On the Robustness of Theoretical Asset Pricing Models

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Abstract

We derive a parsimonious returns-based stochastic discount factor that is robust to model misspecification. We consider a general equilibrium model with heterogeneous agents who can invest their wealth in many assets. As long as (i) agents have (individual-, time-, and state-dependent) recursive preferences that are homothetic in current consumption and continuation value with a *common* relative risk aversion coefficient γ and (ii) asset returns and individual state variables are conditionally independent (*e.g.*, GARCH processes), we prove that the $(-\gamma)$-th power of market return is a valid stochastic discount factor. Within this class of models, asset prices are determined by relative risk aversion and technology alone, and "returns-based asset pricing" is robust to model misspecification as opposed to the consumption-based approach. We recast the equity premium puzzle as a consumption/saving puzzle, not as an asset pricing puzzle.

Keywords: asset pricing puzzles, heterogeneous-agent model, model misspecification, recursive preferences.

JEL codes: D53, D58, D91, G11, G12.

1 Introduction

In the absence of arbitrage a stochastic discount factor (SDF) exists (Ross, 1976). As economists we would like to use economic theory to derive stochastic discount factors, but finding an SDF that is robust to model misspecification is not trivial. In asset pricing theory, it is well known that the "returns-based approach" is equivalent to the "consumption-based approach" *given the model.*[1] However, the consumption-based approach has performed fairly poorly and

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[1]Let p, m, x denote the asset price, stochastic discount factor, and asset payoff. The "consumption-based approach" proceeds as follows: (1) form a statistical model of the consumption process, (2) use a model to calculate the implied SDF m given optimal consumption and portfolio allocations, (3) calculate asset prices and returns directly from the basic pricing equation $p = \mathrm{E}[mx]$. In contrast, the "returns-based approach" proceeds: (1) form a model of asset returns, (2) solve the optimal consumption-portfolio decisions given asset returns, (3) use the equilibrium consumption value in $p = \mathrm{E}[mx]$. The definitions of the two approaches are cited from Cochrane (2005), p. 40. Classic examples of the returns-based approach are Markowitz (1952), Tobin (1958), Sharpe (1964), Lintner (1965a,b), Samuelson (1969), Merton (1969, 1971, 1973), and Fama (1970), and examples of the consumption-based approach are Lucas (1978) and Mehra and Prescott (1985), just to name a few.

practitioners predominantly use the returns-based approach (Campbell and Cochrane, 2000). In this paper we argue that the returns-based approach is fairly robust to model misspecification, while the consumption-based approach is not.We derive, for a general class of models with heterogeneous agents, a stochastic discount factor that is both model-robust and trivial to calculate. The key is to "bypass consumption data altogether, and instead look directly at asset returns" (Ludvigson, 2013, p. 801).

To illustrate our result in the simplest possible way, consider the following example. There is an investor who lives for two periods with the additive constant relative risk aversion (CRRA) utility function

$$\frac{1}{1-\gamma}\left(c_0^{1-\gamma}+\beta\,\mathrm{E}[c_1^{1-\gamma}]\right).$$

Suppose that the investor is endowed with initial wealth $w>0$ today and nothing tomorrow, but can invest in K assets indexed by $k=1,\dots,K$. Asset k has gross return $R^k\geq 0$, which is a random variable. Letting ϕ^k be the fraction of the remaining wealth invested in asset k, $\phi=(\phi^1,\dots,\phi^K)\in\mathbb{R}^K$ (where $\sum_k \phi^k=1$) the portfolio, and $R(\phi)=\sum_k R^k\phi^k$ the gross return on portfolio ϕ, the budget constraint is $c_1=R(\phi)(w-c_0)$.

Substituting the budget constraint into the utility function, the optimal consumption-portfolio problem becomes

$$\max_{c,\phi}\frac{1}{1-\gamma}\left(c^{1-\gamma}+\beta\,\mathrm{E}[R(\phi)^{1-\gamma}](w-c)^{1-\gamma}\right),$$

Because utility is homothetic, this problem can be broken into separable portfolio-allocation and consumption-savings problems:

$$\frac{F}{1-\gamma}:=\max_{\phi}\frac{1}{1-\gamma}\,\mathrm{E}[R(\phi)^{1-\gamma}], \tag{1.1a}$$

$$U:=\max_{c}\frac{1}{1-\gamma}(c^{1-\gamma}+\beta F(w-c)^{1-\gamma}). \tag{1.1b}$$

Let ϕ^* be the solution to the optimal portfolio problem (1.1a). It is well-known that $\beta(c_1/c_0)^{-\gamma}$ is a valid SDF given this utility function, but so is (a constant multiple of) $R(\phi^*)^{-\gamma}$. To see this, consider investing ϵ more (less) in asset k and ϵ less (more) in the optimal portfolio ϕ^*. Taking the first order condition with respect to ϵ and setting $\epsilon=0$, we obtain

$$\mathrm{E}[R(\phi^*)^{-\gamma}(R^k-R(\phi^*))]=0$$

for any asset k. Hence, $R(\phi^*)^{-\gamma}$ (times a constant) is also a valid stochastic discount factor. Note that the only property used to derive this SDF is the homotheticity of the utility function, not its particular functional form; the returns-based SDF $R(\phi^*)^{-\gamma}$ is robust. In contrast, the consumption-based SDF $\beta(c_1/c_0)^{-\gamma}$ is *not* robust to model misspecification: if the utility function changes, so does the SDF. (See Hansen and Renault (2010) for several examples.) The rest of the paper is an elaboration of this simple idea.

In this paper we consider an economy with many agents who have individual, time, and state-dependent recursive preferences that are homothetic in current consumption and continuation value. We show that if asset returns and individual state variables are conditionally independent (*e.g.*, GARCH processes), then the portfolio and the consumption/savings decisions can be disentangled. If, in addition, (i) agents have a *common* relative risk aversion coefficient γ and (ii) the efficient market hypothesis holds, then agents make the same portfolio choice, and consequently the individually optimal portfolio must be the market portfolio. A corollary is that the $(-\gamma)$-th power of the gross return on the market portfolio (market return) is a valid stochastic discount factor.

This result has three important implications. First, since its validity does not depend on the functional form beyond homotheticity and hence on the consumption process, the "returns-based asset pricing" approach is robust to model misspecification as opposed to the consumption-based approach. Since in our model consumption is not directly connected to asset prices, the low volatility of consumption growth (or the low covariance between consumption growth and asset returns) needed in order to explain asset prices is not an asset pricing puzzle but a consumption/saving puzzle; the "consumption volatility puzzle" belongs to macroeconomics, not to finance. Second, since the asset pricing formula contains only asset returns data, which are available in high frequency and high accuracy, the '$(-\gamma)$-th power of market return' SDF can be used in practice. Third, the relative risk aversion γ can be estimated using only asset returns data; consumption data (aggregate or individual) contain no more information than the asset returns data for estimating the relative risk aversion coefficient. We do this exercise and obtain an estimate of $\gamma = 2$, which suggests a satisfactory "resolution" to asset pricing puzzles.

Although these results concern relative pricing, we also consider absolute pricing. By assuming further that (iii) agents have access to constant-returns-to-scale stochastic saving technologies (AK model, *e.g.*, Levhari and Srinivasan (1969)) and (iv) technological shocks and individual state variables are conditionally independent, we derive an asset pricing formula which depends only on fundamentals (technology and relative risk aversion).

The rest of the paper is organized as follows. After a brief discussion of the related literature, Section 2 presents the model and solves the single agent optimal consumption-portfolio problem. Section 3 derives relative asset pricing formulas that do not depend on consumption in a partial equilibrium setting. Section 4 characterizes the general equilibrium with many heterogeneous agents and constant-returns-to-scale stochastic saving technologies, and derives absolute asset pricing formulas.

1.1 Related literature

A few papers are related to our work. Rubinstein (1976) derived the '$(-\gamma)$-th power of market return' SDF under the assumption of a representative agent with additive CRRA utility and serially independent returns. We obtain the same SDF, but under much weaker assumptions listed above. Most importantly, in Rubinstein's model aggregate consumption is proportional to wealth and hence consumption growth and market return have the same volatility (which is counterfactual, hence the "consumption volatility puzzle"), but in our model aggregate consumption is not connected to market return. Campbell (1993) obtained an asset pricing formula without using consumption in a representative agent setting by log linearizing the intertemporal budget constraint. One result of this paper is that relative risk aversion is the only preference parameter that matters for asset prices (in particular, the EIS is irrelevant). In our model there are many heterogeneous agents with more general preferences and the asset pricing formula is exact, not an approximation.

Cass and Stiglitz (1970) showed in a static setting that the only utility functions for which the mutual fund theorem holds are the quadratic and power utility functions (if there is no risk-free asset) and the linear risk tolerance (LRT) utility $-\frac{u'(w)}{u''(w)} = A + Bw$ (if there is a risk-free asset). Rubinstein (1974) showed a similar result in a two period economy with additive utility functions. Our result extends theirs to a dynamic setting with recursive preferences.

Chabi-Yo et al. (2014) study the effect of incomplete markets and skewness of returns for stochastic discount factors. They show that with a representative agent, or with complete markets, an SDF which is quadratic in the market return is valid. However when markets are incomplete so that agents cannot properly hedge skewness risk, skewness leads to tracking error in the SDF; the SDF is not spanned by a polynomial of the market return. In our model markets are complete, which is why the $(-\gamma)$-th power of market return is valid.

Guvenen (2009) and Gârleanu and Panageas (2015) both use parsimonious heterogeneous agent models in attempts to resolve asset pricing puzzles. Guvenen (2009) presents a two-agent model in which agents have the same degree of relative risk aversion ($\gamma = 6$) but heterogeneous elasticities of intertemporal substitution (EIS) and limited stock market participation. The model generates a countercyclical equity premium, but consumption is still too volatile compared to the data; furthermore, the model does not admit a parsimonious consumption-based SDF. Gârleanu and Panageas (2015) present a two-agent stochastic endowment economy in which agents have heterogeneous EIS and heterogeneous RRA (99% of agents have $\gamma = 10$ and the rest have $\gamma = 1.5$). With these parameters the model is able to capture the volatility of asset returns, the equity premium, and the risk-free rate. Their paper shows that heterogeneous risk aversion (together with heterogeneity in other dimensions) is useful for matching the behavior of asset returns; our result is complementary since we use a common risk aversion but heterogeneity in other dimensions.

2 Individual decision

All random variables are defined on a probability space $(\Omega, \mathcal{F}, P)$. Time is discrete and finite,[2] $t = 0, 1, \dots, T$. An agent starts with initial wealth $w > 0$ and has no income other than those obtained by investing in assets.[3]

2.1 Assets, information, and preference

Assets There are K assets indexed by $k \in K = \{1, \dots, K\}$. Let P_t^k, D_t^k be the price and dividend of asset k at time t. The gross return of asset k between the end of time t and the beginning of time $t+1$ is denoted by $R_{t+1}^k = (P_{t+1}^k + D_{t+1}^k)/P_t^k$, and the vector of gross asset returns is denoted by

$$\mathbf{R}_{t+1} = (R_{t+1}^1, \dots, R_{t+1}^K).$$

[2]The model can be easily generalized to infinite horizon if we assume that each agent lives for only finite periods or if we make distributional assumptions (such as Markov shocks) to guarantee the convergence of the value function. See Toda (2014b) for such an example.

[3]Since this paper is concerned with frictionless complete asset markets, the agent can sell off his future endowments and incorporate into the initial wealth.

Let ϕ_t^k be the fraction of wealth invested in asset k at time t and $\phi_t = (\phi_t^1, \dots, \phi_t^K)$ be the portfolio, so $\sum_k \phi_t^k = 1$. Of course, $\phi_t^k > 0 (< 0)$ means a long (short) position in asset k. The agent can be constrained in the choice of portfolio: let $\Pi_t \subset \mathbb{R}^K$ be the set of feasible portfolios. The gross return on portfolio $\phi_t \in \Pi_t$ is denoted by

$$R_{t+1}(\phi_t) := \mathbf{R}'_{t+1}\phi_t = \sum_{k=1}^{K} R_{t+1}^k \phi_t^k.$$

The sequential budget constraint of the agent is therefore

$$(\forall t)\ w_{t+1} = R_{t+1}(\phi_t)(w_t - c_t) \geq 0.$$

Information and preference The agent's information is represented by the filtration (an increasing sequence of σ-algebras) $\{\mathcal{F}_t\}_{t=0}^T \subset \mathcal{F}$. Let w_t be the agent's wealth at the beginning of time t and $\mathbf{X}_t = (X_t^1, X_t^2, \dots)$ be the vector of state variables at time t *other than wealth*. What we have in mind for the state variables are public information such as past returns and volatility, but it may also include private information such as past consumption (in the case of habit formation). To obtain the results it is unnecessary to specify $\mathbf{X}_t$ explicitly. The conditional expectation with respect to time t information is denoted by $\mathrm{E}[\cdot \mid \mathcal{F}_t]$ or more compactly $\mathrm{E}_t[\cdot]$, which are functions of $\mathbf{X}_t$ and w_t because by assumption these are all the state variables. Let $c_t, U_t \in \mathbb{R}$ be the consumption and the continuation utility at time t. We make the following assumptions.

Assumption 1 (Irrelevance of wealth)**.** *For any $\mathcal{F}_{t+1}$-measurable function f, we have $\mathrm{E}[f(\mathbf{X}_{t+1}) \mid \mathcal{F}_t] = g(\mathbf{X}_t)$ for some g, that is, agent's wealth is irrelevant for predicting a function of next period's state variables other than wealth.*

Assumption 1 simply means that the agent is so small compared to the market that his wealth level does not affect asset returns, *i.e.*, the agent is a price taker.

Assumption 2 (Constant relative risk aversion and homotheticity)**.** *The continuation utilities $\{U_t\}_{t=0}^T$ satisfy the recursion $U_T = a_T(\mathbf{X}_T)c_T$ and*

$$U_t = f_t\left(c_t, \mathrm{E}\left[U_{t+1}^{1-\gamma} \,\middle|\, \mathcal{F}_t\right]^{\frac{1}{1-\gamma}}, \mathbf{X}_t\right), \quad t = 0, \dots, T-1, \tag{2.1}$$

where $a_T > 0$ is some function of the state variables $\mathbf{X}_T$, $\gamma > 0$ is the relative risk aversion coefficient, and

$$f_t : \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}^{\dim \mathbf{X}_t} \to \mathbb{R}_+$$

is strictly increasing and homogeneous of degree 1 in the first two arguments.

f_t is called the *aggregator* (Epstein and Zin, 1989; Boyd, 1990). Since the risk aversion is over the continuation utility, not consumption, it is the correct notion of risk aversion (Swanson, 2012). At this point it is helpful to provide concrete examples.

Example 1 (Additive CRRA utility)**.** If $a_T(\mathbf{X}_T) = 1$ and the aggregator is given by

$$f_t(c, v, \mathbf{X}_t) = (c^{1-\gamma} + \beta v^{1-\gamma})^{\frac{1}{1-\gamma}}$$

(so the state variables do not directly enter the aggregator), then iterating (2.1) and using the law of iterated expectations, we obtain

$$U_t = \mathrm{E}\left[\sum_{s=t}^{T} \beta^{t-s} c_s^{1-\gamma} \,\middle|\, \mathcal{F}_t\right]^{\frac{1}{1-\gamma}},$$

which is ordinally equivalent to the standard additive CRRA utility

$$\mathrm{E}_t \sum_{s=t}^{T} \beta^{t-s} \frac{c_s^{1-\gamma}}{1-\gamma}$$

with discount factor β and relative risk aversion γ.

Example 2 (Recursive CRRA/CEIS utility)**.** If $a_T(\mathbf{X}_T) = 1$ and the aggregator is given by

$$f_t(c, v, \mathbf{X}_t) = (c^{1-\sigma} + \beta v^{1-\sigma})^{\frac{1}{1-\sigma}}$$

(so the state variables do not directly enter the aggregator), then U_t is the constant relative risk aversion (CRRA), constant elasticity of intertemporal substitution (CEIS) recursive utility (Epstein and Zin, 1989) with discount factor β, relative risk aversion γ, and elasticity of intertemporal substitution $1/\sigma$.

Example 3 (Habit formation)**.** In Examples 1 and 2, the aggregator f_t did not explicitly depend on the state variables $\mathbf{X}_t$, but (2.1) allows such dependence. For example, if $\mathbf{X}_t$ consists of past consumption and the aggregator explicitly depends on $\mathbf{X}_t$, the recursive utility (2.1) depends on past consumption and hence we can incorporate some form of habit formation (Abel, 1990). One such example that satisfies Assumption 2 is

$$f_t(c, v, x) = \left[(c/x)^{1-\sigma} + \beta v^{1-\sigma}\right]^{\frac{1}{1-\sigma}},$$

where x is the habit stock.

2.2 Optimal portfolio problem

To solve the optimal consumption-portfolio problem we further need an assumption on asset returns and state variables.

Assumption 3 (Conditional independence)**.** *For each t, the next period's state variables $\mathbf{X}_{t+1}$ and asset returns $\mathbf{R}_{t+1}$ are independent conditional on time t information $\mathcal{F}_t$.*

Conditional independence implies, in particular, that the most recent asset return is not a state variable: $\mathbf{R}_t \notin \mathbf{X}_t$, which is clearly a restriction. An obvious case in which conditional independence holds is when returns are i.i.d. and independent of state variables. However, the assumption is still weak enough to be useful. For example, suppose that returns are lognormal with time-varying expected return and volatility: $\log R_{t+1} \sim N(\mu_t, \sigma_t^2)$. Here the state variable is $\mathbf{X}_t = (\mu_t, \sigma_t)$. Conditional independence holds if, for instance, the expected return-volatility pair $\{\mathbf{X}_t\}$ is a Markov process and $\log R_{t+1} = \mu_t + \sigma_t z_{t+1}$, where $\{z_t\}$ is a Gaussian white noise that is independent from the process $\{\mathbf{X}_t\}$.[4]

[4] In this example we implicitly assumed that there is a single risky asset, but the argument clearly holds for any number of assets.

Another example is the GARCH process with no leverage effect. Let $\log R_{t+1} = \mu + \epsilon_{t+1}$ and consider the GARCH(p,q) process

$$\begin{aligned}\epsilon_{t+1} &= \sigma_t z_{t+1},\\ \sigma_t^2 &= \alpha_0 + \alpha_1 \epsilon_t^2 + \cdots + \alpha_q \epsilon_{t-q+1}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_p \sigma_{t-p}^2,\end{aligned}$$

where $\{z_t\}$ is a white noise. Then the state variables are

$$\mathbf{X}_t = (\epsilon_t, \dots, \epsilon_{t-q+1}, \sigma_{t-1}, \dots, \sigma_{t-p}),$$

and the conditional independence assumption does not necessarily hold because ϵ_{t+1} (part of next period's state variables) and $R_{t+1} = \exp(\mu + \epsilon_{t+1})$ are not independent conditional on $\mathbf{X}_t$. However, if $\alpha_1 = 0$ (no leverage effect), then ϵ_t is no longer a state variable, and conditional independence holds.

The following theorem shows that the optimal portfolio problem can be disentangled from the optimal consumption/saving problem, and that the former depends only on risk aversion and asset returns.

Theorem 2.1. *Under Assumptions 1–3, the value function*

$$\begin{aligned}&V_t(w, \mathbf{X}_t)\\ &= \sup \{U_t \mid w_t = w, (\forall s \geq t)\ w_{s+1} = R_{s+1}(\phi_s)(w_s - c_s) \geq 0, \phi_s \in \Pi_s\} \end{aligned} \tag{2.2}$$

is linear in wealth w and the optimal portfolio problem at time t reduces to

$$\max_{\phi \in \Pi_t} \frac{1}{1-\gamma} \mathrm{E}\left[R_{t+1}(\phi)^{1-\gamma} \,\middle|\, \mathcal{F}_t\right]. \tag{2.3}$$

If the portfolio constraint Π_t is nonempty, compact, and

$$\mathrm{E}\left[\sup_{\phi \in \Pi_t} R_{t+1}(\phi)^{1-\gamma} \,\middle|\, \mathcal{F}_t\right] < \infty,$$

then the optimal portfolio problem (2.3) *has a solution.*

Proof. The proof is by induction. If $t = T$, then $U_T = a_T(\mathbf{X}_T) c_T$, so

$$V_T(w, \mathbf{X}_T) = \sup \{a_T(\mathbf{X}_T) c_T \mid c_T \leq w\} = a_T(\mathbf{X}_T) w$$

is linear in wealth and there are no portfolio decisions to make. Suppose the claim is true for time $s = t+1, \dots, T$ and let $V_s(w, \mathbf{X}_s) = a_s(\mathbf{X}_s) w$. Then we obtain

$$\begin{aligned}&V_t(w, \mathbf{X}_t)\\ &= \sup_{\substack{0 \leq c \leq w\\ \phi \in \Pi_t}} f_t\left(c, (w-c) \mathrm{E}_t[a_{t+1}(\mathbf{X}_{t+1})^{1-\gamma} R_{t+1}(\phi)^{1-\gamma}]^{\frac{1}{1-\gamma}}, \mathbf{X}_t\right)\\ &= \sup_{0 \leq c \leq w} f_t\left(c, (w-c) \mathrm{E}_t[a_{t+1}(\mathbf{X}_{t+1})^{1-\gamma}]^{\frac{1}{1-\gamma}} \sup_{\phi \in \Pi_t} \mathrm{E}_t[R_{t+1}(\phi)^{1-\gamma}]^{\frac{1}{1-\gamma}}, \mathbf{X}_t\right)\\ &= \sup_{0 \leq c \leq w} f_t\left(c, (w-c) b_t(\mathbf{X}_t) \sup_{\phi \in \Pi_t} \mathrm{E}_t[R_{t+1}(\phi)^{1-\gamma}]^{\frac{1}{1-\gamma}}, \mathbf{X}_t\right)\\ &= \sup_{0 \leq \tilde{c} \leq 1} w f_t\left(\tilde{c}, (1-\tilde{c}) b_t(\mathbf{X}_t) \sup_{\phi \in \Pi_t} \mathrm{E}_t[R_{t+1}(\phi)^{1-\gamma}]^{\frac{1}{1-\gamma}}, \mathbf{X}_t\right) =: a_t(\mathbf{X}_t) w,\end{aligned}$$

where we used backward induction in the first equality, conditional independence (Assumption 3) and monotonicity of f_t in the second, the irrelevance of wealth (Assumption 1) in the third, and the homogeneity of f_t (Assumption 2) in the last, where we set $\tilde{c} = c/w$. Therefore the value function is linear in wealth. Since f_t is increasing in the second argument, the optimal portfolio problem at time t is

$$\max_{\phi \in \Pi_t} \mathrm{E}\left[R_{t+1}(\phi)^{1-\gamma} \,\middle|\, \mathcal{F}_t\right]^{\frac{1}{1-\gamma}},$$

which is equivalent to (2.3) because $x \mapsto \frac{x^{1-\gamma}}{1-\gamma}$ is monotone.

If $\mathrm{E}\left[\sup_{\phi \in \Pi_t} R_{t+1}(\phi)^{1-\gamma} \,\middle|\, \mathcal{F}_t\right] < \infty$,[5] then by the Dominated Convergence Theorem $\phi \mapsto \mathrm{E}_t[R_{t+1}(\phi)^{1-\gamma}]$ is continuous. Therefore if the portfolio constraint Π_t is nonempty and compact, the optimal portfolio problem (2.3) has a solution. □

Theorem 2.1 is related to Kocherlakota (1990), where he proves in a representative agent, complete markets, endowment economy setting that the CRRA/CEIS recursive utility model (Example 2) is observationally equivalent to the standard additive CRRA utility model if consumption growth is i.i.d. His irrelevance result can be generalized as in the following proposition.

Proposition 2.2. *Consider the recursive utility model satisfying Assumption 2 with a time-homogeneous aggregator $f(c, v)$ with no state variables. If asset returns are i.i.d. and U_0 defined by (2.1) converges as $T \to \infty$, then the recursive utility model is observationally equivalent to the standard additive CRRA utility model.*

Proof. It suffices to show that the optimal portfolio choice and consumption are observationally equivalent in the two models. By Theorem 2.1, the portfolio choice is the same. If the recursive utility converges as time periods tends to infinity, the Bellman equation becomes time-homogeneous. Since the aggregator $f(c, v)$ is homogeneous of degree 1, the optimal consumption is a constant fraction of wealth, which is observationally equivalent to the additive CRRA case. □

3 Partial equilibrium

Having solved the single agent problem, in this section we consider an economy with many agents. In a partial equilibrium setting, we derive a relative asset pricing formula that depends only on the market portfolio and the relative risk aversion.

3.1 Description of the economy

The financial market is the same as in Section 2, so asset k has (per share) price P_t^k, dividend D_t^k, and gross return $R_{t+1}^k = (P_{t+1}^k + D_{t+1}^k)/P_t^k$. As in any partial equilibrium analysis, the stochastic processes of price and dividend are exogenous in the model, not just exogenous from the point of view of individual

[5]This condition is not very stringent. For example, it holds if $\gamma > 1$ (< 1) and the portfolio return is bounded away from zero (bounded above).

agents.[6] Let W_t^k be the market capitalization (per share price P_t^k times the number of shares outstanding) of asset k.

The economy is populated by I agents indexed by $i \in I = \{1, \dots, I\}$ with recursive preferences defined by (2.1), where the aggregators $\{(f_{it})_{i\in I}\}_{t=0}^{T-1}$ and the state variables $\{(\mathbf{X}_{it})_{i\in I}\}_{t=0}^{T}$ are potentially different but the relative risk aversion $\gamma > 0$ and the portfolio constraint $\Pi_t \subset \mathbb{R}^K$ are *common* across agents. Agent i is endowed with initial wealth $w_{i0} > 0$ but nothing thereafter. Let $\mathcal{F}_{it}$ be the private information of agent i at time t and $\mathcal{F}_t = \bigcap_i \mathcal{F}_{it}$ be the public information.

The sequential partial equilibrium is defined by agent optimization and market clearing.

Definition 3.1 (Sequential partial equilibrium)**.** Given asset prices and dividends $\left\{(P_t^k, D_t^k)_{k\in K}\right\}_{t=0}^{T}$, the profile of individual consumption, wealth, and portfolio $\{(c_{it}, w_{it}, \phi_{it})_{i\in I}\}_{t=0}^{T}$ and market capitalization $\left\{(W_t^k)_{k\in K}\right\}_{t=0}^{T}$ constitute a sequential partial equilibrium if

1. given asset returns $R_{t+1}^k = (P_{t+1}^k + D_{t+1}^k)/P_t^k$, the portfolio ϕ_{it} solves

$$\max_{\phi\in\Pi_t} \frac{1}{1-\gamma} \mathrm{E}\left[R_{t+1}(\phi)^{1-\gamma} \,\middle|\, \mathcal{F}_{it}\right], \tag{3.1}$$

2. given the portfolio choice, c_{it} solves the optimal consumption problem (2.2),

3. asset markets clear, *i.e.*, for each asset k and time t we have $\sum_{i=1}^{I} \phi_{it}^k (w_{it} - c_{it}) = W_t^k$, and

4. individual wealth evolves according to the budget constraint

$$w_{i,t+1} = R_{t+1}(\phi_{it})(w_{it} - c_{it}).$$

Since prices and dividends are exogenous in a partial equilibrium, the problem of finding the equilibrium is merely a collection of individual optimization problems. Once we solve for the optimal rules (c_{it}, ϕ_{it}), we obtain the market capitalization $\left\{W_t^k\right\}$ by adding up the individual holdings of each asset.

3.2 Relative asset pricing

In order to prove the main result, we need one more assumption. We assume markets are efficient in the sense that private information is useless for predicting asset returns.

Assumption 4 (Efficient market hypothesis)**.** *For each i and t, the distribution of asset returns $\mathbf{R}_{t+1} = (R_{t+1}^k)_{k=1}^K$ conditional on private information $\mathcal{F}_{it}$ is the same as the distribution conditional on public information $\mathcal{F}_t$.*

This definition of market efficiency is taken from the first definition in Bewley (1982). The following proposition shows that if there is an equilibrium, there is also an equivalent symmetric equilibrium (common portfolio choice).

[6] We can think of a partial equilibrium as a small open economy.

Proposition 3.2. *Let everything be as above. Suppose that*

1. *agents have information and recursive preferences satisfying Assumptions 1 and 2,*

2. *for each agent conditional independence (Assumption 3) holds, and*

3. *the efficient market hypothesis (Assumption 4) holds.*

If there is a partial equilibrium, then there is also an equilibrium with a common portfolio choice ϕ_t^* *(market portfolio) and the same consumption and wealth as in the original equilibrium* $\{(c_{it}, w_{it})_{i\in I}\}_{t=0}^T$.

Proof. By the efficient market hypothesis (Assumption 4), we can replace the private information $\mathcal{F}_{it}$ in (3.1) by the public information $\mathcal{F}_t$. Then the optimal portfolio problem becomes common across all agents, which is (2.3).

Suppose that $\{(c_{it}, w_{it}, \phi_{it})_{i\in I}\}_{t=0}^T$ is a sequential partial equilibrium. Define the value weighted average portfolio by

$$\bar{\phi}_t := \sum_{i=1}^I \phi_{it}(w_{it} - c_{it}) / \sum_{i=1}^I (w_{it} - c_{it}).$$

By the definition of $\bar{\phi}_t$ and the market clearing condition, we have

$$\sum_{i=1}^I \bar{\phi}_t^k (w_{it} - c_{it}) = \sum_{i=1}^I \phi_{it}^k (w_{it} - c_{it}) = W_t^k$$

for each k, so the common portfolio $\bar{\phi}_t$ (market portfolio) clears the market. Since the function $\frac{1}{1-\gamma} R_{t+1}(\phi)^{1-\gamma}$ is quasi-concave in ϕ and ϕ_{it} solves (2.3) for each i, so does $\bar{\phi}_t$. Therefore $\left\{(c_{it}, w_{it}, \bar{\phi}_t)_{i\in I}\right\}_{t=0}^T$ (same consumption and wealth as in the original equilibrium with common portfolio $\bar{\phi}_t$) is also an equilibrium. □

Let $\phi_t^* := \bar{\phi}_t$ be the market portfolio, which is also an individually optimal portfolio. The following theorem, which is the main result of this paper, shows that the $(-\gamma)$-th power of the return on the market portfolio is a valid stochastic discount factor.

Theorem 3.3. *Let everything be as in Proposition 3.2 and* $\{(c_{it}, w_{it}, \phi_t^*)_{i\in I}\}_{t=0}^T$ *be a symmetric sequential partial equilibrium, where* ϕ_t^* *is the market portfolio. If the portfolio constraint* $\phi \in \Pi_t$ *does not bind at the market portfolio* ϕ_t^* *for asset* k, *letting* $R_{m,t+1} = R_{t+1}(\phi_t^*)$ *be the return on the market portfolio, we have*

$$\mathrm{E}\left[R_{m,t+1}^{-\gamma}(R_{t+1}^k - R_{m,t+1}) \,\middle|\, \mathcal{F}_t\right] = 0, \tag{3.2a}$$

$$P_t^k = \frac{\mathrm{E}\left[R_{m,t+1}^{-\gamma}(P_{t+1}^k + D_{t+1}^k) \,\middle|\, \mathcal{F}_t\right]}{\mathrm{E}\left[R_{m,t+1}^{1-\gamma} \,\middle|\, \mathcal{F}_t\right]}, \tag{3.2b}$$

i.e., *the* $(-\gamma)$*-th power of the return on the market portfolio (more precisely,* $R_{m,t+1}^{-\gamma} / \mathrm{E}\left[R_{m,t+1}^{-\gamma} \,\middle|\, \mathcal{F}_t\right]$*) is a valid stochastic discount factor. In particular, the* *one period risk-free rate is*

$$R_{f,t} = \frac{\mathrm{E}\left[R_{m,t+1}^{1-\gamma} \,\middle|\, \mathcal{F}_t\right]}{\mathrm{E}\left[R_{m,t+1}^{-\gamma} \,\middle|\, \mathcal{F}_t\right]}. \tag{3.3}$$

Furthermore, the equity premium satisfies the covariance pricing formula

$$\mathrm{E}\left[R_{t+1}^k \,\middle|\, \mathcal{F}_t\right] - R_{f,t} = -\frac{\mathrm{Cov}\left[R_{m,t+1}^{-\gamma}, R_{t+1}^k \,\middle|\, \mathcal{F}_t\right]}{\mathrm{E}\left[R_{m,t+1}^{-\gamma} \,\middle|\, \mathcal{F}_t\right]}. \tag{3.4}$$

Proof. Consider investing the fraction of wealth $1-\alpha$ in the market portfolio ϕ_t^* and α in asset k. Clearly $\alpha = 0$ is optimal by the definition of ϕ_t^*, so

$$0 \in \arg\max_{\alpha} \frac{1}{1-\gamma} \mathrm{E}\left[\left[(1-\alpha)R_{m,t+1} + \alpha R_{t+1}^k\right]^{1-\gamma} \,\middle|\, \mathcal{F}_t\right]. \tag{3.5}$$

Since by assumption the portfolio constraint $\phi \in \Pi_t$ does not bind, by taking the first-order condition of the maximization (3.5) at the optimum $\alpha = 0$, we obtain (3.2a). Substituting $R_{t+1}^k = (P_{t+1}^k + D_{t+1}^k)/P_t^k$ into (3.2a) and rearranging terms, we obtain (3.2b). Setting $P_{t+1}^k = 0$ and $D_{t+1}^k = 1$ in (3.2b), we obtain the price of the one period risk-free bond $1/R_t^f$, and hence (3.3). The derivation of (3.4) is completely standard. □

Theorem 3.3 may appear completely standard at first glance, but it is not. In a consumption-based representative agent setting (with a standard additive CRRA utility function), the growth rate of consumption is proportional to the return on the market portfolio, and (3.2a) is trivial (it is the Euler equation). What is surprising is that despite the presence of many agents with heterogeneous preferences, we derived a simple stochastic discount factor, $R_m^{-\gamma}$, which depends only on the relative risk aversion and the market portfolio.

This result has three important implications. First, since this result does not depend on any particular utility function and hence on the aggregate or individual consumption process, the "returns-based asset pricing" approach is robust to misspecification of the model as opposed to the consumption-based approach. Since in our model consumption is not directly connected to asset prices, the low volatility of consumption growth (or the low covariance between consumption growth and asset returns) needed in order to explain asset prices ("consumption volatility puzzle") is not an asset pricing puzzle (that belongs to finance) but a consumption/saving puzzle (that belongs to macroeconomics).

Second, since the asset pricing formula contains only asset returns data, which are available in high frequency and high accuracy, our model can be used in practice. The CAPM can also be interpreted as an approximation to our model. To see this, note that the (conditional) CAPM implies the existence of numbers a_t, b_t such that (3.2a) holds by replacing $R_{m,t+1}^{-\gamma}$ with $a_t - b_t R_{m,t+1}$. By Taylor expanding $R^{-\gamma}$ around $R = 1$, we get

$$R^{-\gamma} \approx 1 - \gamma(R-1) = 1 + \gamma - \gamma R,$$

so setting $a_t = 1+\gamma$ and $b_t = \gamma$, CAPM is a linear approximation of our discount factor.

Third, the relative risk aversion γ can be estimated by GMM using only asset returns data, which (unlike consumption) are highly accurate and available in high frequency. The commonly used Euler equation, for example, does not contain more information than (3.2a) for estimating γ even if the Euler equation is true (*i.e.*, the model is correctly specified). This means that the rejection of a particular model using consumption data should be interpreted as the rejection of the particular specification of the model rather than the rejection of the asset pricing implications of the model.

To the best of our knowledge, documenting the robustness of the '$(-\gamma)$-th power of market return' SDF seems to be new. The closest expression we found in the literature is Rubinstein (1976), in which he obtains the same discount factor, but assuming (i) a representative agent with an additive CRRA utility function, (ii) single asset, and (iii) independent returns. In testing the CRRA/CEIS recursive utility model of Example 2, Epstein and Zin (1991) derived the following equation:

$$\mathrm{E}\left[(c_{t+1}/c_t)^{-\frac{\sigma(1-\gamma)}{1-\sigma}} R_{m,t+1}^{\frac{\sigma-\gamma}{1-\sigma}} (R_{t+1}^k - R_{m,t+1}) \,\middle|\, \mathcal{F}_t\right] = 0, \tag{3.6}$$

where $1/\sigma$ is the elasticity of intertemporal substitution and we have changed their notation to be compatible with mine. Since (3.2a) obtains by setting $\sigma = 0$ in (3.6), (3.2a) is a stronger implication. However, (3.2a) holds with much more general preferences than CRRA/CEIS recursive utility (in particular, (3.2a) is true with any σ). Therefore our result is sharper despite the assumption being weaker.

Dittmar (2002) stresses the importance of a nonlinear pricing kernel, but his specification is based on a representative agent model and the coefficients are hard to interpret.

3.3 Empirical Implications

The conditional moment restriction (3.2a) as well as the unconditional moment restriction

$$(\forall k = 1, \dots, K) \;\; \mathrm{E}[R_m^{-\gamma}(R^k - R_m)] = 0, \tag{3.7}$$

are testable implications of the partial equilibrium model and also allow us to estimate the relative risk aversion γ. Using data from 1926-2011, we obtain RRA estimates around 2 with standard errors around 0.6, and our results are virtually identical across the choice of test assets and instruments. In addition, the conditional and unconditional moment restrictions are not rejected. This is satisfactory since any proposed solution of asset pricing puzzles that "does not explain the premium for $\gamma < 2.5$ ... is ... likely to be widely viewed as a resolution that depends on a high degree of risk aversion" (Lucas, 1994, p. 335). Complete details are in Section A.

4 General equilibrium

This section deals with absolute pricing in a general equilibrium setting, which is similar to Cox et al. (1985). We introduce firms and financial assets (assets that are in zero net supply) and derive asset pricing formulas.

4.1 Description of the economy

Firms and assets There is a single perishable good which can be consumed or invested as capital. There are J firms indexed by $j \in J = \{1, \dots, J\}$. Production takes time and exhibits constant returns to scale. If firm j employs capital K at the end of period t, it produces $A^j_{t+1}K$ at the beginning of period $t+1$, where A^j_{t+1} is the (random) productivity as well as the total return of capital after depreciation. In particular, if an agent invests one unit of capital in firm j at time t, he will receive A^j_{t+1} at the beginning of the next period. We can think of firms as stochastic saving technologies. Let $\mathbf{A}_{t+1} = (A^1_{t+1}, \dots, A^J_{t+1})$ be the vector of productivities.

There are K assets in zero net supply indexed by $k \in K = \{1, \dots, K\}$, with dividend D^k_t at period t (which is, of course, a random variable). Letting P^k_t be the price of asset k in period t (determined in equilibrium), the gross return between periods t and $t+1$ is defined by $R^k_{t+1} = (P^k_{t+1} + D^k_{t+1})/P^k_t$. Let $\mathbf{D}_t = (D^1_t, \dots, D^k_t)$ be the vector of dividends.

Let $(\theta, \phi) \in \mathbb{R}^J_+ \times \mathbb{R}^K$ be the portfolio of investment and asset holdings, so θ^j and ϕ^k are the fraction of wealth invested in firm j and asset k. As before, there might be a portfolio constraint denoted by $\Pi_t \subset \mathbb{R}^J_+ \times \mathbb{R}^K$ at time t. The portfolio $(\theta, \phi) \in \Pi_t$ defines the return on portfolio

$$R_{t+1}(\theta, \phi) = \sum_{j=1}^{J} A^j_{t+1}\theta^j + \sum_{k=1}^{K} R^k_{t+1}\phi^k. \tag{4.1}$$

Equilibrium As usual the sequential general equilibrium is defined by agent optimization and market clearing.

Definition 4.1. $\left\{(c_{it}, w_{it}, \theta_{it}, \phi_{it})_{i \in I}, (P^k_t)_{k \in K}\right\}_{t=0}^{T}$ constitute a sequential general equilibrium if

1. given asset returns $R^k_{t+1} = (P^k_{t+1} + D^k_{t+1})/P^k_t$, the portfolio (θ_{it}, ϕ_{it}) solves

$$\max_{(\theta,\phi) \in \Pi_t} \frac{1}{1-\gamma} \mathrm{E}\left[R_{t+1}(\theta, \phi)^{1-\gamma} \,\middle|\, \mathcal{F}_{it}\right], \tag{4.2}$$

2. given the portfolio choice, c_{it} solves the optimal consumption problem (2.2),

3. markets for assets in zero net supply clear, *i.e.*, for each asset k and time t we have $\sum_{i=1}^{I} \phi^k_{it}(w_{it} - c_{it}) = 0$, and

4. individual wealth evolves according to the budget constraint

$$w_{i,t+1} = R_{t+1}(\theta_{it}, \phi_{it})(w_{it} - c_{it}).$$

4.2 Absolute asset pricing

Theorem 4.2. *Let* $\Theta_t = \left\{\theta \in \mathbb{R}^J_+ \,\middle|\, (\theta, 0) \in \Pi_t\right\}$ *be the portfolio constraint on investment with holdings in assets in zero net supply restricted to be zero. Suppose that*

1. *agents have information and recursive preferences satisfying Assumptions 1 and 2,*

2. *for each agent conditional independence (Assumption 3) holds,* i.e., *the distributions of the individual state variables* $\mathbf{X}_{i,t+1}$ *and the productivities and dividends* $(\mathbf{A}_{t+1}, \mathbf{D}_{t+1})$ *are independent conditional on private information* $\mathcal{F}_{it}$*,*

3. *the efficient market hypothesis (Assumption 4) holds,*

4. *the aggregators* (f_{it}) *are sufficiently regular so that the optimal consumption always exists,*[7] *and*

5. Θ_t *is nonempty, compact, convex, and*

$$\mathrm{E}\left[\sup_{\theta\in\Theta_t} R_{t+1}(\theta,0)^{1-\gamma} \,\middle|\, \mathcal{F}_t\right] < \infty.$$

Then there exists a symmetric equilibrium with a common portfolio of investment θ_t^* *and no trade in zero net supply assets, where*

$$\theta_t^* \in \operatorname*{arg\,max}_{\theta\in\Theta_t} \frac{1}{1-\gamma} \mathrm{E}\left[R_{t+1}(\theta,0)^{1-\gamma} \,\middle|\, \mathcal{F}_t\right]. \tag{4.3}$$

Proof. By Theorem 2.1, the optimal portfolio problem (4.3) has a solution θ_t^*. Let c_{it} be the optimal consumption corresponding to θ_t^*, which exists by assumption. Define the price of asset k, P_t^k, by iterating (3.2b), where $R_{m,t+1} = R_{t+1}(\theta_t^*, 0)$. Then by construction the first-order condition for the maximization (4.2) (with $\mathcal{F}_t$ instead of $\mathcal{F}_{it}$) holds for every asset $k \in K$. By the definition of θ_t^*, the first-order condition for the maximization (4.2) holds for every investment $j \in J$. Hence the first-order condition holds for every returns j and k. Since the first-order condition is sufficient for maximum because the objective function in (4.2) is quasi-concave, $(\theta_t^*, 0)$ is optimal in Π_t. Since the individual asset holdings is zero by construction, the markets of assets in zero net supply clear. Therefore we obtain a sequential equilibrium. ☐

Remark. Since

$$R_{t+1}(\theta,0) = \sum_{j=1}^{J} A_{t+1}^{j}\theta$$

by the definition of returns on portfolio (4.1), the symmetric equilibrium portfolio θ_t^* in (4.3) can be computed without knowing the asset prices. The asset prices can then be computed using (3.2b) with $R_{m,t+1} = R_{t+1}(\theta_t^*, 0)$.

Combining Theorems 3.3 and 4.2, we obtain an absolute asset pricing formula.

Corollary 4.3. *Let everything be as in Theorem 4.2. Then the conclusion of Theorem 3.3 holds.*

[7] For instance, the upper semi-continuity of the aggregator $f(c, v, \mathbf{X})$ with respect to the first two arguments on $\mathbb{R}_+^2$ suffices.

As in Theorem 3.3, the $(-\gamma)$-th power of the return on the market portfolio $(\theta_t^*, 0)$ is a valid stochastic discount factor. In order to build a general equilibrium model (*i.e.*, *not* a partial equilibrium model), in Theorem 4.2 we assumed that firms are AK type technologies and ignored inputs other than capital, for example labor or raw materials. It is not easy to solve for the general equilibrium if we make the model more realistic by introducing other inputs. However, we can obtain the same results even if there are multiple capital types and firms have nonlinear production functions as long as there is no capital adjustment costs and the production functions are constant-returns-to-scale (see Toda (2014a) for such an example).

Corollary 4.3 is surprising in that any preference characteristics other than risk aversion have no asset pricing implications: asset prices are determined by the technologies and relative risk aversion alone. In particular, the interest rate is completely pinned down, no matter how patient or impatient agents are. How could this be true? The intuition is simple: if there is no uncertainty, because a linear production technology between today and tomorrow determines the relative price between today and tomorrow, it is obvious that the interest rate is determined only by the technology. The risk-free rate formula (3.3) is the generalization to the case with uncertainty.

5 Concluding remarks

This paper can be summarized as follows: (i) we found a simple, yet economically motivated stochastic discount factor ($R_m^{-\gamma}$) (ii) we theoretically showed the robustness of this SDF (iii) we tested the SDF and failed to reject it: a relative risk aversion coefficient of around 2 is consistent with the historical asset returns data. (Being primarily a theoretical paper, we kept the empirical analysis to the bare minimum.) Although this SDF has been already known (Rubinstein, 1976), it has not captured much attention.[8] Given the robustness of this SDF, it deserves a serious consideration. Our paper presents further support for the returns-based approach.

A clear lesson from this paper is the usefulness of AK models. Combined with homothetic preferences, AK models admit full analytical tractability, even with many agents with heterogeneous preferences as long as agents have a common relative risk aversion. This is true even in an incomplete markets setting, as shown by Toda (2014b), which derives a similar stochastic discount factor. AK models have investment as a key element and hence are more realistic, unlike pure exchange models that can only proxy hunter-gatherer economies.

Our main result is that the $(-\gamma)$-th power of market return is robust to model specification so long as agents have common risk aversion. Naturally, our proposed SDF is *not* robust to heterogeneity in risk aversion. The value of our contribution should nonetheless be evident: heterogeneity in dimensions besides risk aversion allows for a great deal of tractability. Since our results suggest that the "asset pricing puzzles" are actually "consumption puzzles", an important direction for future research is to emphasize heterogeneity in these dimensions.

[8] Cochrane (2005) mentions only the log utility case ($\gamma = 1$) briefly.

A Testing the asset pricing implications

In this appendix we estimate the relative risk aversion γ and test the conditional moment restriction (3.2a) as well as the unconditional moment restriction

$$(\forall k = 1, \dots, K) \;\; \mathrm{E}[R_m^{-\gamma}(R^k - R_m)] = 0, \tag{A.1}$$

which are testable implications of the partial equilibrium model of Section 3. Using monthly data from 1926 to 1981, Brown and Gibbons (1985) estimated γ from the unconditional moment condition (A.1) with only one asset (the risk-free asset) and obtained $\hat{\gamma} = 1.79$, but they did not test the moment condition (since γ is exactly identified). The focus of this section is in testing both the conditional and unconditional moment restrictions, not just estimating the relative risk aversion coefficient. Testing the general equilibrium model of Section 4 (possibly using firm data or data on national wealth, GDP, and investment) would be certainly interesting but is beyond the scope of this paper.

A.1 Data

For nominal asset returns data, we use the monthly and quarterly returns of NYSE value-weighted portfolio (total market as well as portfolios sorted by size and book-to-market value) for stocks and Treasury Index, both available from the Center for Research in Security Prices (CRSP). Nominal returns are converted to real returns by adjusting with the Consumer Price Index (CPI). As instrumental variables for testing the conditional moment restriction (3.2a), we consider past annual dividend yields because they are known to predict returns (Fama and French, 1988). The data on annual dividend yields is taken from Robert Shiller's website.[9] More specifically, we consider two sets of test assets:

S10 30 day T-bill rate and 10 portfolios of stocks sorted by size, and

FF25 30 day T-bill rate and Fama-French 25 portfolios of stocks sorted by size and book-to-market value.

The sample period is January 1926–December 2011 for the 10 stock size portfolios and July 1931–December 2011 for the Fama-French 25 portfolios because in the latter case some returns data is unavailable for 1926–1931.

We assume that the gross return on any agent's wealth portfolio is proportional to the stock market return. Of course we are aware that the stock market is not the portfolio of total wealth (Roll, 1977; Stambaugh, 1982), but this is not a bad first approximation. We can justify this assumption as follows. Assets are priced by asset market participants, who are typically wealthy and hold a large number of stock shares. Therefore it is reasonable to expect that the stock market return is a proxy of the total return on the wealth portfolio of asset market participants.

A.2 Identification

Let $\mathbf{R}_t = (R_t^1, \dots, R_t^K)$ be the vector of gross asset returns, $\mathbf{z}_t \in \mathbb{R}^L$ be the vector of instruments (a constant ($\mathbf{z}_t = 1$) for testing the unconditional moment

[9] `http://www.econ.yale.edu/~shiller/data.htm`

restriction (A.1) and the vector of a constant and past dividend yields for testing the conditional moment restriction (3.2a)),

$$u_t(\gamma) = R_{m,t}^{-\gamma}(\mathbf{R}_t - R_{m,t}\mathbf{1}_K) \otimes \mathbf{z}_t \in \mathbb{R}^{KL}$$

be the pricing error ("$\otimes$" denotes the Kronecker product), and

$$g_T(\gamma) = \frac{1}{T}\sum_{t=1}^{T} u_t(\gamma)$$

be its sample average. Let $\nabla u_t(\gamma), \nabla g_T(\gamma)$ be the derivative of u_t, g_T with respect to γ, which are also $KL \times 1$ vectors.

It has been recognized that weak identification can be a source of poor performance of the GMM estimator, especially in nonlinear models (Stock and Wright, 2000). We follow Wright (2003) for testing the lack of identification. In order to identify γ_0 (the true parameter value), $\mathrm{E}[\nabla u_t(\gamma_0)]$ must have full column rank, which is equivalent to $\mathrm{E}[\nabla u_t(\gamma_0)] \neq 0$ since ∇u_t is a vector, not a matrix. For a fixed γ, under the null that $\mathrm{E}[\nabla u_t(\gamma)] = 0$, the test statistic

$$T\nabla g_T(\gamma)'\hat{C}(\gamma)^{-1}\nabla g_T(\gamma)$$

is asymptotically $\chi^2(KL)$ distributed, where $\hat{C}(\gamma)$ is a consistent estimator of the long run variance of $\nabla u_t(\gamma)$.[10] For $\hat{C}(\gamma)$ we take the Newey and West (1987) heteroskedasticity and autocorrelation consistent covariance matrix with truncation parameter $m = \lfloor T^{1/3} \rfloor$. Table 1 shows the range of γ for which the lack of identification is not rejected at significance level 0.05 for each combination of test assets and number of lagged dividend yields used as instrument. According to Table 1, γ may be unidentified if either (i) we use quarterly data, (ii) we use more than one year of past lagged dividend yields as instrument, or (iii) $\gamma \notin [-15, 5]$. Therefore in what follows we only use monthly data with either no lagged dividend yields (unconditional model) or the previous year's dividend yield (conditional model), and assume that the true parameter value γ_0 is in the range $[-15, 5]$ and estimate γ over this interval.

Table 1. Range of γ for which lack of identification is not rejected. # lags indicates the number of lagged dividend yields used as instrument.

Test assets	10 size portfolios (S10)		Fama-French 25 (FF25)	
Period	1926–2011		1931–2011	
Frequency	monthly	quarterly	monthly	quarterly
# lags				
0	$[-26.5, 13.9]$	$[-33.6, 5.0]$	$[-34.2, 12.2]$	$\emptyset$
1	$[-37.6, 16.0]$	$[-48.3, 1.4]$	$[-14.8, 5.1]$	$\emptyset$
2	$[-27.7, 9.0]$	$\emptyset$	$\emptyset$	$\emptyset$

[10] In Wright (2003) the test statistic is more complicated because he develops a general theory for any number of parameters (one has to perform a minimization over all matrices of rank less than the number of parameters). Since in our model there is only one parameter, this step is unnecessary.

A.3 Estimation

Since weak identification does not seem to be an issue at least for monthly data with no or one year of lagged dividend yields and $\gamma \in [-15, 5]$, we obtain the estimate $\hat{\gamma}$ by minimizing the continuously updated optimal GMM criterion

$$J_T(\gamma) = T g_T(\gamma)' \hat{\Omega}(\gamma)^{-1} g_T(\gamma),$$

where $\hat{\Omega}(\gamma)$ is the Newey-West HAC estimator of the long run variance of $u_t(\gamma)$.

Table 2 presents the estimate $\hat{\gamma}$ of the relative risk aversion (RRA) coefficient, its standard error, the number of periods and moment restrictions, and the P value of the J-test for overidentifying restrictions with monthly data. The results are virtually identical across the choice of test assets (10 stock size portfolios or Fama-French 25 portfolios) and instruments (no or previous year's dividend yield). The RRA estimates are around 2 with standard errors around 0.6 for all specifications. Therefore the log-utility CAPM ($\gamma = 1$) is not rejected. The moment restriction is not rejected except the unconditional model with FF25, which implies that there is no equity premium puzzle or risk-free rate puzzle. This is satisfactory since any proposed solution of asset pricing puzzles that "does not explain the premium for $\gamma < 2.5$ ... is ... likely to be widely viewed as a resolution that depends on a high degree of risk aversion" (Lucas, 1994, p. 335). Our RRA estimate of around 2 is also in line with estimates using the Consumption Expenditure Survey (CEX). For example, Brav et al. (2002) and Vissing-Jørgensen (2002) report RRA of 3–4 and 2.5–3.3, respectively.[11]

Table 2. GMM estimation results of $\mathrm{E}[R_m^{-\gamma}(R^k - R_m)] = 0$.

Test assets	10 size portfolios (S10)		Fama-French 25 (FF25)	
Period	1926–2011		1931–2011	
Conditional?	no	yes	no	yes
RRA, $\hat{\gamma}$	2.0	2.15	2.05	1.55
S.E.	0.65	0.64	0.58	0.52
T	1032	1032	966	966
# moments	11	22	26	52
P (J test)	0.123	0.125	0.0015	0.106

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[11]Vissing-Jørgensen (2002) reports the elasticity of intertemporal substitution (EIS) to be 0.3–0.4 for stock holders. With additive CRRA preferences, EIS is the inverse of RRA, so the range of RRA is 2.5–3.3. Given the irrelevance result of Kocherlakota (1990) or Proposition 2.2, it seems more appropriate to interpret her result as an estimate of RRA.

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