Groups of Special Units

Amanda Beeson

1 Introduction

My research centers on explicit class field theory but, more generally, I have a taste for explicit constructive number theory. This means my research is quite amenable to undergraduate work. Every one of the problems described in Section 4 have pieces that a student could work on independently! For example, I would love to work with a student on getting my software working to compute translates of modular units (a more or less combinatorial program). I would also love to have a student work with me on using the new trigonometric identities in Section 2.2 and the relations satisfied by gamma-monomials to give an independent construction of the squares cited in Section 2.2. An advanced student with some abstract algebra background would even be able to work on a further study of the almost abelian extensions introduced at the end of this section.

Leopold Kronecker (1823-1891) first observed that every field extension of the rational numbers with abelian Galois group is contained in a pure cyclotomic field. For example, $\mathbb{Q}(\sqrt{-3})/\mathbb{Q}$ has Galois group $\mathbb{Z}/2\mathbb{Z}$ and is contained in $\mathbb{Q}(\zeta_3)$ for $\zeta_3 = e^{2\pi i/3} = \frac{1+\sqrt{-3}}{2}$, a third root of unity. He then noted that the single holomorphic function, $f(z) = e^{2\pi iz}$, evaluated at rational numbers is responsible for every such extension. Inspired by this observation, he dreamt that a function might be found to thus classify abelian extensions of imaginary quadratic base fields. His dream came true in the form of complex multiplication, hailed by David Hilbert (1862-1943) as "not only the most beautiful part of mathematics but also of all science"[9]. In fact, the natural generalization, namely to find a function or family of functions whose special values generate the maximal abelian extension of any extension of the rationals, or *number field*, became Hilbert's twelfth problem.

In [10] Harold Stark put forth a conjectural program about units in number fields and the special values of L-functions that, in some settings, offers a solution to Hilbert's twelfth problem. I will motivate and describe the abelian conjecture where the order of vanishing at $s = 0$ is one, but there is a conjecture about the first non-vanishing coefficient in the non-abelian and higher order of vanishing settings as well.

I will then map out the related work of Anderson and Kubert-Lang whose work has inspired my own.

The mingling of algebra and analysis that Hilbert so revered is typified by the analytic class number formula, which relates the first nonzero Taylor series coefficient of the Dedekind zeta function to algebraic invariants of the number field K. Let K/k be a Galois extension of number fields. The Dedekind zeta function of K is defined by its Euler product in the right half-plane $\Re(s) > 1$ and there it factors as the product of its constituent Artin L-functions

$$\zeta_K(s) = \prod_p (1 - \mathcal{N}p^{-s})^{-1} = \prod_\chi L_{K/k}(s, \chi)$$

where $\mathcal{N}(p)$ is the absolute norm of p, the first product is taken over prime ideals p of the ring of integers of K, and the second is the product over irreducible characters of the Galois group of K/k. Dirichlet's analytic class number formula then says that the Dedekind zeta function has a Taylor expansion about $s = 0$ that looks like

$$\zeta_K(s) = -\frac{hR}{W} s^{r_1+r_2-1} + O(s^{r_1+r_2}),$$

where r_1 and r_2 are the number of real and half the number of complex embeddings of K, respectively, h is the class number, R the regulator, and W the number of roots of unity in K. Stark wanted to split the regulator matrix into character components in correspondence with the L-function decomposition of the zeta function. Furthermore, he wanted to see whether the leading terms of the L-functions factor into a transcendental part corresponding to the regulator and a rational (or at worst algebraic) part corresponding to $-h/W$ in the class number formula.

We now state Stark's conjecture for K/k an abelian extension of number fields. Let S be a finite set of primes in k containing all infinite and ramified primes of k, and at least one prime, say v, that splits completely from k to K, and at least two primes overall. Let $L_S(s, \chi)$ be the Dirichlet L-function associated to $\chi \in \widehat{G}$ with Euler-factors associated to primes in S removed. Then Stark's conjecture predicts the existence of an S-unit $\epsilon \in K$, unique up to roots of unity, such that

$$L'_S(0, \chi) = -\frac{1}{W} \sum_\sigma \overline{\chi}(\sigma) \log |\epsilon|_{w^\sigma}, \quad \forall \chi \in \widehat{G}.$$

The sum on the right is over σ in the Galois group of K/k and w is a fixed prime above v. In this setting, Stark further conjectures that the W^{th} root of ϵ generates an abelian extension not only over K, but over k, and experimental data confirms this.

Stark's conjecture as formulated above is proved only for $k = \mathbb{Q}$, k imaginary quadratic, and a handful of special cases. In the rank one abelian case, the conjecture

remains open for other number fields. The higher rank and non-abelian cases remain largely untouched.

In [1], Anderson defines an *almost abelian group* to be one such that every commutator is central and squares to the identity. He defines $G^{ab+\epsilon}$ to be the quotient of $G(\overline{\mathbb{Q}}/\mathbb{Q})$ universal for continuous homomorphisms to almost abelian profinite groups. He shows that the corresponding Galois extension of $\mathbb{Q}$ is, in fact G^{ab}, the maximal abelian extension of $\mathbb{Q}$ with the fourth roots of rational primes and certain gamma-monomials adjoined (hence, $+\epsilon$!). As he remarks,

"The relations standing between the Main Formula, the index formulas of Sinnott, Deligne reciprocity, the theory of Fröhlich, the theory of Das, the theory of the group cohomology of the universal ordinary distribution and Stark's conjecture and its variants deserve to be thoroughly investigated. We have only scratched the surface here. Stark's conjecture is relevant in view of the well known expansion

$$\sum_{n=0}^{\infty} \frac{1}{(n+x)^s} = \frac{1}{2} - x + s \log \frac{\Gamma(x)}{\sqrt{2\pi}} + O(s^2)$$

of the Hurwitz zeta function at $s = 0$"[1].

2 Thesis Research

2.1 Units in the field of modular functions

Kubert and Lang prove in [4] that if U is the group of units in the field of modular functions on congruence subgroups of $SL_2(\mathbb{Z})$, and S is the subgroup generated by the Siegel units, then U/S has exponent two. Their work begs the question of how to construct explicitly elements in S whose square roots are in $U \setminus S$. This relates to Anderson's construction over $\mathbb{Q}$ because the special values of these functions and their square roots might together generate almost abelian extensions of an imaginary quadratic extension of $\mathbb{Q}$.

I was able to find combinations of Siegel units analogous to Anderson's gamma-monomials that are conjectured to have a square root in U. I have proved the following theorem and its corollary. To state them, we first define the level of a modular function. Let $\pi : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ be the natural projection map. Then $\Gamma(N) = \ker \pi$ and a meromorphic function on the complex upper half-plane $\mathcal{H}$ is a modular function of level N if $f(\gamma(z)) = f(z)$ for all $\gamma \in \Gamma(N)$ and its magnitude is bounded at the limit points of $\Gamma(N)\backslash\mathcal{H}$.

Theorem. If $f(z)$ is a modular unit and $\sqrt{f(z)}$ is a modular function of level M for some $M \in \mathbb{N}$ with $N|M$ then, in fact, $\sqrt{f(z)}$ is a modular function of level $2N$.

Corollary. Let $\mathcal{F}_N$ denote the full field of modular functions of level N. If $f(z)$ is a modular unit of level N then either $\sqrt{f(z)} \in \mathcal{F}_N(\sqrt{j(z)-1728})$ or $\sqrt{f(z)}$ is not level M for any $M \in \mathbb{N}$.

In order to better understand the behavior of these square roots I followed up on Anderson's suggestion regarding the relation of his work to the Stark's conjecture. What follows is a discussion of that study.

2.2 Explicit squares in cyclotomic fields

Anderson's result discussed above includes the construction of gamma-monomials that, together with roots of unity, generate almost abelian extensions of $\mathbb{Q}$. Let k be the cyclotomic field $\mathbb{Q}(e^{2\pi i/pq})$ and let $\alpha \in k$ be a unit such that $k(\sqrt{\alpha})/\mathbb{Q}$ is almost abelian. Let σ be an element of the Galois group of $k/\mathbb{Q}$. Then $\frac{\alpha}{\sigma(\alpha)}$ is the square of a unit in k and, thus, has a square root in k. By considering the group of relations satisfied by gamma-monomials, I can find this square root explicitly. Thus, finding a multiplicative basis for the units of index class number extending these squares would prove certain class numbers are even.

My work in this area led to two results. First, I found a family of new trigonometric identities indexed by products of distinct primes pq, exemplified by $pq = 15$:

$$\frac{4\sin(\pi/15)\sin(4\pi/15)\sin(9\pi/15)}{\sin(3\pi/15)} = 1.$$

Second, in light of the expansion of the Hurwitz zeta function, and the fact that

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)},$$

these identities show that certain sums of Hurwitz zeta functions vanish to first order at $s = 0$. For example,

$$\zeta(s,\frac{3}{15}) - \zeta(s,\frac{1}{15}) - \zeta(s,\frac{4}{15}) - \zeta(s,\frac{9}{15}) + \zeta(s,\frac{12}{15}) - \zeta(s,\frac{14}{15}) - \zeta(s,\frac{11}{15}) - \zeta(s,\frac{6}{15}) =$$

$$s\log\frac{4\sin(\pi/15)\sin(4\pi/15)\sin(9\pi/15)}{\sin(3\pi/15)} + O(s^2)$$

vanishes to first order at $s = 0$. Therefore, the coefficient of s^2 is of interest as we shall see below.

2.3 L-functions attached to characters of conductor p

The vanishing of these Hurwitz zeta functions is, in turn, connected to the known first-order vanishing of L-functions associated to even characters of conductor p. Let χ_p be a non-trivial even character of conductor p, and χ_{pq} its inflation to the group $(\mathbb{Z}/pq\mathbb{Z})^\times$. Suppose $S = \{p, q\}$, and recall that $L_S(s, \chi)$ has Euler factors associated to primes in S removed. Then the second order vanishing of $(pq)^s L(s, \chi_p)(1 - q^{-s})$ at $s = 0$ can be expressed in terms of the second order vanishing of Hurwitz zeta functions as in the following example for $S = \{3, 5\}$

$$15^s L(s, \chi_5)(1 - 3^{-s}) =$$
$$15^s L(s, \chi_5)(1 - \chi_5(3)3^{-s}) - 35^s L(s, \chi_5)[3^{-s}(1 - \chi_5(3))] =$$
$$15^s L_S(s, \chi_{15}) - 5^s L(s, \chi_5) + 5^s \chi_5(3) L(s, \chi_5) =$$
$$\sum_{\substack{a=1 \\ (a,15)=1}}^{14} \zeta(s, \frac{a}{15})\chi_{15}(a) - \sum_{b=1}^{4} \zeta(s, \frac{3b}{15})\chi_5(b) + \sum_{c=1}^{4} \zeta(s, \frac{3c}{15})\chi_5(3c).$$

If we now use the character orthogonality relations to isolate the $a = 1, 4, 11, 14$, $b = 2, 3$, and $c = 1, 4$ terms then we see that the first non-vanishing coefficient from the previous section is, in fact, the lead term of $-2 \cdot 15^s L(s, \chi_5)(1 - 3^{-s})$, which is known (because the Stark's conjecture is proved in this setting) to be

$$-2\log(3)L'(0, \chi_5) = -\log(3)[\log|(1 - \zeta_5)(1 - \zeta_5^{-1})| + \log|(1 - \zeta_5^2)(1 - \zeta_5^{-2})|] =$$
$$-2\log(3)\log|(1 - \zeta_5)(1 - \zeta_5^2)^{-1}|,$$

where ζ_5 is a primitive fifth root of unity.

3 Current Research

Anderson says "Perhaps there is an analogue of the Main Formula over an imaginary quadratic field involving elliptic units. This possibility seems especially intriguing."[1] My current research goal is to write down this analogue and thereby give explicit generators and Galois action for the maximal almost abelian extension of an imaginary quadratic base field.

Let $\mathcal{A}$ be the free group on symbols of the form $[a]$, where $[a]$ is the class of $a \in \mathbb{Q}$ modulo the relation $[a] = [b] \iff b - a \in \mathbb{Z}$. Define the map

$$\sin : \mathcal{A} \to (\mathbb{Q}^{ab})^\times$$

to be the unique homomorphism such that

$$\sin[a] = \begin{cases} 2\sin(\pi a) = |1 - e^{2\pi i a}| & \text{if } 0 < a < 1 \\ 1 & \text{if } a = 0. \end{cases}$$

Now define the symbol $\mathbf{a}_{pq}$ for $2 < p < q$ to be

$$\mathbf{a}_{pq} = \sum_{i=1}^{\frac{p-1}{2}} \left(\left[\frac{i}{p} \right] - \sum_{k=0}^{\frac{q-1}{2}} \left[\frac{i}{pq} + \frac{k}{q} \right] \right) - \sum_{j=1}^{\frac{q-1}{2}} \left(\left[\frac{j}{q} \right] - \sum_{l=0}^{\frac{p-1}{2}} \left[\frac{j}{pq} + \frac{l}{p} \right] \right)$$

and for $p = 2$ to be

$$\mathbf{a}_{pq} = \left(\left[\frac{1}{4} \right] - \sum_{k=0}^{\frac{q-1}{2}} \left[\frac{1}{4q} + \frac{k}{q} \right] \right) - \sum_{j=1}^{\frac{q-1}{2}} \left(\left[\frac{j}{q} \right] + \left[-\frac{1}{2q} + \frac{j}{q} \right] - \left[\frac{j}{2q} \right] - \left[-\frac{1}{4q} + \frac{j}{2q} \right] \right)$$

Let $S = \{-1\} \cup \{\text{rational prime numbers }\}$, which is a $\mathbb{Z}/2\mathbb{Z}$-basis for $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$. For $p \in S$ define $e_p \in \mathrm{H}^1(G^{ab}, \mathbb{Z}/2\mathbb{Z})$ by

$$\sigma(\sqrt{p}) = (-1)^{e_p(\sigma)}\sqrt{p}$$

for $\sigma \in G^{ab}$. By Kummer Theory, $\{e_p\}_{p \in S}$ is a $\mathbb{Z}/2\mathbb{Z}$-basis for $\mathrm{H}^1(G^{ab}, \mathbb{Z}/2\mathbb{Z})$ so $e_p \wedge e_q$ for $p > q$ is a basis for $\wedge^2 \mathrm{H}^1(G^{ab}, \mathbb{Z}/2\mathbb{Z})$

Anderson's Main Formula is

$$\mathrm{D}(\mathbf{a}_{pq} \mod \mathbb{Q}^{\times 2}) = e_p \wedge e_q$$

for $p > q$, which gives an explicit inversion of the isomorphism

$$\mathrm{D} : \mathrm{H}^0(G^{ab}, \mathbb{Q}^{ab\times}/\mathbb{Q}^{ab\times 2}) \to \wedge^2 \mathrm{H}^1(G^{ab}, \mathbb{Z}/2\mathbb{Z}).$$

on a basis of $\wedge^2 \mathrm{H}^1(G^{ab}, \mathbb{Z}/2\mathbb{Z})$. Explicitly inverting this map is the key step in Anderson's construction of $\mathbb{Q}^{ab+\epsilon}$.

Let k be an imaginary quadratic field. I am presently working on explicitly inverting the isomorphism

$$\mathrm{D} : \mathrm{H}^0(G_k^{ab}, k^{ab\times}/k^{ab\times 2}) \to \wedge^2 \mathrm{H}^1(G_k^{ab}, \mathbb{Z}/2\mathbb{Z}).$$

This will give a basis for the maximal almost abelian extension of k.

4 Future Directions

1. Further investigation of the behavior of the proposed square root under the action of $SL_2(\mathbb{Z})$?

2. Find a multiplicative basis for the subgroup of cyclotomic units with index class number in the full unit group of $\mathbb{Q}(\zeta_m)^+$.

3. Let $k = \mathbb{Q}(\zeta_m)^+$ and $G = G(k/\mathbb{Q})$. Let S be the set containing the infinite prime of $\mathbb{Q}$ and all the finite primes dividing m. Find a $\mathbb{Z}[G]$-submodule of U_S, the S-units in k, call it C, that contains the cyclotomic numbers and is of index class number in U_S. Show that $\widehat{H}^{q-2}(G, X) \cong \widehat{H}^q(G, C)$, where X is the $\mathbb{Z}[G]$-module of degree zero divisors of primes of k supported at the primes dividing m.

To accomplish (1), I will compute the transformation of my proposed square root using software I am developing. There are three possibilities for the outcome. First, the square root could fail to be in U. Second, the square root could transform under $\Gamma(N)$ for some N, but fail to have cyclotomic coefficients. Third, the square root could transform under $\Gamma(N)$ for some N and have cyclotomic coefficients. Let the imaginary quadratic base field be k. In the third case, special values of the function would generate an abelian extension K/k and its square root would generate a quadratic extension of K that is abelian over k. In this case, there would exist units that are squares. Together with (2), this could be used to show certain class numbers are even. In the second case, the square root would generate quadratic extensions of the maximal abelian extension of k in analogy with Anderson's construction. In this case, the next step would be to investigate to what extent these square roots generate the maximal almost abelian extension of k.

For (2), I will start with $m = pq$, for p and q distinct odd primes. Let $k = \mathbb{Q}(\zeta_{pq})^+$. Sinnott proves in [9] that the cyclotomic units have index in the full group of units equal to the class number of k. I have been working on finding a multiplicative basis for these units with an eye towards proving certain class number are even using results of Anderson. I have used Stark's conjecture in conjunction with explicit group-ring determinants and successfully accomplished this in examples. There are two avenues that I plan to explore more thoroughly: using Tate's representation-theoretic reformulation of Stark's conjecture [11] to break up the group determinant into character components; and using Sinnott's proof to break the construction of a basis into steps correlated to his intermediate index calculations.

(3) is a natural question that arises in the theory of Tate sequences and was pointed out to me by Popescu. In the case that m is a prime power, the cyclotomic units, call them C_S, are already index class number in U_S. In this setting, C_S is isomorphic to

$\mathbb{Z}[G]\epsilon$, where ϵ is the Stark unit in k. Furthermore, the group of divisors on primes above p and infinity, which we shall call Y_S, is simply $Y_S \cong \mathbb{Z} \oplus \mathbb{Z}[G]$. Thus, the short exact sequence

$$0 \to X_S \to Y_S \stackrel{deg}{\to} \mathbb{Z} \to 0$$

splits and we see that X_S is isomorphic to $\mathbb{Z}[G]$. Hence, both C_S and X_S are cohomologically trivial. The next case to treat is $m = pq$.

5 Bibliography

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