endogenous, reflecting in part expected default rates).

(ii) Since debt collateralization can also be a metaphor for capital structure, the price of firms whose capital structure implicitly provides debt collateralization (through greater use of subordinated tranches, for example) should be more volatile. Since capital structure is endogenous, testing this prediction would require using some exogenous variation in capital structure (perhaps driven by corporate governance or regulation).

(iii) The model with only leverage suggests that when there is greater dispersion in payoffs conditional on default (M and D are more different) then price crashes after really bad news becomes worse because agents make more risky promises. Measures of uncertainty should correlate with larger price crashes, holding fixed expectations about worst-case scenarios.

6 Conclusion

Securitized markets and senior-subordinate capital structures implicitly and explicitly give investors the ability to use debt contracts as collateral for further promises (“debt collateralization”). The ability to collateralize debt backed by a risky asset decreases margins on the risky asset (increases leverage) and decreases the risk premia for risky debt: agents exclusively use maximal leverage, which is also cheaper because risk premia fall. When debt serves as collateral, the price of debt increases, giving leveraged investors an incentive make larger promises to use more leverage. The key insight of our results is that debt collateralization is a way of stretching scarce collateral. Our results suggest that one motivating factor for senior-subordinated capital structures is to provide a way to stretch scarce collateral.

Our analysis offers several testable implications: (i) capital structures are affected by funding markets and ease of financing, designed in part to stretch collateral; (ii) debt collateralization decreases economy-wide margins as borrowers shift to using high-leverage, risky contracts; (iii) debt collateralization decreases risk premia, increases defaults, and tends to increase asset prices and volatility; (iv) these effects are strongest when the economy has larger demand for negative-beta assets.

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Appendices

A Solving for Equilibrium in the Static Model

A.1 Leverage Economy

Marginal investors are indifferent between two different options, defined by equalizing the expected returns (defined as the expected marginal utility divided by price) on the different investments. Agent h_M is indifferent between buying asset with leverage promising M and buying asset with leverage promising D ,

$$\frac{\gamma_U(h_M)(1-M)}{p-\pi^M} = \frac{\gamma_U(h_M)(1-D) + \gamma_M(h_M)(M-D)}{p-D}. \quad (4)$$

Agent h_D is indifferent between buying asset promising D and holding risky debt j_M ,

$$\frac{\gamma_U(h_D)(1-D) + \gamma_M(h_D)(M-D)}{p-D} = \frac{(1-\gamma_D(h_D))m + \gamma_D(h_D)d}{\pi^M}. \quad (5)$$

Agent h_π is indifferent between holding risky debt j_M and holding safe assets,

$$\frac{(1-\gamma_D(h_\pi))M + \gamma_D(h_\pi)D}{\pi^M} = 1. \quad (6)$$

Market clearing for the risky asset Y requires

$$(1 - h_M) \frac{1 + p}{p - \pi^M} + (h_M - h_D) \frac{1 + p}{p - D} = 1, \quad (7)$$

and market clearing for the risky debt j_M requires

$$(1 - h_M) \frac{1 + p}{p - \pi^M} = (h_D - h_\pi) \frac{1 + p}{\pi^M}. \quad (8)$$

Equation (7) states that the agents buying the risky asset, $h \in (h_D, 1)$, will spend all of their endowment, $(1 + p)$, to purchase the risky asset, which costs price p , borrowing either π^M or D to leverage their purchases, and that the demand is equal to the supply of the risky asset, 1. Equation (8) states that the amount of risky debt demanded by agents $h \in (h_M, 1)$ is equal to the amount of risky debt supplied by agents $h \in (h_\pi, h_D)$.

A.2 Debt Collateralization

Thus, with collateralization we have the following equations defining the marginal investors (again given by equalizing expected returns on two investment options). Agent \hat{h}_M is indifferent between holding the risky asset with leverage and the risky debt with leverage,

$$\frac{\gamma_U(\hat{h}_M)(1 - M)}{p - \hat{\pi}^M} = \frac{\gamma_U(\hat{h}_M)(M - D) + \gamma_M(h_i)(M - D)}{\hat{\pi}^M - D}. \quad (9)$$

Agent \hat{h}_π is indifferent between holding the risky debt with leverage and the safe asset,

$$\frac{\gamma_U(\hat{h}_\pi)(M - D) + \gamma_M(\hat{h}_\pi)(M - D)}{\hat{\pi}^M - D} = 1. \quad (10)$$

Market clearing for the risky asset Y requires

$$\frac{(1 - \hat{h}_M)(1 + \hat{p})}{\hat{p} - \hat{\pi}^M} = 1, \quad (11)$$

and market clearing for the risky debt j_M requires

$$\frac{(1 - \hat{h}_M)(1 + \hat{p})}{\hat{p} - \hat{\pi}^M} = \frac{(\hat{h}_M - \hat{h}_\pi)(1 + \hat{p})}{\hat{\pi}^M - D}. \quad (12)$$

A.3 Generalization to N States and L Levels of Collateralization

With three states, the economy with leverage features only one risky contract in equilibrium. Once this contract can be used as collateral, no agent invests in risky debt and makes a risky promise. In other words, allowing the first risky contract to be securitized was sufficient so that no further contracts could be securitized. With more than 3 states this result is no longer true. In a leverage economy, multiple *risky* contracts will be traded in equilibrium. If agents can use these initial debt contracts as collateral, in equilibrium some agents will invest in risky debt contracts and make *risky* promises. These second-level debt contracts (backed by debt backed by the asset) could in principal be used as collateral to make further promises. Equilibrium will thus depend on how many “levels of debt” can be used as collateral (i.e., how many stages removed are promises from the risky asset ultimately backing those promises).

This issue is precisely what we analyze in this section. We show that every level of debt collateralization increases the minimum promise made by agents buying the asset, and with “complete collateralization”—when any existing risky debt contract can be used as collateral—agents make the maximum (natural) promise available for every investment, risky asset or risky debt.

We will now generalize the above lemmas and proposition to a static model with N states and L levels of collateralization. Considering multiple levels of collateralization requires introducing some new notation for debt contracts issued at each level of collateralization. Let the states be given by $S = \{S_1, \dots, S_N\}$ with the payout of the asset Y being s_n in the state S_n . For convenience, we well-order the states so that $s_1 < s_2 < \dots < s_N$ and normalize so that $s_N = 1$. Each agent h assigns probability $\gamma_n(h)$ to the state S_n and we have that

$$\sum_{n=1}^N \gamma_n(h) = 1, \quad \gamma_n(h) \geq 0 \quad \forall n, h.$$

We assume that for all $M \in [2, N]$, the ratio

$$\frac{\gamma_M(h)}{\sum_{n=1}^M \gamma_n(h)}$$

is monotonically increasing in h . This condition implies that the subjective conditional probability of state S_M given states $\{S_1, \dots, S_M\}$ is increasing in h , denoting that optimism increases uniformly with h .

In equilibrium with leverage, agents can buy the risky asset leveraged with any promise s_1, \dots, s_N by selling the promise $j_n = (s_n, Y)$. That is, the agent promises to pay s_n at time 1 and uses Y as collateral. Note that the full payout of the promise can only be realized in states S_n, \dots, S_N . For all states S_l with $l < n$, the seller of j_n defaults and the buyer of j_n seizes the asset as collateral, which is worth $s_l < s_n$. Thus, each j_n pays $(s_n, s_n, \dots, s_2, s_1)$ in the states $(S_N, S_{N-1}, \dots, S_2, S_1)$. Additionally, note that $j_1 = (s_1, Y)$ is safe debt.

We write Y/j_n to denote the act of holding Y and selling the debt contract j_n , and denote the price of the debt contract j_n by π_n . In the absence of debt collateralization, agents will do one of the following in equilibrium:

1. hold Y/j_n , where $1 \leq n \leq N - 1$,
2. hold risky debt j_n with $2 \leq n \leq N - 1$,
3. hold safe debt j_1 or the safe asset.

Definition A.1. We say the *first level of debt collateralization* is the creation of promises j_n^1 using $j_k \in J^Y$ as collateral. Denote the set of contracts at the first level of debt collateralization by J^1 . We write $j_n^1(j_k) = (s_n, j_k)$ to denote the *debt contract* that is traded when an agent holding j_k and sells the promise j_n^1 . Note that s_n is the amount promised and we must have $k > n$. Again, an agent holding j_k and selling j_n^1 is denoted by j_k/j_n^1 .

For a contract j_k to be meaningful collateral for a promise s_n it must be that $s_k > s_n$ because otherwise the payoff to j_k would always be less than the promise (and equality would render the new promise redundant). Thus, in what follows we will only consider when agents use meaningful collateral to make new promises, requiring that $k > n$ for any contract $j_n^1(j_k)$. Given this restriction, the payoffs to $j_n^1(j_k)$ are the same for every $k > n$, and so we can denote the price of a contract $j_n^1(j_k)$ by π_n^1 .

Definition A.2. Denote the set of contracts at the L -th level of debt collateralization by J^L . The L -th level of debt collateralization is the creation of the promises j_n^L , backed by contracts $j_k^{L-1} \in J^{L-1}$, where $1 < n < N - L$ and $1 < k < N - L + 1$. In other words, the buyer of the promise j_k^{L-1} is able to sell the promise j_n^L , using j_k^{L-1} as collateral. Again, we must have $n < k$. We denote the promise of j_n^L with j_k^{L-1} as collateral by writing $j_n^L(j_k^{L-1}) = (s_n, j_k^{L-1})$. We denote an agent buying j_k^{L-1} and selling j_n^L by j_k^{L-1} / j_n^L .

With L levels of debt collateralization, the set of financial contracts is given by $J = J^Y \cup J^1 \cup \dots \cup J^L$. Thus, each additional level of collateralization involves the creation of new bonds, and allows all previously existing, risky bonds to be purchased with leverage. So long as the backing collateral is meaningful, given the monotonicity of payoffs for debt contracts, the payoff of any contract is defined by the promise. We use π_n^L to denote the price of any debt security $j_n^L(j_k^{L-1}) \in J^L$ with $k > n$. Note that for all l , $j_1^l(j_j^{l-1}) = (s_1, j_j^{l-1})$. Thus, the price of $j_1^l(j_j^{l-1})$ is s_1 for all l because it is risk-free debt. The following theorem describes equilibrium. We can explicitly characterize equilibrium for any level of collateralization.

Theorem A.1. *Consider an economy in which, when agents can leverage, $N - 1$ contracts are traded in equilibrium. At the L -th level of debt collateralization, at most the following leveraged positions exist in the economy*⁴⁵

- Y / j_n , where $L < n < N$
- j_j^l / j_k^{l+1} , where $0 \leq l < L$, $L - l < j < N - l$, $L - l \leq k < j$
- j_ℓ^L , where $1 \leq \ell < N - L$.

Additionally, more optimistic investors invest in assets with larger face values, and within each asset-class investors are ordered by the amount of leverage they use.

The intuition for this result is that each level of collateralization increases the collateral value of new promises and of every debt contract that could already be used as collateral. As collateralization increases, more debt contracts have collateral value, which increases the price of the newly securitized debt as well as the values of all the “upstream” debt contracts that can back

⁴⁵With $N > 3$ states, it is not always the case that all of the listed contracts will be traded. Fixing agents’ beliefs, the contracts traded in equilibrium depend on the state payoffs; however, there exist robust sets of parameters such that all of the contracts are traded. We have not been able to characterize which contracts are traded under which conditions, but we characterize all of the contracts that could possibly be traded.

those promises. As a result, when a security can be used to back promises that serve as collateral L times, making a smaller promise than stipulated by the theorem would not maximize the collateral value of debt contracts. Thus, investors make the largest promise that maximizes the collateral value of “downstream” promises.

We state a few implications of the theorem to provide more meaning. The second and third corollaries follow immediately from Theorem A.1.

Corollary A.1. *In an N -state model with $N \geq 3$, there can be at most $N - 2$ levels of debt collateralization.*

Proof. At $N - 2$ levels of debt collateralization, we have that only the agents holding the risky asset Y are holding it leveraged against state S_{N-1} . That is, they hold Y/j_{N-1} . Since we must have a Lebesgue-measurable set of agents holding the asset, it must be the case that $N - 2$ is the maximum level of collateralization. \square

Corollary A.2. *With each additional level of debt collateralization, there is one fewer marginal buyer of the risky asset Y .*

Corollary A.3. *Consider the continuum of agents in the economy. At the maximum $N - 2$ levels of debt collateralization, the interval $(0,1)$ is broken up into $N + 1$ sub-intervals, denoted $(1, \hat{a}_1), (\hat{a}_1, \hat{a}_2), \dots, (\hat{a}_{n+1}, 0)$. The first interval, $(1, \hat{a}_1)$ consists entirely of agents holding Y/j_{N-1} . The second interval, (\hat{a}_1, \hat{a}_2) consists only of agents holding j_{N-1}/j_{N-2}^1 . In general, the k^{th} interval, where $k > 1$, consists of agents holding $j_{N+1-k}^{N+2-k}/j_{N-k}^{N+1-k}$. In other words, every level of agents in the economy is lending directly to the level above and maximally leveraging the asset or contract in which they invest.*

A.4 Securitization Does Not Necessarily Increase Prices

We demonstrate that prices do not necessarily increase with the introduction of securitization when beliefs are weakly monotonic. As before, we let $M = .3$ and $D = .1$, be the payouts of asset Y in states M and D . We also define the following marginal agents: $h_1 = .6, h_2 = .65, h_3 = .69$. Beliefs are given as follows:

$$\text{For } h \leq h_1: \gamma_U(h) = 1 - (1 - h_1)^2, \gamma_M(h) = h_1(1 - h_1)^2, \gamma_D(h) = (1 - h_1)^3.$$

$$\text{For } h \in (h_1, h_2): \gamma_U(h) = 1 - (1 - h)^2, \gamma_M(h) = h(1 - h)^2, \gamma_D(h) = (1 - h)^3.$$

For $h \in [h_2, h_3)$: $\gamma_U(h) = 1 - (1 - h_2)^2$, $\gamma_M(h) = h_2(1 - h_2)^2$, $\gamma_D(h) = (1 - h_2)^3$.

Finally, for $h > h_3$: $\gamma_U(h) = 1 - (1 - (h - (h_3 - h_2)))^2$, $\gamma_M(h) = (h - (h_3 - h_2))(1 - (h - (h_3 - h_2)))^2$, $\gamma_D(h) = (1 - (h - (h_3 - h_2)))^3$.

Equilibrium in the economy with leverage has price $p = 0.894$ while equilibrium in the debt collateralization economy has price $\hat{p} = 0.888$.⁴⁶ Introducing debt collateralization in this case causes the price to *decrease*. The reason is that the leveraged return on debt has increased sufficiently, which increases the required return for investing in the risky asset, decreasing its price.

B Proofs

Proof of Lemma 1. Suppose for contradiction that there exists an h_i who prefers to hold the risky debt with some amount of leverage L , $0 \leq L < D$, less than the maximum. Since $L < D$ it is risk-free and thus $\hat{\pi}^L = L$. The marginal utilities from investing in j_M against promise L , from investing in j_M against promise D , and from holding safe assets are:

$$\text{debt with leverage } L: \frac{(\gamma_U(h_i) + \gamma_M(h_i))(M - L) + \gamma_D(h_i)(D - L)}{\hat{\pi}^M - L} \quad (13)$$

$$\text{debt with leverage } D: \frac{\gamma_U(h_i)(M - D) + \gamma_M(h_i)(M - D)}{\hat{\pi}^M - D} \quad (14)$$

$$\text{safe asset: } 1. \quad (15)$$

Since by assumption h_i strictly prefers the first option, it must be the case that (13) $>$ (14) and (13) $>$ (15). That is, the investor is optimistic enough to prefer the risky debt to safe debt but not so optimistic as to want zero payoff in D . Hence,

$$\frac{(\gamma_U(h_i) + \gamma_M(h_i))(M - L) + \gamma_D(h_i)(D - L)}{\hat{\pi}^M - L} > \frac{\gamma_U(h_i)(M - D) + \gamma_M(h_i)(M - D)}{\hat{\pi}^M - D}, \quad (16)$$

$$\frac{(\gamma_U(h_i) + \gamma_M(h_i))(M - L) + (1 - \gamma_U(h_i) - \gamma_M(h_i))(D - L)}{\hat{\pi}^M - L} > 1 \quad (17)$$

⁴⁶Furthermore, $h_M = 0.715$, $h_D = 0.669$, $h_\pi = 0.534$, $\pi^M = 0.287$ and $\hat{h}_M = 0.682$, $\hat{h}_\pi = 0.583$, $\hat{\pi}^M = 0.287$.

Simplifying 16, we obtain

$$\hat{\pi}^M - (\gamma_U(h_i) + \gamma_M(h_i))M - \gamma_D(h_i)D > 0 \implies \hat{\pi}^M > (\gamma_U(h_i) + \gamma_M(h_i))M + \gamma_D(h_i)D$$

Simplifying 17, we obtain

$$\hat{\pi}^M - \gamma_D(h_i)D - (\gamma_U(h_i) + \gamma_M(h_i))M < 0 \implies \hat{\pi}^M < \gamma_D(h_i)D + (\gamma_U(h_i) + \gamma_M(h_i))M$$

Note that the above gives us $\hat{\pi}^M > \hat{\pi}^M$. This is a contradiction. Thus, in equilibrium, all agents holding risky debt will do so with maximal leverage. \square

Proof of Lemma 2. In equilibrium, each unit of the leveraged risky asset must be backed by one unit of debt, either safe or risky and leveraged. By previous lemma, we have shown that all agents holding risky debt will be maximally leveraged. We therefore know that agents holding the risky asset must either be leveraged against state D or state M and not something in-between.

Suppose for contradiction that there is some agent h_i who prefers to hold the risky asset leveraged against state D and the price of debt is D . That is, the investor is optimistic enough to prefer the risky asset with low leverage to the leveraged risky debt, but not so optimistic as to want to maximally leverage the asset and get zero payoff in M . Note that returns from investment strategies are:

$$\text{Marginal utility from risky asset with debt } D : \frac{\gamma_U(h_i)(1-D) + \gamma_M(h_i)(M-D)}{\hat{p} - D} \quad (18)$$

$$\text{Marginal utility from risky asset with debt } M : \frac{\gamma_U(h_i)(1-M)}{\hat{p} - \hat{\pi}^M} \quad (19)$$

$$\text{Marginal utility from risky debt : } \frac{(\gamma_U(h_i) + \gamma_M(h_i))(M-D)}{\hat{\pi}^M - D} \quad (20)$$

Since by assumption h_i strictly prefers the first option, it must be the case that (18) > (19) and (18) > (20). That is,

$$\frac{\gamma_U(h_i)(1-D) + \gamma_M(h_i)(M-D)}{\hat{p} - D} > \frac{\gamma_U(h_i)(1-M)}{\hat{p} - \hat{\pi}^M} \quad (21)$$

$$\frac{\gamma_U(h_i)(1-D) + \gamma_M(h_i)(M-D)}{\hat{p} - D} > \frac{(\gamma_U(h_i) + \gamma_M(h_i))(M-D)}{\hat{\pi}^M - D} \quad (22)$$

Simplifying 21, we obtain that

$$\gamma_M(h_i)(M\hat{p} + D\hat{\pi}^M - M\hat{\pi}^M - D\hat{p}) > \gamma_U(h_i)(\hat{\pi}^M + MD + D\hat{p} - D - D\hat{\pi}^M - M\hat{p})$$

Simplifying 22, we obtain

$$\gamma_U(h_i)(\hat{\pi}^M + MD + D\hat{p} - D - D\hat{\pi}^M - M\hat{p}) + \gamma_M(h_i)(M\hat{\pi}^M + D\hat{p} - D\hat{\pi}^M - M\hat{p}) > 0$$

For convenience, let

$$\alpha := \gamma_M(h_i)(M\hat{p} + D\hat{\pi}^M - M\hat{\pi}^M - D\hat{p}), \quad \beta := \gamma_U(h_i)(\hat{\pi}^M + MD + D\hat{p} - D - d\hat{\pi}^M - M\hat{p})$$

Notice that the above two equations simplify to $\alpha > \beta$ and $\beta - \alpha > 0$. This is clearly a contradiction since we cannot have both $\alpha > \beta$ and $\beta > \alpha$. Thus, all investors holding the risky asset will be maximally leveraged against state M . \square

B.1 Proof of Theorem A.1

We proceed with the proof by induction, and break the proof into the following two parts. (1) The equilibrium at the first level of collateralization. (2) Equilibrium at the L^{th} level of collateralization.

B.1.1 Equilibrium at the first level of collateralization

To prove the base case, we will show that agents will hold one of the following assets in equilibrium:

(i) Y/j_i , where $2 \leq i \leq N-1$; (ii) j_i/j_j^1 , where $2 \leq i \leq N-1$, $1 \leq j < i$; (iii) j_j^1 , where $1 \leq j \leq N-2$.

That is, agents will hold the risky asset, Y , leveraged against states S_2, \dots, S_{N-1} ; the risky debt contract, j_n (backed by the risky asset), leveraged against some state S_j with $j < n$; or a debt security j_k^1 ($1 \leq k \leq N-2$), which is backed by risky debt.

Note that the j_k^1 contracts are just securities created in the first round of debt collateralization. Apart from this, the only difference from equilibrium with leverage is that all risky debt contracts backed directly by Y are now bought with leverage, and no agent holds Y , leveraged against S_1 . Thus, to prove the base case, it suffices to prove the following two lemmas:

Lemma 3. *In the first level of collateralization, no agent will hold Y/j_1 .*

Proof of Lemma 3. The intuition for this lemma is nearly identical to the intuition for lemmas 1 and 2. Suppose for contradiction that some agent, h prefers to hold Y/j_1 . Then, it must be the case that the expected return of holding Y/j_1 is greater than holding Y/j_{N-1} . This implies that we must have

$$\frac{\sum_{i=1}^N \gamma_i(h)(s_i - s_1)}{\hat{p} - \hat{\pi}_1} > \frac{\gamma_N(h)(s_N - s_{N-1})}{\hat{p} - \hat{\pi}_{N-1}} \quad (23)$$

Rearranging and simplifying, we obtain

$$\sum_{i=1}^{N-1} \gamma_i(h)(s_i - s_1)(\hat{p} - \hat{\pi}_{N-1}) + \gamma_N(h)[(s_N - s_1)(\hat{p} - \hat{\pi}_{N-1}) - (s_1 - s_{N-1})(\hat{p} - \hat{\pi}_{N-1})] > 0. \quad (24)$$

Furthermore, we know that the expected return of holding Y/j_1 is greater than holding j_{N-1}/j_1^1 , which gives

$$\frac{\sum_{i=1}^N \gamma_i(h)(s_i - s_1)}{\hat{p} - \hat{\pi}_1} > \frac{\sum_{i=1}^{N-1} \gamma_i(h)(s_i - s_1) + \gamma_N(h)(s_{N-1} - s_1)}{\hat{\pi}_{N-1} - \hat{\pi}_1^1} \quad (25)$$

Rearranging and simplifying, we obtain

$$\sum_{i=1}^{N-1} \gamma_i(h)(s_i - s_1)(\hat{\pi}_{N-1} - \hat{p}) + \gamma_N(h)[(s_N - s_1)(\hat{\pi}_{N-1} - \hat{\pi}_1^1) - (s_{N-1} - s_1)(\hat{p} - \hat{\pi}_1^1)] > 0 \quad (26)$$

A quick check will assure readers that equations (24) and (26) provide a contradiction because the expressions to the left of the $>$ sign are additive inverses and therefore cannot be both strictly greater than 0. □

Lemma 4. *In the first level of collateralization, any agent buying the promise j_j with $j > 1$ will also sell a promise j_k^1 with $1 \leq k < j$.*

Proof of Lemma 4. Now suppose that some agent h prefers to hold a promise j_ℓ with $j > 1$, but not sell a debt security. Then, it must be the case that the expected return of holding j_ℓ^0 is greater than the expected return of holding j_ℓ^0/j_1^1 . That is,

$$\frac{\sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) + \sum_{i=\ell}^N \gamma_i(h)(s_\ell)}{\hat{\pi}_\ell^0} > \frac{\sum_{i=1}^{\ell-1} \gamma_i(h)(s_i - s_1) + \sum_{i=\ell}^N \gamma_i(h)(s_\ell - s_1)}{\hat{\pi}_\ell^0 - \hat{\pi}_1^1} \quad (27)$$

Note that $\hat{\pi}_1^1 = s_1$, since j_1^1 promises s_1 in all states and is therefore safe debt. Thus, rearranging and simplifying 33, we obtain

$$s_1 \left(\hat{\pi}_\ell^0 - \sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) - \sum_{i=\ell}^N \gamma_i(h)(s_\ell) \right) > 0 \implies \hat{\pi}_\ell^0 - \sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) - \sum_{i=\ell}^N \gamma_i(h)(s_\ell) > 0$$

We also know that the expected return of holding j_ℓ^0 must be greater than holding the safe asset. Consequently,

$$\frac{\sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) + \sum_{i=\ell}^N \gamma_i(h)(s_\ell)}{\hat{\pi}_\ell^0} > 1 \quad (28)$$

Rearranging and simplifying 34, we obtain

$$\sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) + \sum_{i=\ell}^N \gamma_i(h)(s_\ell) > \hat{\pi}_\ell^0 \implies \sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) + \sum_{i=\ell}^N \gamma_i(h)(s_\ell) - \hat{\pi}_\ell^0 > 0$$

The above clearly cannot happen because we have that the two equations are additive inverses of each other and therefore cannot both be strictly greater than 0. Thus, no agent holding a risky debt contract will prefer to hold the contract unleveraged.

□

B.1.2 Induction Hypothesis

We now assume that the theorem holds for all levels of collateralization T with $T < L$. Specifically, this means that the theorem holds with $L - 1$ levels of collateralization. Looking at this level, we have that agents will hold one of the following assets in equilibrium: (i) Y/j_i , with $L - 1 < i \leq N - 1$, (ii) j_j^l/j_k^{l+1} , with $0 \leq l < L - 1$, $L - 1 - l < j < N - l$, and $L - 1 - l \leq k < j$, (iii) j_ℓ^{L-1} , with $1 \leq \ell < N - L + 1$.

B.1.3 Equilibrium at the L^{th} level of collateralization

At the L^{th} level of collateralization, we allow all agents holding j_ℓ^{L-1} (with $1 \leq \ell < N - L + 1$) to sell the promise $j_n^L(j_\ell^{L-1}) = (s_n, j_\ell^{L-1})$ where $1 \leq p < \ell$.

We will prove the following: (i) No agent holds Y/j_M . This implies that the asset j_M/j_{M-1}^1 no longer exists; (ii) No agent holding the debt security j_ℓ^{L-1} with $1 < \ell < N - L + 1$ will do so without leveraged; (iii) No agent will hold j_j^l/j_{L-1-l}^{l+1} , for $0 \leq l < L - 1$ and $L - 1 - l < j < N - l$. This implies that all $j_{L-1-l}^{l+1}/A_{L-2-l}^{l+2}$ no longer exist in equilibrium.

Note that the above are the changes between the $L - 1$ and L^{th} levels of collateralization given by the theorem. We break up the proof into three lemmas, corresponding to the three claims listed above.

Lemma 5. *At the L^{th} level of collateralization, no agent will hold Y/j_L .*

Proof of Lemma 5. Suppose for contradiction that some investor h wants to hold Y/j_L . Then the leveraged expected return to this asset must be strictly greater than the expected return to holding Y/j_{N-1} . This means that

$$\frac{\sum_{i=L}^N \gamma_i(h)(s_i - s_L)}{\hat{p} - \hat{\pi}_L} > \frac{\gamma_N(h)(s_N - s_{N-1})}{\hat{p} - \hat{\pi}_{N-1}}. \quad (29)$$

Rearranging and simplifying the above, we obtain

$$\sum_{i=L}^{N-1} \gamma_i(h)(s_i - s_L)(\hat{p} - \hat{\pi}_{N-1}) + \gamma_N(h)(s_N - s_L)(\hat{p} - \hat{\pi}_{N-1}) - (s_N - s_{N-1})(\hat{p} - \hat{\pi}_L) > 0 \quad (30)$$

Additionally, holding Y/j_L must have a higher expected return than holding j_{N-1}/j_L^1 . Note that at the L^{th} level of collateralization, both j_L and j_L^1 are fully securitized so they have the same price. That is $\hat{\pi}_L = \hat{\pi}_L^1$.

$$\frac{\sum_{i=L}^N \gamma_i(h)(s_i - s_L)}{\hat{p} - \hat{\pi}_L} > \frac{\sum_{i=L}^{N-1} \gamma_i(h)(s_i - s_L) + \gamma_N(h)(s_{N-1} - s_L)}{\hat{\pi}_{N-1} - \hat{\pi}_L} \quad (31)$$

Rearranging and simplifying, we have

$$\sum_{i=L}^{N-1} \gamma_i(h)(s_i - s_L)(\hat{\pi}_{N-1} - \hat{p}) + \gamma_N(h)[(s_N - s_L)(\hat{\pi}_{N-1} - \hat{\pi}_L) - (s_{N-1} - s_L)(\hat{p} - \hat{\pi}_L)] > 0 \quad (32)$$

The expressions on the left side of the $>$ sign in equations (30) and (32) are additive inverses, and therefore cannot both be strictly greater than 0. Thus, we have a contradiction and no agent will hold Y/j_L .

□

Lemma 6. *At the L^{th} level of collateralization, every agent holding j_i^{L-1} with $1 < i < N - L + 1$ will sell a promise j_m^L , where $1 \leq m < i$.*

Proof of Lemma 6. Suppose that there exists an agent, h , holding j_i^{L-1} with $1 < \ell < N - L + 1$ and prefers not to sell any promises. Then, it must be the case that the expected return of holding j_ℓ^{L-1} is greater than the expected return of holding j_ℓ^{L-1}/j_1^L . That is,

$$\frac{\sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) + \sum_{i=\ell}^N \gamma_i(h)(s_\ell)}{\hat{\pi}_\ell^{L-1}} > \frac{\sum_{i=1}^{\ell-1} \gamma_i(h)(s_i - s_1) + \sum_{i=\ell}^N \gamma_i(h)(s_\ell - s_1)}{\hat{\pi}_\ell^{L-1} - \hat{\pi}_1^L} \quad (33)$$

Note that $\hat{\pi}_1^L = s_1$, since j_1^L promises s_1 in all states and is therefore safe debt. Thus, rearranging and simplifying 33, we obtain

$$s_1 \left(\hat{\pi}_\ell^{L-1} - \sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) - \sum_{i=\ell}^N \gamma_i(h)(s_\ell) \right) > 0 \implies \hat{\pi}_\ell^{L-1} - \sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) - \sum_{i=\ell}^N \gamma_i(h)(s_\ell) > 0$$

We also know that the expected return of holding j_ℓ^{L-1} must be greater than holding the safe asset. Consequently,

$$\frac{\sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) + \sum_{i=\ell}^N \gamma_i(h)(s_\ell)}{\hat{\pi}_\ell^{L-1}} > 1 \quad (34)$$

Rearranging and simplifying 34, we obtain

$$\sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) + \sum_{i=\ell}^N \gamma_i(h)(s_\ell) > \hat{\pi}_\ell^{L-1} \implies \sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) + \sum_{i=\ell}^N \gamma_i(h)(s_\ell) - \hat{\pi}_\ell^{L-1} > 0$$

The above clearly cannot happen because we have that the two equations are additive inverses of each other and therefore cannot both be strictly greater than 0. Thus, no agent holding a risky debt contract will prefer to hold the contract unleveraged at the L^{th} level of collateralization. \square

Lemma 7. *At the L^{th} level of collateralization, for all $0 \leq l < L - 1$, no agent will hold j_k^l / j_{L-1-l}^{l+1} , where $L - l \leq k < N - l$.*

Proof of Lemma 7. Suppose for contradiction that there exist some agent h who prefers to be in the position stated above. Then, it must be the case that the is greater than the expected return of holding j_k^l / j_{L-1-l}^{l+1} . Thus,

$$\frac{\sum_{i=L-l}^{k-1} \gamma_i(h)(s_i - s_{L-1-l}) + \sum_{i=k}^N \gamma_i(h)(s_k - s_{L-1-l})}{\hat{\pi}_k^l - \hat{\pi}_{L-1-l}^{l+1}} > \frac{\sum_{i=L-l}^{k-1} \gamma_i(h)(s_i - s_{L-l}) + \sum_{i=k}^N \gamma_i(h)(s_k - s_{L-l})}{\hat{\pi}_k^l - \hat{\pi}_{L-l}^{l+1}}. \quad (35)$$

We rearrange and simplify the above to obtain

$$\sum_{i=L-l}^{k-1} \gamma_i(h)\Omega + \sum_{i=k}^N \gamma_i(h)\Psi > 0 \quad (36)$$

where

$$\Omega := (s_i - s_{L-1-l})(\hat{\pi}_k^l - \hat{\pi}_{L-1-l}^{l+1}) - (s_i - s_{L-l})(\hat{\pi}_k^l - \hat{\pi}_{L-1-l}^{l+1}) \quad (37)$$

and

$$\Psi := (s_k - s_{L-1-l})(\hat{\pi}_k^l - \hat{\pi}_{L-1-l}^{l+1}) - (s_k - s_{L-l})(\hat{\pi}_k^l - \hat{\pi}_{L-1-l}^{l+1}) \quad (38)$$

Furthermore, it must also be the case that the expected return of holding j_k^l / j_{L-1-l}^{l+1} is greater than the expected return from holding $j_{L-l}^{l+1} / j_{L-1-l}^{l+2}$. It is important to note here that the price of j_{L-1-l}^{l+2} is the same as the price of j_{L-1-l}^{l+1} because at the L^{th} level of collateralization, both have been securitized to the exact same degree, so the two have the same value. Thus, abusing notation,

we can write $\hat{\pi}_{L-1-l}^{l+2} = \hat{\pi}_{L-1-l}^{l+1}$. This give us

$$\frac{\sum_{i=L-l}^{k-1} \gamma_i(h)(s_i - s_{L-1-l}) + \sum_{i=k}^N \gamma_i(h)(s_k - s_{L-1-l})}{\hat{\pi}_k^l - \hat{\pi}_{L-1-l}^{l+1}} > \frac{\sum_{i=L-l}^N \gamma_i(h)(s_{L-l} - s_{L-1-l})}{\hat{\pi}_{L-l}^{l+1} - \hat{\pi}_{L-1-l}^{l+1}} \quad (39)$$

Rearranging and simplifying the above inequality, we obtain

$$\sum_{L-l}^{k-1} \gamma_i(h)\Upsilon + \sum_{i=k}^N \gamma_i(h)\Phi > 0, \quad (40)$$

where

$$\Upsilon := (s_i - s_{L-1-l})(\hat{\pi}_{L-l}^{l+1} - \hat{\pi}_{L-1-l}^{l+1}) - (s_{L-l} - s_{L-1-l})(\hat{\pi}_k^l - \hat{\pi}_{L-1-l}^{l+1}), \quad (41)$$

and

$$\Phi := (s_k - s_{L-1-l})(\hat{\pi}_{L-l}^{l+1} - \hat{\pi}_{L-1-l}^{l+1}) - (s_{L-l} - s_{L-1-l})(\hat{\pi}_k^l - \hat{\pi}_{L-1-l}^{l+1}). \quad (42)$$

A quick check will assure the readers that $\Upsilon = -\Omega, \Phi = -\Psi$, a contradiction, meaning equations (36) and (40) cannot both be true. Thus, no agent will hold j_k^l / j_{L-1-l}^{l+1} , where $0 \leq l < L-1$ and $L-l \leq k < N-l$.

□

B.2 Proof of Proposition 2

The full statement of Proposition 2: *The senior-subordinated tranching equilibrium is equivalent to equilibrium with complete debt collateralization. That is, there exists a bijective mapping of assets and prices from the debt collateralization equilibrium to the senior-subordinated tranching equilibrium such that the buyers of assets remain the same. Specifically,*

1. Any agent buying Y / j_{N-1} (collateralization) will buy T_N (senior-subordinated tranching).
2. Any agent holding j_n^l / j_{n-1}^{l+1} with $N > n > 1$ (collateralization) will buy T_n (senior-subordinated tranching).
3. Any agent holding j_1^l (collateralization) will buy T_1 (down-tranching).

4. Letting q_N denote the price of T_N , senior-subordinated tranching equilibrium will have (i) $q_N = \hat{p} - \hat{\pi}_{N-1}$, (ii) $q_n = \hat{\pi}_n^{N-n-1} - \hat{\pi}_{n-1}^{N-n}$, (iii) $q_1 = \hat{\pi}_1^{N-2} = s_1$, where \hat{p} and $\hat{\pi}_j^k$, are the equilibrium prices for the asset and debt securities in the complete collateralization equilibrium, respectively.

Proof. This follows because the expected return of holding T_n in the senior-subordinated tranching equilibrium is the same as holding j_n^l/j_{n-1}^{l+1} in the collateralization equilibrium, when $N > n > 1$. Similarly, the expected return of Y/j_{N-1} is identical to that of T_N ; the expected return to holding q_1 is exactly the return of j_1^{N-2} \square

C Equilibrium and Default in the Dynamic Model

C.1 Equilibrium Conditions with Leverage: Section 4.1

For a few marginal investors, we have no doubt about their course of action. The equations defining them are as follows.

Marginal Investors, known

h_{M0} : indifferent between leveraging against p_M and p_D at time $t = 0$. If at time $t = 1$ we are in state D , then h_{M0} is no longer in the market. If at $t = 1$ we are in state M , h_{M0} will choose to hold the risky asset because he is the most optimistic investor in the market. Thus, we have

$$\frac{\gamma_U(h_{M0})(1 - p_M)}{p_0 - \pi_0} = \frac{\gamma_U(h_{M0})(1 - p_D)}{p_0 - p_D} + \left(\frac{\gamma_M(h_{M0})(p_M - p_D)}{p_0 - p_D} \right) \left(\frac{\gamma_U(h_{M0})(1 - d_{MD}^Y)}{p_M - d_{MD}^Y} \right).$$

The above equates the marginal utility divided by payoff of holding the risky asset leveraged against p_M and the marginal utility divided by the payoff of holding the asset leveraged against p_D .

h_{MM} : Indifferent between risky asset and riskless asset given the realization of state M at $t = 1$. Since this marginal investor only exists at time $t = 1$ in state M . There is no ambiguity.

$$\frac{\gamma_U(h_{MM})(1 - d_{MD}^Y)}{p_M - d_{MD}^Y} = 1$$

h_{DD} : Indifferent between risky asset and riskless asset given the realization of state D at $t = 1$.

h_{DD} also only exists at time $t = 1$ in state D .

$$\frac{\gamma_U(h_{DD})(1 - d_{DD}^Y)}{p_D - d_{DD}^Y} = 1$$

Marginal Investors, unknown

h_{D0} : indifferent between leveraging against p_D and holding risky debt at time $t = 0$. If at time $t = 1$ the world is in state D , h_{D0} will choose to hold the risky asset. But, at time $t = 1$ in state M , h_{D0} can either choose to hold the risky asset or the safe asset. Let h_{D0} hold the risky asset. To simplify notation, let $\gamma_U^{D0} = \gamma_U(h_{D0})$, $\gamma_M^{D0} = \gamma_M(h_{D0})$ and $\gamma_D^{D0} = \gamma_D(h_{D0})$. Then, equating payoffs (multiplied by continuation values, we have:

$$\begin{aligned} & \frac{\gamma_U^{D0}(1 - p_D)}{p_0 - p_D} + \left(\frac{\gamma_M^{D0}(p_M - p_D)}{p_0 - p_D} \right) \left(\frac{\gamma_U^{D0}(1 - d_{MD}^Y)}{p_M - d_{MD}^Y} \right) \\ &= \frac{\gamma_U^{D0} p_M}{\pi_0} + \left(\frac{\gamma_M^{D0}(p_M)}{\pi_0} \right) \left(\frac{\gamma_U^{D0}(1 - d_{MD}^Y)}{p_M - d_{MD}^Y} \right) + \left(\frac{\gamma_D^{D0} p_D}{\pi_0} \right) \left(\frac{\gamma_U^{D0}(1 - d_{DD}^Y)}{p_D - d_{DD}^Y} \right) \end{aligned}$$

When h_{D0} holds the safe asset, we have:

$$\frac{\gamma_U^{D0}(1 - p_D) + \gamma_M^{D0}(p_M - p_D)}{p - p_D} = \frac{(\gamma_U^{D0} + \gamma_M^{D0})p_M}{\pi_0} + \left(\frac{\gamma_D^{D0} p_D}{\pi_0} \right) \left(\frac{\gamma_U^{D0}(1 - d_{DD}^Y)}{p_D - d_{DD}^Y} \right)$$

$h_{\pi 0}$: indifferent between holding risky debt and holding riskless asset at time $t = 0$. There are several possibilities for this agent. At time $t = 1$ in state M , the agent can either hold the safe or risky asset. At time $t = 1$ in state D , the agent can either hold the safe or risky asset. To simplify notation, let $\gamma_U^{\pi 0} = \gamma_U(h_{\pi 0})$, $\gamma_M^{\pi 0} = \gamma_M(h_{\pi 0})$ and $\gamma_D^{\pi 0} = \gamma_D(h_{\pi 0})$. Thus, we have four possible equations defining this agent:

M safe, D safe.

$$\frac{(\gamma_U^{\pi 0} + \gamma_M^{\pi 0})p_M + (\gamma_D^{\pi 0})p_D}{\pi_0} = 1$$

M risky, *D* risky.

$$\begin{aligned} & \frac{\gamma_U^{\pi_0} p_M}{\pi_0} + \left(\frac{\gamma_M^{\pi_0} p_M}{\pi_0} \right) \left(\frac{\gamma_U^{\pi_0} (1 - d_{MD}^Y)}{p_M - d_{MD}^Y} \right) + \left(\frac{\gamma_D^{\pi_0} p_D}{\pi_0} \right) \left(\frac{\gamma_U^{\pi_0} (1 - d_{DD}^Y)}{p_D - d_{DD}^Y} \right) \\ & = \gamma_U^{\pi_0} + \gamma_M^{\pi_0} \left(\frac{\gamma_U^{\pi_0} (1 - d_{MD}^Y)}{p_M - d_{MD}^Y} \right) + \gamma_D^{\pi_0} \left(\frac{\gamma_U^{\pi_0} (1 - d_{DD}^Y)}{p_D - d_{DD}^Y} \right) \end{aligned}$$

M safe, *D* risky.

$$\frac{(\gamma_U^{\pi_0} + \gamma_M^{\pi_0}) p_M}{\pi_0} + \left(\frac{\gamma_D^{\pi_0} p_D}{\pi_0} \right) \left(\frac{\gamma_U^{\pi_0} (1 - d_{DD}^Y)}{p_D - d_{DD}^Y} \right) = \gamma_U^{\pi_0} + \gamma_M^{\pi_0} + \gamma_D^{\pi_0} \left(\frac{\gamma_U^{\pi_0} (1 - d_{DD}^Y)}{p_D - d_{DD}^Y} \right)$$

M risky, *D* safe.

$$\frac{\gamma_U^{\pi_0} p_M}{\pi_0} + \left(\frac{\gamma_M^{\pi_0} p_M}{\pi_0} \right) \left(\frac{\gamma_U^{\pi_0} (1 - d_{MD}^Y)}{p_M - d_{MD}^Y} \right) + \frac{\gamma_D^{\pi_0} p_D}{\pi_0} = \gamma_U^{\pi_0} + \gamma_M^{\pi_0} \left(\frac{\gamma_U^{\pi_0} (1 - d_{MD}^Y)}{p_M - d_{MD}^Y} \right) + \gamma_D^{\pi_0}$$

The known market clearing conditions are for the asset and debt at time 0:

Time t = 0, Asset:

$$(1 - h_{M0}) \frac{(1 + p_0)}{(p_0 - \pi_0)} + (h_{M0} - h_{D0}) \frac{(1 + p_0)}{(p_0 - p_D)} = 1$$

Time t = 0, Risky Debt:

$$(1 - h_{M0}) \frac{(1 + p_0)}{(p_0 - \pi_0)} = (h_{D0} - h_{\pi_0}) \frac{(1 + p_0)}{\pi_0}$$

Market Clearing, unknown

Time t = 1, State M Asset: We are unsure whether $h_{MM} \in (0, h_{\pi_0})$, $h_{MM} \in (h_{\pi_0}, h_{D0})$, or $h_{MM} \in (h_{D0}, h_{M0})$. This issue can be resolved by considering all cases, solving for equilibrium, and checking that h_{MM} is indeed in the specified interval. $h_{MM} \in (0, h_{\pi_0})$:

$$\frac{(h_{M0} - h_{D0}) \left(\frac{1+p_0}{p_0 - p_D} \right) (p_M - p_D)}{p_M - d_{MD}^Y} + \frac{(h_{D0} - h_{\pi_0}) \left(\frac{1+p_0}{\pi_0} \right) (p_M)}{p_M - d_{MD}^Y} + \frac{(h_{\pi_0} - h_{MM}) (1 + p_0)}{p_M - d_{MD}^Y} = 1$$

$h_{MM} \in (h_{\pi_0}, h_{D0})$:

$$\frac{(h_{M0} - h_{D0}) \left(\frac{1+p_0}{p_0 - p_D} \right) (p_M - p_D)}{p_M - d_{MD}^Y} + \frac{(h_{D0} - h_{MM}) \left(\frac{1+p_0}{\pi_0} \right) (p_M)}{p_M - d_{MD}^Y} = 1$$

$h_{MM} \in (h_{D0}, h_{M0})$:

$$\frac{(h_{M0} - h_{MM}) \left(\frac{1+p_0}{p_0-p_D} \right) (p_M - p_D)}{p_M - d_{MD}^Y} = 1$$

Time $t = 1$, State D Asset: We do not know whether $h_{DD} \in (0, h_{\pi 0})$ or $h_{DD} \in (h_{\pi 0}, h_{D0})$. In the first case, we have:

$$\frac{(h_{D0} - h_{\pi 0}) \left(\frac{1+p_0}{\pi_0} \right) p_D}{p_D - d_{DD}^Y} + \frac{(h_{\pi 0} - h_{DD})(1 + p_0)}{p_D - d_{DD}^Y} = 1.$$

In the second case, we have

$$\frac{(h_{D0} - h_{DD}) \left(\frac{1+p_0}{\pi_0} \right) p_D}{p_D - d_{DD}^Y} = 1.$$

Thus, we obtain the following possible cases in equilibrium:

1. h_{D0} holds risky asset at time 1 in state M . (a) $h_{\pi 0}$ holds **risky** asset in state M and **risky** asset in state D . This implies that $h_{MM}, h_{DD} \in (0, h_{\pi 0})$. (b) $h_{\pi 0}$ holds **safe** asset in state M and **safe** asset in state D . This implies that $h_{MM}, h_{DD} \in (h_{\pi 0}, h_{D0})$. (c) $h_{\pi 0}$ holds **risky** asset in state M and **safe** asset in state D . This implies that $h_{MM} \in (0, h_{\pi 0})$ and $h_{DD} \in (h_{\pi 0}, h_{D0})$. (d) $h_{\pi 0}$ holds **safe** asset in state M and **risky** asset in state D . This implies that $h_{MM} \in (h_{\pi 0}, h_{D0})$ and $h_{DD} \in (0, h_{\pi 0})$.
2. h_{D0} holds safe asset at time 1 in state M . This implies that $h_{MM} \in (h_{D0}, h_{M0})$ and $h_{\pi 0}$ holds safe asset in state M . (a) $h_{\pi 0}$ holds **safe** asset in state D . This implies that $h_{DD} \in (h_{\pi 0}, h_{D0})$. (b) $h_{\pi 0}$ holds **risky** asset in state D . This implies that $h_{DD} \in (0, h_{\pi 0})$.

C.2 Equilibrium Conditions with Collateralization: Section 4.2

The equations defining equilibrium are as follows

Marginal Investors, known

\hat{h}_{M0} : indifferent between holding risky asset, leveraged against state M and risky debt leveraged against state D at time 0

$$\frac{\gamma_U(\hat{h}_{M0})(1 - \hat{p}_M)}{\hat{p}_0 - \hat{\pi}_0} = \frac{\gamma_U(\hat{h}_{M0})(\hat{p}_M - \hat{p}_D)}{\hat{\pi}_0 - \hat{p}_D} + \frac{\gamma_M(\hat{h}_i)(\hat{p}_M - \hat{p}_D)}{\hat{\pi}_0 - \hat{p}_D} \left(\frac{\gamma_U(1 - d_{MD}^Y)}{\hat{p}_M - d_{MD}^Y} \right)$$

\hat{h}_{MM} : Indifferent between holding risky asset and safe asset at time 1, state M .

$$\frac{\gamma_U(\hat{h}_{MM})(1 - d_{MD}^Y)}{\hat{p}_M - d_{MD}^Y} = 1$$

\hat{h}_{DD} : Indifferent between holding risky asset and safe asset at time 1, state D .

$$\frac{\gamma_U(\hat{h}_{DD})(1 - d_{DD}^Y)}{\hat{p}_D - d_{DD}^Y} = 1$$

Marginal Investors, unknown

$\hat{h}_{\pi 0}$: Indifferent between holding risky debt with leverage and holding the safe asset. There are two possibilities for this agent: at time $t = 1$ in state M , the agent can either hold the safe or risky asset; at time $t = 1$ in state D , $\hat{h}_{\pi 0}$ will be the most optimistic agent still in the market, forcing him to hold the risky asset. Thus, we have the following two possibilities

M safe.

$$\frac{(\gamma_U(\hat{h}_{\pi 0}) + \gamma_M(\hat{h}_{\pi 0}))(\hat{p}_M - \hat{p}_D)}{\hat{\pi}_0 - \hat{p}_D} = \gamma_U(\hat{h}_{\pi 0}) + \gamma_M(\hat{h}_{\pi 0}) + \gamma_D(\hat{h}_{\pi 0}) \left(\frac{\gamma_U(\hat{h}_{\pi 0})(1 - d_{DD}^Y)}{p_D - d_{DD}^Y} \right)$$

M risky.⁴⁷

$$\begin{aligned} & \frac{\gamma_U(\hat{h}_{\pi 0})(\hat{p}_M - \hat{p}_D)}{\hat{\pi}_0 - \hat{p}_D} + \left(\frac{\gamma_M(\hat{h}_{\pi 0})(\hat{p}_M - \hat{p}_D)}{\hat{\pi}_0 - \hat{p}_D} \right) \left(\frac{\gamma_U(\hat{h}_{\pi 0})(1 - d_{MD}^Y)}{\hat{p}_M - d_{MD}^Y} \right) \\ & = \gamma_U(\hat{h}_{\pi 0}) + \gamma_M(\hat{h}_{\pi 0}) \left(\frac{\gamma_U(\hat{h}_{\pi 0})(1 - d_{MD}^Y)}{\hat{p}_M - d_{MD}^Y} \right) + \gamma_D(\hat{h}_{\pi 0}) \left(\frac{\gamma_U(\hat{h}_{\pi 0})(1 - d_{DD}^Y)}{p_D - d_{DD}^Y} \right) \end{aligned}$$

Market Clearing, known

Time $t = 0$, Risky Asset:

$$(1 - \hat{h}_{M0}) \left(\frac{1 + \hat{p}_0}{\hat{p}_0 - \hat{\pi}_0} \right) = 1$$

Time $t = 1$, state D , Risky Asset:

$$(\hat{h}_{\pi 0} - \hat{h}_{DD}) \left(\frac{1 + \hat{p}_0}{\hat{p}_D - d_{DD}^Y} \right) = 1$$

⁴⁷the payout of the asset in state M at time 1 is multiplied by the continuation value of the asset in time 2 since we have specified that $\hat{h}_{\pi 0}$ will hold the risky asset.

Time $t = 0$, Risky Debt:

$$(1 - \hat{h}_{M0}) \left(\frac{1 + \hat{p}_0}{\hat{p}_0 - \hat{\pi}_0} \right) = (\hat{h}_{M0} - \hat{h}_{\pi 0}) \left(\frac{1 + \hat{p}_0}{\hat{\pi}_0 - \hat{p}_D} \right)$$

Market Clearing, unknown

Time $t = 1$, State M Asset: We are unsure whether $\hat{h}_{MM} \in (0, \hat{h}_{\pi 0})$ or $\hat{h}_{MM} \in (\hat{h}_{\pi 0}, \hat{h}_{M0})$. This issue can be resolved by considering both cases, solving for equilibrium, and checking that \hat{h}_{MM} is indeed in the specified interval.

$$\hat{h}_{MM} \in (0, \hat{h}_{\pi 0}):$$

$$\frac{(\hat{h}_{M0} - \hat{h}_{\pi 0}) \left(\frac{1 + \hat{p}_0}{\pi - \hat{p}_D} \right) (\hat{p}_M - \hat{p}_D)}{\hat{p}_M - d_{MD}^Y} + \frac{(\hat{h}_{\pi 0} - \hat{h}_{MM})(1 + \hat{p}_0)}{\hat{p}_M - d_{MD}^Y} = 1$$

$$\hat{h}_{MM} \in (\hat{h}_{\pi 0}, \hat{h}_{M0}):$$

$$\frac{(\hat{h}_{M0} - \hat{h}_{MM}) \left(\frac{1 + \hat{p}_0}{\pi - \hat{p}_D} \right) (\hat{p}_M - \hat{p}_D)}{\hat{p}_M - d_{MD}^Y} = 1$$

C.3 Default in the Dynamic 3-state Model

This section isolates the role of default in the dynamic model in two ways. First, it considers a surprise injection of wealth to bailout agents who lost money to default. Second, it maps the 3-state model onto binomial models, in which there is no default, and compares equilibrium in each case.

We solve our model numerically with $\gamma_U(h) = h^2$, $\gamma_M(h) = h^2(1 - h^2)$, and payoffs $d_{MD}^Y = .3$ and $d_{DD}^Y = .1$. For these parameters, h_π holds the safe asset in state M and the risky asset in state D . Notice that the probability of down states (M or D) is $1 - h$, similar to parametrization in binomial models. The marginal investors and prices in equilibrium are:

$$\begin{aligned} h_{M0} &= 0.963, & h_{D0} &= 0.823, & h_{\pi 0} &= 0.719, & p_0 &= 0.805, & \pi_0 &= 0.595, \\ h_{MM} &= 0.720, & h_{DD} &= 0.610, & p_M &= 0.663, & p_D &= 0.434, & i_0 &= 2.25\%. \end{aligned}$$

The percent price drop, given by $1 - \frac{p_s}{p}$, is 17.6% in state M , and 46% in state D .

C.3.1 The Default Mechanism

We demonstrate the impact that default has on asset prices at time 1 by considering the counterfactual scenario. We now suppose that in the down state, holders of risky debt receive an unexpected, exogenous wealth increase at time 1: suppose that the holders of risky debt are compensated the difference between p_M and p_D . This wealth shock is unexpected; it does not change the equilibrium at time 0 and only at time 1. In the down-state, we now have $h \in (h_{\pi_0}, h_{D_0})$ holding p_M units of wealth and $h \in (0, h_{\pi_0})$ holding $1 + p_0$ units of wealth. The marginal investor and market clearing equations defining equilibrium are:

$$\frac{\gamma_U(h_{DD})(1 - d_{DD}^Y)}{p_D - d_{DD}^Y} = 1, \text{ and } \frac{(h_{D_0} - h_{\pi_0}) \left(\frac{1+p}{\pi_0} \right) p_M}{p_D - d_{DD}^Y} + \frac{(h_{\pi_0} - h_{DD})(1 + p_0)}{p_D - d_{DD}^Y} = 1.$$

Using the same specifications as before, as well as the results for $h_{D_0}, h_{\pi_0}, \pi_0, p_0$ and p_M from the previous subsection, we find that $h_{DD} = 0.598, p_D = 0.638, \text{down crash} = 33.08\%$.

Without this injection of cash, $p_D = 0.607$ and the crash was 36.33%. The wealth increase, offsetting the default mechanism, increases the asset price in D and lowers the volatility. However, it is important to note that this result only occurs if agents do not expect the wealth shock. If agents anticipated the wealth increase at time $t = 0$, then the increase in the p_D price will lead to higher margins at the initial time period and the expectation that all debt is actually safe.

C.3.2 Comparison With Dynamic Two-State Model

We can compare the volatility in the dynamic three-period model to the dynamic 2-period model in several ways. We can normalize by the expected payoff of the asset, normalize by belief in the up state, as well as normalize by the belief in the downstate.

We first consider normalization by the expected payoff of the asset. We use $\gamma_U(h) = h$ and $\gamma_M(h) = h(1 - h)$ to for the probabilities in the three-state model. We want to set the belief of the upstate, $\varphi(h)$, in the two-state model so that for every agent, the ultimate expected payout of the risky asset is the same in the two models. That is,

$$h + h^2(1 - h) + h(1 - h)^2 + h(1 - h)^2 \alpha d_{DD}^Y + (1 - h)^3 d_{DD}^Y = \varphi(h) + (1 - \varphi(h))\varphi(h) + (1 - \varphi(h))^2 d_{DD}^Y,$$

where $d_{MD}^Y = \alpha d_{DD}^Y$. Solving the above, we find that

$$\varphi_i = 1 + \frac{\sqrt{(d_{DD}^Y - 1)(1 - h)^2(d_{DD}^Y - 1 + (\alpha - 1)d_{DD}^Y h)}}{d_{DD}^Y - 1}.$$

Letting $\alpha = 3$, we can solve for equilibrium in the two-state dynamic model.

When we normalize by belief about the up state, we have $\varphi(h) = \gamma_U(h) = h$. Alternatively, if we normalize by belief about the down state, we need $1 - \varphi(h) = \gamma_D(h)$. In summary, we obtain the following differences by normalization.

	Dynamic 3-State		Dynamic 2-State		
	Collateralized debt	No Collateralization	Expected Payout	Belief Up	Belief Down
p_0	.970	0.9531	0.9470	0.9351	0.9876
p_D	.602	0.6069	0.6339	0.6116	0.7314
crash	37.91%	36.33%	33.06%	34.593 %	25.94%

Note that for every normalization, the price crash in the down state of the three state world is greater than the price crash in any of the two-state models and that the biggest price crash is obtained in the case of debt collateralization. Furthermore, the price of the asset in the down state is lowest in the three-state world with debt collateralization. This phenomenon occurs precisely because the three-state model with collateralization has more bankrupt agents at time 1 in the down state when compared to the two-state models. Thus, the agents who are left to buy the asset are more pessimistic and do not value the asset as highly.

C.4 Comparative statistics

We now briefly explore how price and marginal investors change depending on difference between the payout in the d_{MD}^Y and d_{DD}^Y states. To do this, we fix the payout of the asset in d_{DD}^Y to be 0.1 and solve for equilibrium for a full range of d_{MD}^Y . These results are presented in the table below.

For high enough values of d_{MD}^Y , the price crash in state M is negative (the price increases). For high values of d_{MD}^Y there is so much divergence in the payoffs of the risky asset that once the possibility of state D occurring is eliminated, agents are more optimistic about the payout of the asset and are thus willing to buy more of the risky asset.

d_{MD}^Y	h_{M0}	h_{D0}	$h_{\pi 0}$	h_{MM}	h_{DD}	p_0	p_M	p_D	π_0	i_0	M crash	D crash
.2	.945	.826	.714	.672	.616	.784	.562	.442	.527	6.58%	28.36%	43.63%
.3	.963	.823	.719	.720	.610	.805	.663	.434	.595	11.44%	17.58%	46.00%
.4	.977	.818	.725	.764	.605	.819	.750	.429	.655	14.45%	8.44%	47.63%
.5	.987	.809	.734	.807	.601	.830	.826	.425	.711	16.23%	0.50%	48.78%
.6	.994	.799	.743	.854	.599	.838	.892	.422	.761	17.19%	-6.43%	49.58%
.7	.998	.787	.752	.892	.597	.844	.939	.420	.799	17.49%	-11.31%	50.16%
.8	.999	.776	.759	.927	.595	.848	.972	.419	.827	17.53%	-14.67%	50.57%
.9	.999	.769	.764	.963	.594	.850	.992	.418	.845	17.48%	-16.73%	50.84%

Table 4: Comparisons of equilibrium for different values of d_{MD}^Y

D A Model With Finitely Many Risk-Averse Agents

We first present a general model with a finite number of risk-averse agents that, subject to some constraints, exhibits the same properties as the baseline model with a continuum of risk-neutral investors; and we provide a numerical example with 3 agents with log-preferences that replicates the results in the paper.

D.1 A General Model

Let there be N states $s_n \in \{s_1, \dots, s_N\}$, and let state s_n occur with objective probability γ_n . The risky asset is in unit supply and pays d_n^Y in state s_n , where states are ordered so higher n implies higher asset payout..

Let there be $H > 2$ agents (households) denoted $h \in \{1, \dots, H\}$, with time-zero endowments of e_0^h of a non-storable consumption good and y_0^h of the risky asset, and with e_s^h units of consumption good in period-1 in state s . Let agents have quasi-linear utility over consumption in periods 0 and 1 with concave utility over consumption in period 1. Letting $c = (c_0, (c_n)_{n=1, \dots, N})$ be a vector of consumption in both periods, agent have expected utility

$$U^h(c) = c_0 + \sum_{n=1, \dots, N} \gamma_n u_h(c_n),$$

where $u^h(c)$ is increasing and concave, and agents have common priors consistent with objective probabilities.

We impose the following restriction (sufficient, but not necessary) on marginal utilities and

endowments:

Marginal Utilities: For every $h \in \{2, \dots, H\}$, for every $m \in \{1, \dots, N\}$, let marginal utilities satisfy:

$$\frac{\gamma_m u'_h(e_m^h + d_m^Y)}{\sum_{n \leq m} \gamma_n u'_h(e_n^h + d_n^Y)} > \frac{\gamma_m u'_{h-1}(e_m^{h-1})}{\sum_{n \leq m} \gamma_n u'_{h-1}(e_n^{h-1})},$$

which implies that high- h agents have higher marginal utilities in states in which the asset pays more (perhaps because endowments are low in those states). Thus high- h agents “correspond” to high- h investors in the risk-neutral model. This condition is the analog to what is in Section A.3.

To ensure that leverage constraints bind requires that high- h agents’ endowments are not so large. (The conditions are too complicated for us to try to list here. Similarly, one could allow for concavity in utility c_0 with additional assumptions on initial endowments/preferences. See Phelan 2015 for a similar example with two-agents.) For debt collateralization to be meaningful, endowments must not be so small that all agents must borrow from $h = 1$ in equilibrium: endowments must be large enough that a “cascade” of borrowing emerges in the leverage economy.

When leverage constraints bind and there are more than 2 different borrowers in the economy, then the main results in the paper go through for one level of debt collateralization. In general, when leverage constraints do not bind, the results continue to hold but weakly. Thus the results of our paper are general. Intuitively, the debt collateralization will have the effects presented in our paper when, in the leverage economy: (i) leverage constraints bind for a sufficient number of agents and (ii) different agents invest in different debt contracts, implying a pricing wedge across agents and demand for some agents to short contracts (use debt as collateral).

A Numerical Example With 3 Agents

The previous conditions on marginal utilities and hedging needs provide a general way to map heterogeneous beliefs into a model with heterogeneous endowments/risk-aversion. Nonetheless, we provide an example (not the only possible one) that replicates the example in the text. There are 3 agents with log-utility over consumption in period-1, $u_i(c) = \log(c)$, and the three states be equiprobable. Agents are endowed with units of the safe asset X and the risky asset Y , and endowments are as follows: $x^1 = .212$, $x^2 = .185$, and $x^3 = .603$ units of the safe asset X , and the

risky asset Y is endowed entirely to agent 1, $y_0^1 = 1$. Agents have future endowments given by: $e^1 = (14.41, 38.18, 100)$, $e^2 = (100, 1, 100)$, and $e^3 = (3.25, 66.67, 100)$. Assets are priced according to the standard marginal analysis using the marginal utilities of the buyers. Two contracts, j_M and j_D , continue to be traded in equilibrium.

Given these parameters, in equilibrium with leverage, agent 3 buys all of the risky asset and issues contracts promising M and D ; agent 2 buys risky contract M ; and agent 1 buys contracts M and D and all of the safe asset X . In equilibrium with debt collateralization, agent 3 buys the asset leveraged with M ; agent 2 buys the risky debt leveraged with D ; agent 1 buys safe assets. Coincidentally, the asset prices p and \hat{p} and debt prices π^M and $\hat{\pi}^M$ are exactly the same as in the economy in Section 3.