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GENERICITY OF RIGID AND MULTIPLY RECURRENT INFINITE MEASURE-PRESERVING AND NONSINGULAR TRANSFORMATIONS

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ABSTRACT. We show that the generic transformation is rigid and hence multiply recurrent for the group of infinite measure-preserving tranformations M and the group of nonsingular transformations G, for both the weak and uniform topologies.

1. INTRODUCTION

As is well-know, Furstenberg [9] has shown that finite measure-preserving transformations are multiply recurrent. However, it is now known that there exist infinite measure preserving transformations, with various dynamical properties, that are not multiply recurrent [7], [1], [3]. In this note we show that multiple recurrence is generic, in both the weak (also called coarse) and uniform topologies, in the group of infinite measure-preserving transformations and the group of nonsingular transformations. In fact, we show a stronger condition that we call power multiple recurrence. Our method is by showing that rigidity, for infinite measure-preserving and nonsingular trasnsformations is generic, and then observing that rigidity implies multiple recurrence.

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After the preliminary definitions we recall the weak and uniform topologies for infinite measure-preserving and nonsingular transformations. In Section 3 we treat genericity in the weak topology and in Section 4 genericity for the uniform topology.

Let $(X, \mathcal{B}, \mu)$ be a nonatomic Lebesgue probability space. A **nonsingular automorphism** or **transformation** is an invertible map $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ such that T and T^{-1} are measurable, and $\mu(A) = 0$ if and only if $\mu(T^{-1}(A)) = 0$. T is **ergodic** if for all $A \in \mathcal{B}$ with $T^{-1}(A) = A$, we have $\mu(A)\mu(A^c) = 0$. T is **conservative** if for every A with $\mu(A) > 0$ there is an integer $n > 0$ such that $\mu(A \cap T^{-n}(A)) > 0$. T is said to have **infinite ergodic index** if for all integers $d > 0$, the d-fold Cartesian products $T \times \ldots \times T$ are ergodic. Let ν be a $\sigma-$finite, infinite, Lebesgue measure equivalent to μ. A nonsingular automorphism T is **ν- measure-preserving** if $\nu(T^{-1}(A)) = \nu(A)$ for all measurable sets A.

Let $G(X, \mu)$ denote the group of all nonsingular automorphisms on $(X, \mathcal{B}, \mu)$ and let $M(X, \nu)$ the subgroup of $G(X, \mu)$ consisting of all automorphisms on (X, ν) that preserve ν; in this paper we will not be concerned with the group of finite measure-preserving automorphisms.

If T on $(X, \mathcal{B}, \mu)$ is a nonsingular automorphism we will denote by ω_i the Jacobian or Radon-Nikodym derivative of T^{-i}:

$$\mu(T^{-i}A) = \int_A \omega_i d\mu \qquad \forall A \in \mathcal{B}.$$

Let $d > 0$ be an integer. A nonsingular automorphism T on $(X, \mathcal{B}, \mu)$ is said to be **d-recurrent** if for all sets of positive measure A there exists an integer $n > 0$ with $\mu(A \cap T^n A \cap \ldots \cap T^{nd} A) > 0$. T is **multiply recurrent** if it is d-recurrent for all $d > 0$. As is well known, Furstenberg [9] has shown that every finite measure preserving transformation is multiply recurrent. It is clear that multiple recurrence implies conservativity, and that there exist infinite measure-preserving ergodic automorphisms (on atomic spaces) that are not conservative. However, Eigen, Hajian and Halverson in [7] showed that there exist ergodic, conservative, infinite measure preseving transformations that are d-recurrent but not $d+1$-recurrent, for every $d > 1$; these transformations, though, do not have infinite ergodic index. Recently, Aaronson and Nakada [1] have shown that if S is an infinite measure-preserving Markov shift, then S is d-recurrent if and only if the Cartesian product of d copies of S is conservative; so in particular an infinite ergodic index Markov shift is multiply recurrent. Aaronson and Nakada also construct an infinite odometer, not of infinite ergodic index, with some additional properties that is not 2-recurrent. More recently, Adams, Friedman and Silva [3] have shown that there exists an infinite measure-preserving, rank one, automorphism T that has infinite ergodic index but that is not 2-recurrent; for this transformation $T \times T^2$ is not conservative. Thus infinite ergodic index, for infinite measure preserving automorphisms, does not imply multiple recurrence.

Our aim in this article is to show that multiple recurrence is generic for the weak and uniform topologies (defined below) in both the group of nonsingular automorphisms and the group of infinite measure-preserving automorphisms. In fact we show something stronger. We define a nonsingular transformation T to be **power multiply recurrent** if for all finite sequences of integers $\{k_1, \ldots, k_r\}$, $T^{k_1} \times \ldots \times T^{k_r}$ is multiply recurrent. We show that power multiple recurrence is generic. We finally note that infinite ergodic index has been shown to be generic in the weak topology for infinite measure-preserving (not necessarily invertible) transformations by Sachdeva [13], and more recently for nonsinglar automorphisms by Choksi and Nadkarni [6].

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2. The weak and uniform topologies

We consider two well-known topologies on the groups $G(X, \mu)$ and $M(X, \nu)$.

Given a nonsingular automorphism T define the operator U_T on $L^1(X, \mu)$ by

$$U_T(f) = f \circ T^{-1} \omega_1(x).$$

(We should note here that sometimes one lets $\omega_i(x)$ denote the Radon-Nikodym derivative of T^i and not of T^{-i} as we are doing here and defines U_T using T rather than T^{-1}; the reason for our choice here is so that U_T preserves composition and is an isomorphism rather than an anti-isimorphism.)

Clearly, U_T is a positive isometry on $L^1(X,\mu)$. The **coarse topology** or **weak topology** on $G(X,\mu)$ is defined by the following metrix. Let $\{E_n\}$ be any countable sufficient family in (X,μ), and for any two nonsingular automorphisms S,T let

$$d(T,S) = \frac{1}{2^n}\sum_{n=1}^{\infty}(||U_T(\chi_{E_n}) - U_S(\chi_{E_n})||_1 + \\ ||U_{T^{-1}}(\chi_{E_n}) - U_{S^{-1}}(\chi_{E_n})||_1).$$

This topology is independent of the choice of the family $\{E_n\}$ and it can be shown that that $T_n \to T$ if and only if $||U_{T_n}(f) - U_T(f)||_1 \to 0$ for all $f \in L^1(X,\mu)$. We will use the characterization of this topology in the lemma below. For a proof and further properties of this topology we refer to A. Ionescu Tulcea [11], Choksi and Kakutani [4], and Hamachi and Osikawa [10]. We note that while in some works this topology is referred to as the coarse topology, we will call it the weak topology as this agrees with what is commonly called the weak topology on the group of finite measure-preserving automorphisms.

Lemma 2.1. *$G(X,\mu)$ is a complete metrizable group under the weak topology, and $M(X,\nu)$ is a closed subgroup of $G(X,\mu)$. The topology is the same for any other finite or σ-finite measure ν equivalent to μ. Furthermore, if $\{T_n\}_{n=1}^{\infty}, T$ are nonsingular automorphisms,*

$T_n \to T$ weakly as $n \to \infty$ in $G(X,\mu)$ if and only if

$$\mu(T_n(A) \triangle T(A)) \to 0 \quad as \quad n \to \infty$$

and

$$\omega_{T_n} \to \omega_T \quad as \quad n \to \infty \quad in \quad L^1.$$

If $\{T_n\}_{n=1}^{\infty}, T$ are infinite measure-preserving automorphisms, $T_n \to T$ weakly as $n \to \infty$ in $M(X,\nu)$ if and only if for all sets A with $\nu(A) < \infty$,

$$\nu(T_n(A) \triangle T(A)) \to 0 \quad as \quad n \to \infty.$$

The **uniform topology** on $G(X,\mu)$ is defined by the metric

$$d(T,S) = \mu\{x : T(x) \neq S(x)\}.$$

This topology is complete metric and finer than the weak topology. It can also be shown that it depends only on the measure class of the measure μ. If $\{T_n\}_{n=1}^{\infty}, T$ are infinite measure preserving automorphisms, $T_n \to T$ uniformly as $n \to \infty$ in $M(X, \nu)$ if and only if for all sets A with $\nu(A) < \infty$,

$$\nu\{x \in A : T_n(x) \neq T(x)\} \to 0,$$

as $n \to \infty$. For properties of this topology we refer the reader to Friedman[8] and Choksi-Kakutani [4].

The phrase *the generic transformation T satisfies property P* in $G(X, \mu)$ (or in $M(X, \nu)$) will mean that the set of transformations in $G(X, \mu)$ (or in $M(X, \nu)$) that satisfy property P contain a dense $\mathcal{G}_\delta$ set, in the appropriate topology.

The long history of genericity results in ergodic theory for finite measure-preserving transformations and homeomorphisms is well known and the reader is referred to [4] and [5] for a survey. The weak topology for the case of nonsingular automorphisms is first considered in Ionescu-Tulcea [11], where she shows that the generic nonsingular transformation does not admit an equivalent σ-finite invariant measure. Krengel [12] starts the study of the weak topology for infinite measure-preserving transformations and Sachdeva [13] shows that those transformations that have infinite ergodic index form a dense $\mathcal{G}_\delta$ set in the group of infinite measure preserving transformations. Choksi and Kakutani [4] studied both the weak topology and the uniform topology and showed that the nonsingular ergodics are generic. Recenlty these results were generalized by Choksi and Nadkarni [6] who showed that under the weak topology, the nonsingular automorphisms that have infinite ergodic index form a dense $\mathcal{G}_\delta$ set.

3. Multiple Recurrence in the Weak Topology

Theorem 3.1. *The generic T is d-recurrent for all $d > 0$ in both $M(X, \nu)$ and $G(X, \mu)$ for the weak topology.*

Proof. In the case of $M(X, \nu)$ the proof is a more or less standard reduction to the case of finite measure and we present it for completeness. Let E denote the identity transformation. We say that T is rigid if and only if $T^{k_i} \to E$ weakly, for some sequence $k_i \to \infty$. It is clear that if T is rigid, then T is multiply recurrent (a proof of this for the nonsingular case is given below). So if suffices to show that the generic transformation is rigid.

By a cyclic permutation of rank $k \geq 0$ we mean any transformation T which is equal to E on $\mathbb{R} \setminus [-2^k, 2^k)$, and the sets $I_k, TI_k, ..., T^{2^{2k+1}-1}I_k$ form a partition of $[-2^k, 2^k)$ consisting of half-open intervals of equal length, where $I_k = [0, 1/2^k)$ and

$$T^{2^{2k+1}} = E.$$

Let O_k denote the set of all such permutations of rank k.

For any transformation S let $B_\delta(S)$ denote the open ball of radius $\delta > 0$ centered at S. For any sequence of positive functions $\delta_k = \delta_k(S)$ defined on O_k, define the sets

$$C_n = \bigcup_{k>n} \bigcup_{S \in O_k} B_{\delta_k}(S);$$

$$C(\delta_1, \delta_2, ...) = \bigcap_{n \geq 1} C_n.$$

For any n, C_n is open and dense, as the set of cyclic permutations is dense (see, for example, [13]). This gives that $C(\delta_1, \delta_2, ...)$ is a dense G_δ set for any sequence of positive functions δ_k.

Fix $\varepsilon_k \searrow 0$. For $S \in O_k$ we choose $\delta_k = \delta_k(S)$, such that

$$T \in B_{\delta_k}(S) \Rightarrow T^{2^{2k+1}} \in B_{\varepsilon_k}(S^{2^{2k+1}}) = B_{\varepsilon_k}(E).$$

Obviously, if $T \in C(\delta_1, \delta_2, ...)$ for this sequence of δ_k's, then T is rigid.

The case for $G(X, \mu)$ needs more careful consideration. Fix $T \in G(X, \mu)$.

Suppose T is rigid. (Rigidity for nonsingular transformations has been defined independently by del Junco and the second author for a different context (unpublished). Partial rigidity was defined in [2].) Let $T^{k_n} \to E$, i.e. $U_T^{k_n} = U_{T^{k_n}} \to E$. Using the isometry of U_T, we have

$$\forall m,\ U_{T^{mk_n}} = U_T^{mk_n} \to E.$$

Finally, by Lemma 2.1,

$$U_{T_n} \to U_{T_0} \Leftrightarrow \omega_n \to \omega_0 \text{ in } L_1 \ \&$$

$$\mu(T_n A \triangle T_0 A) \to 0 \text{ for any measurable set} A,$$

and thus

$$\forall A \forall m, \mu(T^{mk_n} A \triangle A) \to 0;$$

Therefore T is multiply recurrent.

In order to prove the genericity of rigid transformations, we need to define a new set $O_n \subset G$. Let $T \in O_n$ if and only if the sequence $\{T^k(x)\}_{0\le k<\infty}$ is periodic with strict period n for a.e. x. The set $\bigcup_{n>k} O_n$ is dense for any integer k (see [11]). It follows that C_n is dense and open for any n and any sequence of positive δ_k, where

$$C_n = \bigcup_{k>n} \bigcup_{S\in O_k} B_{\delta_k}(S).$$

It is clear that

$$U_{T_n} \to U_S \Rightarrow \forall k U^k_{T_n} \to U^k_S.$$

If $S \in O_k$, then $U^k_S = E$. It remains to repeat, for $C = \bigcap_{n\ge 1} C_n$, the end of the proof for $M(\mathbb{R}, \nu)$, where δ_k are sufficiently small. □

Theorem 3.2. *The generic T is power multiply recurrent in both $M(\mathbb{R}, \nu)$ and $G(X, \mu)$.*

Proof. It is enough to prove that if T is rigid, then T is power multiply recurrent. But the property $T^{l_n} \to E$ for T gives the same for $T^{k_1} \times T^{k_2} \times ... \times T^{k_m}$ in both $M(\mathbb{R}, \nu)$ and $G(X, \mu)$. (We use here that for any isometry U on L_1, $U^{l_n} \to E \Rightarrow U^{kl_n} \times U^{ml_n} \to E$ for any $k, m \in \mathbb{Z}$). Therefore, $T^{k_1} \times T^{k_2} \times ... \times T^{k_m}$ is rigid also, and so multiply recurrent. □

Remark 3.3. It is not difficult to show that the set of rigid transformations, say C_r, is exactly a dense G_δ set in both $M(X, \nu)$ and $G(X, \mu)$. Indeed, it is

$$\bigcap_{n=1} \bigcap_{l=1} \bigcup_{k>l} \{T : T^k \in B_{1/n}(E)\}.$$

The density of C_r follows from the fact that $C \subset C_r$ (see proof of Theorem 3.1), or directly if we add the obvious note that if $T \in O_n$ for some n, then T is rigid, i.e. $T \in C_r$.

4. Multiple Recurrence in the Uniform Topology

Theorem 4.1. *The generic T is power multiply recurrent in both $M(X, \nu)$ and $G(X, \mu)$ with respect to the uniform topology.*

Proof. We will use the same approach. Let T be a rigid transformation. Rewrite this condition in the following form:

$$\forall A(\mu(A) < \infty)\mu\{x \in A : T^{n_i}x \ne x\} \to 0 \text{ as } n_i \to \infty.$$

This also gives an analogous statement for $T^{k_1} \times T^{k_2} \times ... \times T^{k_m}$. Therefore T is power multiply recurrent.

Now consider

$$C_r = \bigcap_{n=1} \bigcap_{l=1} \bigcup_{k>l} \{T : T^k \in B_{1/n}(E)\}.$$

We use here that the uniform topology in $M(X, \nu)$, as it is independent of the measure class, may be defined by the metric

$$d(T, S) = \sum_{n=1}^{\infty} \frac{\nu\{x \in E_n : Tx \neq Sx\}}{2^n n},$$

where $E_n = [-n, n]$. It remains to prove that C_r is dense, because it is clear that $\{T : T^k \in B_{1/n}(E)\}$ is open. Note that the set of all periodic transformations is dense in $G(X, \mu)$ ([11], [8]). This is also well-known for $M(X, \nu)$, but, in principle, it is possible to have from the proof of Propositions 2 and 3 in [4]. Finally, every periodic T is rigid with respect to the uniform topology, or, equivalently, belongs to C_r. □

5. The Case of $\mathbb{Z}^n$ Actions

All theorems mentioned above admit a natural extension to the case of more general group actions, but it is not clear what the analog of T^k should be. Therefore, we will work only with a weaker version of power multiple recurrence. More precisely, we say that a **G-action T has infinite multiple recurrence index** if and only if the G-action which consists of elements $T_g \times ... \times T_g$ (k times) ($g \in G$) is multiply recurrent for any positive k. (A sequence of G-actions T^n converges to a G-action T if for all $g \in G$, T_g^n converges to T_g.)

We announce state here the following theorems which can be proved by the methods outlined above.

Theorem 5.1. *The generic $\mathbb{Z}^n$-action T is d-recurrent for any d and n in both $M_{\mathbb{Z}^n}(\mathbb{R}, \nu)$ and $G_{\mathbb{Z}^n}(X, \mu)$ with respect to the weak and uniform topologies.*

Theorem 5.2. *The generic $\mathbb{Z}^n$-action T has infinite multiply recurrent index for any n in both $M_{\mathbb{Z}^n}(\mathbb{R}, \nu)$ and $G_{\mathbb{Z}^n}(X, \mu)$ with respect to the weak and uniform topologies.*

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