On Factors of Non-singular Cartesian Products *

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Abstract

We classify all factors of the Cartesian product of any two non-singular type III_λ, $0 < \lambda \leq 1$, or type II_1 Chacon transformations, as well as the centralizer of finite Cartesian products of such transformations.

1 Introduction

In [R79], Rudolph introduced the property of minimal self-joinings (MSJ) for finite measure-preserving transformations; this is a strong property that led to a series of important examples. In particular, this property implies that the factors of Cartesian products of T with itself are just the obvious ones (those obtained by fixing a co-ordinate and the symmetric factors). Rudolph also constructed examples of mixing finite measure-preserving transformations satisfying the MSJ property using an extension the "random spacers" method of Ornstein [O72]. Later it was shown by the first author, Rahe, and Swanson [JRS80] that the much simpler weak mixing (finite measure-preserving) Chacon transformation has the MSJ property. This transformation has in turn been used as a source of many examples in ergodic theory.

Rudolph and the second author in [RS89] generalized the notion of minimal self-joinings to non-singular transformations, and constructed examples of non-singular transformations, both with no equivalent σ-finite invariant measure and with equivalent infinite σ-finite invariant measure, with the (non-singular) MSJ property. However, while the MSJ theory in Rudolph [R79] considers n-fold (measure-preserving)

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self-joinigs of T, the non-singular theory in [RS89] was generalized only to 2-fold self-joinings. The reasons for this were technical problems with extending the notion of rational joinings from 2-fold to n-fold self-joinings. (It is shown in [RS89] that the class of all non-singular joinings is too broad and that one must restrict them to a subclass such as the rational joinings.) However, while the 2-fold (non-singular) MSJ property is sufficient to imply primeness (i.e., no non-trivial invertible factors) and trivial centralizer (i.e., commuting only with its powers) [RS89], it is not clear whether it implies anything about the factors or centralizer of $T \times T$. In fact, *a priori* it seems that one needs to know the 4-fold joinings of T to control the factors of $T \times T$, and it is an open problem even in the finite measure-preserving case whether 2-fold MSJ implies n-fold MSJ. We also note that while finite measure-preserving odometers have uncountable centralizers, non-singular type III_λ odometers with trivial centralizer were constructed by Hamachi [H81] and later and independently by Aaronson [Aa87] (see also [Aa97]). However these maps have non-ergodic Cartesian squares and non-trivial factors, and the methods of proof are different from ours and the MSJ theory.

In this paper we approach the study of factors and centralizers of Cartesian products of non-singular transformations using coding techniques. Coding techniques were used by the first author in [J78] to show that the classic Chacon automorphism is prime and has trivial centralizer. Non-singular coding was later introduced by the authors in [JS95], where it was used to show that the type III_λ, $0 < \lambda < 1$, non-singular Chacon automorphisms are prime, and where many of the results of this paper were announced. Here we extend our methods to study the centralizer and factors for Cartesian products of these maps.

The paper is organized as follows. Section 2 starts with some preliminary definitions and a self-contained presentation of non-singular codes. In section 3 we define the non-singular Chacon maps. We start with the geometric definition of a map T_λ, $0 < \lambda \leq 1$, on the unit interval with Lebesgue measure μ; we then show that (T_λ, μ) is isomorphic to (T, μ_λ), where T is the shift on a symbolic space and μ_λ is a non-singular Borel measure on this symbolic space. In our proofs we use the symbolic version (T, μ_λ) but sometimes we refer to some properties that are easier to see in the geometric version (T_λ, μ). Furthermore, to simplify notation, in some cases we may write T_λ for the symbolic version (T, μ_λ). For each $0 < \lambda < 1$, T_λ is a type III_λ non-singular map, and for $\lambda = 1$ it is the classical type II_1 map. We also show how to obtain type III_1 Chacon maps for which our proofs also apply (see

Remark 1).

In section 4 we classify the centralizer of any finite Cartesian product of non-singular Chacon maps and prove the following theorem.

Theorem A. *(cf. Theorem 3) Let* $0 < \lambda_1 < \ldots < \lambda_k \leq 1$ *and* $n_1, \ldots, n_k$ *be integers. Then the centralizer of the Cartesian product* $T_{\lambda_1}^{\otimes n_1} \times \ldots \times T_{\lambda_k}^{\otimes n_k}$ *is generated by maps of the form* $U_1 \times \ldots \times U_k$*, where each* U_i*, acting on the* n_i*-dimensional product space* X^{n_i}*, is a Cartesian product of powers of* T_{λ_i}*, or a co-ordinate permutation on* X^{n_i}*.*

Finally, in section 5 we prove the following theorem that classifies of all the factors of the Cartesian product of any two non-singular type III_λ, $0 < \lambda \leq 1$, or type II_1 Chacon maps.

Theorem B. *(cf. Theorem 6) Let* $\mathbf{X} = (X, \mathcal{B}, \mu_{\lambda_1}, T)$ *and* $\mathbf{X} = (X, \mathcal{B}, \mu_{\lambda_2}, T)$ *be two non-singular Chacon systems. Let* $\mathcal{F}$ *be a factor algebra of* $(T \times T, \mu_{\lambda_1} \times \mu_{\lambda_2})$*.*

(a) If $\lambda_1 \neq \lambda_2$ *then* $\mathcal{F}$ *is equal* $\bmod \mu$ *to one of the four algebras* $\mathcal{B} \otimes \mathcal{C}$, $\mathcal{B} \otimes \mathcal{N}$, $\mathcal{N} \otimes \mathcal{C}$*, or* $\mathcal{N} \otimes \mathcal{N}$ *(where* $\mathcal{N}$ *is the trivial algebra).*

(b)If $\lambda_1 = \lambda_2$ *then* $\mathcal{F}$ *is equal* $\bmod \mu$ *to one of the following algebras* $\mathcal{B} \otimes \mathcal{C}$, $\mathcal{B} \otimes \mathcal{N}$, $\mathcal{N} \otimes \mathcal{C}$, $\mathcal{N} \otimes \mathcal{N}$*, or* $(T^m \times Id)\mathcal{B}^{2\odot}$ *for some integer* m*.*

While in the finite measure-preserving case our results are not new, as they follow from the MSJ property of Chacon's map, our proofs are new also in the finite measure-preserving case and provide another approach to controlling the centralizer and factors of Cartesian products, independent of the MSJ theory. The reader interested in our proof for the finite measure-preserving case may assume that our $\bar{d}$-distance is the classic one and that the Radon-Nikodym derivative (defined below) is always 1.

Lastly, we mention a partial answer to a question in [CEP89], where Choksi, Eigen and Prasad asked whether there exists a zero entropy, finite measure-preserving mixing automorphism S, and a non-singular type III automorphism T, such that $T \times S$ has no Bernoulli factors. This question motivated in part the work in [RS89], but the original question remained open. It follows from Theorem 1 that if S is the finite measure-preserving mildly mixing Chacon automorphism and T is any non-singular Chacon automorphism as defined here, the factors of $T \times S$ are only the trivial ones, and so $T \times S$ has no Bernoulli factors, partially answering the question in [CEP89] with S mildly mixing instead of mixing.

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2 Preliminaries

2.1 Non-singular Transformations

All the spaces we consider are standard (Borel) probability spaces, i.e., they consist of a standard Borel space $(X, \mathcal{B})$ and a probability measure μ on $\mathcal{B}$. A set is *co-null* if it is Borel and its complement is of measure 0. A map $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is a *non-singular automorphism* if there exists a co-null set $X' \subset X$ such that $T : X' \to X'$ is $1:1$ and measurable, and for every $A \in \mathcal{B} \cap X'$, $\mu(T^{-1}A) = 0$ if and only if $\mu(A) = 0$. If T is as above but not necessarily $1:1$ then it is a *non-singular endomorphism.* A *non-singular (dynamical) system* consists of a non-singular automorphism T defined on a standard space $(X, \mathcal{B}, \mu)$; we sometimes denote such a non-singular system by $\mathbf{X} = (X, \mathcal{B}, \mu, T)$. T is *ergodic* if whenever $T^{-1}(A) = A$ then $\mu(A)\mu(A^c) = 0$, and it is *conservative* if for all sets of positive measure A there exists an integer $n > 0$ such that $\mu(T^{-n}(A) \cap A) > 0$. We denote by $C(\mathbf{X})$ or $C(T)$ the *centralizer* of T, that is, all non-singular endomorphisms S of $(X, \mathcal{B}, \mu)$ such that $S \circ T = T \circ S$ a.e. We denote by $F(\mathbf{X})$ the class of *factor algebras* $\mathcal{F}$ of T, that is $\mathcal{F}$ is a sub-σ-algebra of $\mathcal{B}$ invariant under T: $T\mathcal{F} = \mathcal{F} \pmod{\mu}$. Let $\mathcal{N}$, or $\mathcal{N}(X)$ denote the trivial sub-σ-algebra.

If $\mathbf{X} = (X, \mathcal{B}, \mu, T)$ is a non-singular system we denote by ω_i the Jacobian or Radon-Nikodym derivative of T^i:

$$\mu(T^iA) = \int_A \omega_i d\mu \qquad \forall A \in \mathcal{B}. \tag{2.1}$$

When μ or T^i needs to be emphasized we write ω_{μ,T^i}. One has the cocycle relation

$$\omega_i(x)\omega_j(T^ix) \ = \ \omega_{i+j}(x) \quad a.e. \tag{2.2}$$

If $\mathbf{X} = (X, \mathcal{B}, \mu, T)$ is non-singular system, a *factor* of $\mathbf{X}$ is a non-singular system $\mathbf{Y} = (Y, \mathcal{C}, \nu, S)$ such that there exist co-null sets $X' \subset X, Y' \subset Y$ with $TX' \subset X', SY' \subset Y'$ and a measurable, measure-preserving $\phi : X' \to Y'$, called a *homomorphism* (or *factor map*), with $\phi \circ T = S \circ \phi$ on X'. (If ϕ is only non-singular, by replacing ν with the equivalent measure $\mu \circ \phi^{-1}$ we can assume ϕ is measure-preserving.) Clearly if ϕ is a homomorphism $\phi^{-1}(\mathcal{C})$ is a factor algebra. Furthermore, if $\mathcal{F}$ is a factor algebra of $\mathbf{X}$ then there is a non-singular system $\mathbf{Y}$ and a homomorphism $\phi : \mathbf{X} \to \mathbf{Y}$ such that $\mathcal{F} = \phi^{-1}\mathcal{C} \mod \mu$, see e.g. [Aa97, 1.0.10]. We call ϕ the *homomorphism corresponding to* the factor algebra $\mathcal{F}$.

If $\phi : \mathbf{X} \to \mathbf{Y}$ is a homomorphism, μ may be disintegrated with respect to ϕ, i.e., there exist a co-null set Y_0 and a measurable map $y \to \mu_y$ so that for all $y \in Y_0$, μ_y is a probability measure supported on $\phi^{-1}\{y\}$ and such that

$$\mu \;=\; \int \mu_y d\nu(y). \tag{2.3}$$

Furthermore, for all $y = \phi(x) \in Y_0$, $\mu_{Sy} \circ T$ is a measure equivalent to μ_y and

$$\omega_{\mu,T}(x) = \omega_{\nu,S}(y) \quad \frac{d\mu_{Sy} \circ T}{d\mu_y}(x) \quad \mu_y\text{-a.e. for all } y \in Y_0. \tag{2.4}$$

See e.g. [Aa97, 1.0.8, 1.0.11] or [RS89, 1.3.2]. One easily checks that

$$\omega_{\nu,S}(y) \;=\; \int_X \omega_{\mu,T}(x) d\mu_y(x), \quad \text{for} \quad \nu \text{ } a.a. \text{ } y. \tag{2.5}$$

As is well-known, all finite measure-preserving (and all infinite measure-preserving) automorphisms are orbit equivalent [D59,63]. The situation is quite different for non-singular automorphisms. Krieger introduced in [Kr70] the ratio set as an invariant for orbit equivalence of nonsingular automorphisms and showed that all type III_λ, $0 < \lambda < 1$, (defined below), and all type III_1 automorphisms are orbit equivalent [Kr76]. Given a non-singular automorphism T on (X, μ), the *ratio set* of T, denoted by $r(T)$, is defined to be the set of non-negative real numbers t such that for all $\varepsilon > 0$ and all measurable sets A of positive measure there exists $n > 0$ such that

$$\mu(A \cap T^{-n}A \cap \{x \colon \omega_{T^n}(x) \in N_\varepsilon(t)\}) > 0,$$

where $N_\varepsilon(t) = \{s \geq 0 : |s - t| < \varepsilon\}$.

The set $r(T) \setminus \{0\}$ is a closed multiplicative subgroup of the reals and if $0 \in r(T)$ then T admits no equivalent σ-finite invariant measure. This allows four possibilities: 1) $r(T) = \{1\}$, 2) $r(T) = \{0, 1\}$, 3) $r(T) = \{0\} \cup \{\lambda^k : 0 < \lambda < 1,\ k \in \mathbb{Z}\}$., 4) $r(T) = [0, \infty)$.

The first case is called type II and these are actions that admit an equivalent σ-finite invariant measure; if the invariant measure is infinite we say it is type II_∞, otherwise it is type II_1. The others are types III_0, III_λ, $0 < \lambda < 1$, and III_1, respectively. For background material we refer to [Kr70],[Kr76], [HO81] and [KW91].

2.2 Relative Product Measure

We use the relative product measure to study the factors of a non-singular system. The relative product, introduced by Furstenberg for finite measure-preserving systems and used in [RS89] in the nonsingular setting, is an example of a joining. Although the only joining we will use is the relative product, we state several of the results below in terms of joinings as that is their natural context.

If $\mathbf{X}_j = (X_j, \mathcal{B}_j, \mu_j, T_j),\ j = 1, 2$ are dynamical systems a (non-singular) *joining* of $\mathbf{X}_1$ and $\mathbf{X}_2$ is a measure $\hat{\mu}$ on $\mathcal{B}_1 \times \mathcal{B}_2$ projecting onto μ_1 and μ_2 and non-singular for $T_1 \times T_2$. We write

$$\omega_i^1 \ = \ \omega_{\mu_1, T_1^i},\ \omega_i^2 = \omega_{\mu_2, T_2^i},\ \hat{\omega}_i = \omega_{\hat{\mu}, T_1^i \times T_2^i} \tag{2.6}$$

for $i \in \mathbb{Z}$. As a special case of (2.5) on has

$$\omega_i^1(x_1) \ = \ \int_{X_2} \hat{\omega}_i(x_1, x_2) d\hat{\mu}_{x_1}(x_2) \tag{2.7}$$

where $\hat{\mu} = \int_{X_1} \hat{\mu}_{x_1} d\mu_1(x_1)$ is the disintegration of $\hat{\mu}$ over the first co-ordinate in $X_1 \times X_2$. Following [RS89], $\hat{\mu}$ is called a *rational joining* if there are measurable functions $c^1(x_1)$ and $c^2(x_2)$ such that

$$\hat{\omega}(x_1, x_2) \ = \ \omega^1(x_1)\omega^2(x_2)c^1(x_1) \ = \ \omega^1(x_1)\omega^2(x_2)c^2(x_2) \quad \hat{\mu}\ a.e. \tag{2.8}$$

Our main use of rational joinings is that the relative product joining is rational as we see below. However, without the need to change any of the arguments, the reader who wishes to may substitute "rational joining" with "relative product joining" in Lemma 2.1 (assuming $\mathbf{X}_1 = \mathbf{X}_2$) and Corollary 3.2.

If $\phi : \mathbf{X} \to \mathbf{Y}$ is a homomorphism, the *relative product* measure $\hat{\mu} = \mu \times_\phi \mu$ on $X \times X$ is defined by

$$\hat{\mu} \ = \ \int \mu_y \times \mu_y d\nu(y), \tag{2.9}$$

where $\mu \ = \ \int \mu_y d\nu(y)$ is the disintegration of μ over $\mathbf{Y}$.

It is straight forward to check that the relative product $\hat{\mu}$ is a non-singular joining of $\mathbf{X}$ with $\mathbf{X}$. To see that is is rational note that

$$\hat{\omega}(x_1, x_2) \ = \ \frac{\omega^1(x_1)\omega^2(x_2)}{\omega_{\nu,S}(\phi(x_1))} \ = \ \frac{\omega^1(x_1)\omega^2(x_2)}{\omega_{\nu,S}(\phi(x_2))}, \tag{2.10}$$

where $\omega^1 = \omega^2 = \omega_{\mu,T}$. Since $\hat{\mu}$ is supported on $\{(x_1, x_2) : \phi(x_1) = \phi(x_2)\}$, then $\phi(x_1) = \phi(x_2)$ $\hat{\mu}$ a.e. If we let $c^1(x_1) = \frac{1}{\omega_{\nu,S}(\phi(x_1))}$ and $c^2(x_2) = \frac{1}{\omega_{\nu,S}(\phi(x_2))}$ then formula (2.8) is satisfied and so the relative product is a rational joining.

The following lemma, implicit in [RS89] (Proposition 4.3.2), plays an important role for our arguments as it is used to bound the Radon-Nikodym derivatives of the relative products. Though we use it only for the relative product, it is stated in its natural context of rational joinings.

Lemma 2.1. ([RS89]) *Suppose that $\hat{\mu}$ is a rational joining of $\mathbf{X}_1$ and $\mathbf{X}_2$, $i \in \mathbb{Z}$, and there exists a constant $C \in (0,1)$ such that*

$$C < \omega_i^j < C^{-1} \qquad \mu_j\text{-a.e.} \quad j = 1,2 \ .$$

Then

$$C^3 < \hat{\omega}_i < C^{-3} \qquad \hat{\mu}\text{-a.e.}$$

Proof. Apply the cocycle relation (2.2) to (2.8) to obtain

$$\hat{\omega}_i(x_1, x_2) \ = \ \omega_i^1(x_1)\omega_i^2(x_2)c_i^1(x_1) \quad \hat{\mu}\text{–a.e.}, \tag{2.11}$$

where $c_i^1(x) = c^1(x)\cdots c^1(T_1^{i-1}(x))$, $i \geq 1$. Let $\hat{\mu} = \int \hat{\mu}_{x_1} d\mu_1(x_1)$ be the disintegration of $\hat{\mu}$ over X_1. Integrating (2.11) with respect to $\hat{\mu}_{x_1}$ we get for μ_1-a.a. x_1

$$\omega_i^1(x_1) \ = \ \omega_i^1(x_1)c_i(x_1)\int \omega_i^2(x_2)d\hat{\mu}_{x_1}(x_2)$$

so

$$C < c_i(x_1) \ = \ \left(\int_{X_2} \omega_i^2(x_2)d\hat{\mu}_{x_1}(x_2)\right)^{-1} < C^{-1} \ ,$$

since for μ_1-a.a. x_1, $C < \omega_i^2(x_2) < C^{-1}$ for $\hat{\mu}_{x_1}$-a.a. x_2. Substitute this inequality back into (2.11) to complete the proof. □

2.3 Non-singular Codes

We now give a self-contained presentation of the main properties of the non-singular $\bar{d}$-distance introduced in [JS95]. Denote intervals in $\mathbb{Z}$ by $[m,n] = \{m, m+1, \ldots, n\}$. Let $\mathbf{Z} = (Z, \mathfrak{D}, \lambda, R)$ be any non-singular system and $z \in Z$. Let $\Omega_{\lambda,z}$ denote the measure on $\mathbb{Z}$ defined by $\Omega_{\lambda,z}(i) = \omega_{\lambda,R^i}(z)$. We sometimes write Ω_z instead of $\Omega_{\lambda,z}$ when λ is clear from the context. Given sequences $\xi, \eta \in A^{\mathbb{Z}}$, A finite, let

$$D(\xi,\eta) = \{i : \xi(i) \neq \eta(i)\}.$$

For $I \subset \mathbb{Z}$, I finite, define

$$\bar{d}_z^I(\xi,\eta) \ = \ \Omega_z\big(D(\xi,\eta)\cap I\big)/\Omega_z(I)$$

and then let

$$\bar{d}_z(\xi,\eta) \;=\; \lim_{m,n\to\infty} \bar{d}_z^{[-m,n]}(\xi,\eta) \ ,$$

if the limit exists; equivalently

$$\bar{d}_z(\xi,\eta) \;=\; \lim_{m\to\infty} \bar{d}_z^{[-m,0]}(\xi,\eta) \;=\; \lim_{n\to\infty} \bar{d}_z^{[0,n]}(\xi,\eta)$$

if the two limits exist and agree. When we want to emphasize the measure λ we will write $\bar{d}_z = \bar{d}_{z,\lambda}$. It is clear that $\bar{d}_z$ satisfies the triangle inequality.

For any interval I let $\sigma(I)$ denote the smallest symmetric interval containing I.

Lemma 2.2. *Suppose $\bar{d}_z(\xi,\eta)$ exists. Let $I_1, I_2, \ldots$ be a sequence of intervals in $\mathbb{Z}$ with $\bigcup_{n=1}^{\infty} \sigma(I_n) = \mathbb{Z}$ and such that there is a constant C' with*

$$\Omega_z\big(\sigma(I_n)\big) \le C'\Omega_z(I_n) \quad \textit{for all integers } n.$$

If $\lim_{n\to\infty} \bar{d}_z^{I_n}(\xi,\eta) = 0$ then $\bar{d}_z(\xi,\eta) = 0$.

Proof. This is just a fact about measures on $\mathbb{Z}$. The hypothesis implies that if $\bar{d}$ is small on I_n then it is small on $\sigma(I_n)$, and so it is 0 on $\mathbb{Z}$ since its value on $\mathbb{Z}$ is the limit of its values on $\sigma(I_n)$. □

$\mathbf{Y} = (Y, \mathcal{C}, \nu,\, S)$ is called a *(non-singular) symbolic system*, with alphabet A, if $Y = A^{\mathbb{Z}}$, A is finite, $\mathcal{C}$ is the Borel σ-algebra on Y, S is the left shift on Y and ν is any measure non-singular for the shift map. If $x \in A^{\mathbb{Z}}$ (i.e. $x : \mathbb{Z} \to A$) and $I = [m,n]$ we denote the restriction $x\,|_I$ by $x[m,n] \in A^I$. When it is clear from the context we will sometimes identify A^I with A^{m-n+1}, i.e., we think of $x[m,n]$ as a word of length $m-n+1$ with no particular indexing. By a *code* $\psi : A^{\mathbb{Z}} \to B^{\mathbb{Z}}$, A and B finite, we mean any measurable shift-commuting map. A *finite code* $\psi : A^{\mathbb{Z}} \to B^{\mathbb{Z}}$ is a map $\psi = \psi_f$, determined by the choice of a funtion $f : A^{2k+1} \to B$, for some k, via the formula

$$\psi_f(x)(i) \;=\; f\big(x[i-k, i+k]\big) \ .$$

Such a ψ automatically commutes with the shifts. $2k+1$ is called the code length of ψ, and denoted $|\psi|$. If ψ and ψ' are any two codes from $A^{\mathbb{Z}}$ to $B^{\mathbb{Z}}$ we define

$$D[\psi,\psi'] = \{x \in A^{\mathbb{Z}} | \psi(x)(0) \ne \psi'(x)(0)\}.$$

(Note that $D(\xi,\eta)$ is used for the set of indexes where two sequences differ, while $D[\psi,\psi']$ is the set of points in $A^{\mathbb{Z}}$ where the codes differ at the 0^{th} place.)

Lemma 2.3. *Suppose $(A^{\mathbb{Z}},\mu)$ is a non-singular symbolic system and $\phi, \phi' : A^{\mathbb{Z}} \to B^{\mathbb{Z}}$ are any two codes. Then for almost all $x \in X$ $\bar{d}_x(\phi(x), \phi'(x))$ exists.*

Proof. Apply the Hurewicz ergodic theorem to the set $D[\phi, \phi']$ (in way a similar to the proof of Proposition 2.4). $\square$

A homomorphism ϕ between non-singular symbolic systems $(A^{\mathbb{Z}}, \mu)$ and $(B^{\mathbb{Z}}, \nu)$ can be approximated by a finite code by approximating the sets $\{x : \phi(x)(0) = b\}$, $b \in B$, by cylinders of length $2k+1$ in $A^{\mathbb{Z}}$ to obtain $f : A^{2k+1} \to B$ and hence a finite code $\phi' = \psi_f$. More precisely, we require $D[\phi, \phi']$ to have small μ-measure. (It is important to be aware that ϕ' will not be a homomorphism from $(A^{\mathbb{Z}}, \mu)$ to $(B^{\mathbb{Z}}, \nu)$ since it will not be non-singular.) An application of the ergodic theorem then leads to the following d-bar approximation.

Proposition 2.4. ([JS95]) *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be non-singular symbolic systems with alphabets* A *and* B*,* $\phi : \mathbf{X} \to \mathbf{Y}$ *a homomorphism and* ϕ' *a finite code such that* $\mu(D[\phi, \phi']) < \epsilon^2$*,* $0 < \epsilon < 1$*. Suppose* $\mathbf{X}'$ *is a third non-singular symbolic system and* λ *is any conservative joining of* $\mathbf{X}$ *and* $\mathbf{X}'$*. Then*

$$\lambda\{z = (x, x') \in X \times X' | \bar{d}_z(\phi(x), \phi'(x)) < \epsilon\} > 1 - \epsilon.$$

Here $\bar{d}_z = \bar{d}_{z,\lambda}$.

Proof. Let $E_\lambda(h|\mathcal{I})$ denote the conditional expectation of h with respect to the algebra of invariant sets $\mathcal{I}$ in $\mathbf{X} \times \mathbf{X}'$. Using the fact that ϕ and ϕ' commute with the shifts one sees that for $x \in X$

$$\{i : T^i x \in D[\phi, \phi']\} \;=\; D\big(\phi(x), \phi'(x')\big) \;. \tag{2.12}$$

Let $D = D[\phi, \phi']$. Apply the Hurewicz ergodic theorem to $T \times T', \lambda$ to get

$$E_\lambda(1_{D\times X'}|\mathcal{I})(x, x') \;=\; \lim_{n\to\infty} \frac{\sum_{i=0}^{n} \hat{\omega}_i(x, x') 1_{D\times X'}(T^i x, {T'}^i x')}{\sum_{i=0}^{n} \hat{\omega}_i(x, x')} \tag{2.13}$$

$$= \lim_{n\to\infty} \frac{\Omega_{(x,x')}\big(\{i : T^i x \in D\} \cap [0, n]\big)}{\Omega_{(x,x')}[0, n]} \tag{2.14}$$

$$= \lim_{n\to\infty} \bar{d}^{[0,n]}_{(x,x')}(\phi(x), \phi'(x)) \;. \tag{2.15}$$

Since the invariant σ-algebra of $T^{-1} \times {T'}^{-1}$ is the same as that of $T \times T'$ we also get

$$E_\lambda(1_{D\times X'}|\mathcal{I})(x, x') \;=\; \lim_{n\to\infty} \bar{d}^{[-n,0]}_{(x,x')}(\phi(x), \phi'(x)) \;.$$

The integral of the expectation is the same as the integral of $1_{D\times X'}$ which is $\mu(D[\phi, \phi'])$, since λ is a joining. By Chebyshev's inequality,

$$\lambda\{z|\bar{d}_z(\phi(x), \phi'(x)) \geq \epsilon\} \leq \frac{1}{\epsilon} \int d_z(\phi(x), \phi'(x)) d\lambda = \frac{1}{\epsilon}\mu(D[\phi, \phi']) < \epsilon,$$

which completes the proof. ◻

The following proposition is standard in the measure-preserving case, but we need it in the more general form below.

Proposition 2.5. ([JS95]) *Let* $\mathbf{X} = (X, \mathcal{B}, \mu, T)$ *be a (not necessarily ergodic) non-singular system and let* $\mathcal{F}$ *be a factor algebra which is ergodic, i.e.,* $TA = A$, $A \in \mathcal{F} \Rightarrow \mu(A) = 0$ *or 1. Let* $\mathcal{I}$ *denote the* σ*-algebra of* T*-invariant sets. Suppose that* h *is a* $\mathcal{F}$*-measurable function such that* $h \geq 0$ *and* $\mu\big\{E(h \mid \mathcal{I}) = 0\big\} > 0$*. Then* $h = 0$ *a.e.*

Proof. We use the following property of the ergodic decomposition of μ: there is a homomorphism $\phi : \mathbf{X} \to \mathbf{Y} = (Y, \mathcal{C}, \nu, id)$ such that $\phi^{-1}(\mathcal{C}) = \mathcal{I}$ and if $\mu = \int \mu_y d\nu(y)$ is the disintegration of μ over Y then for ν-a.e. y μ_y is non-singular and ergodic for T.

Let $E = \{y : \int_X h d\mu_y = 0\}$. Our hypothesis is that $\nu(E) > 0$, and we need to show that $\nu(E) = 1$. Define the measure

$$\bar{\mu} \;=\; \int_E \mu_y d\nu(y)$$

and let $\mu_{\mathcal{F}}$ and $\bar{\mu}_{\mathcal{F}}$ denote the restrictions to $\mathcal{F}$. Then $\bar{\mu}_{\mathcal{F}}$ is absolutely continuous with respect to $\mu_{\mathcal{F}}$, and by ergodicity of $\mu_{\mathcal{F}}$ it follows that $\bar{\mu}_{\mathcal{F}}$ is equivalent to $\mu_{\mathcal{F}}$. The definition of $\bar{\mu}$ implies that $h = 0$ $\bar{\mu}_{\mathcal{F}}$-a.e., so $h = 0$ $\mu_{\mathcal{F}}$-a.e., which completes the proof as h is $\mathcal{F}$-measurable. ◻

3 Non-singular Chacon maps

3.1 Geometric Construction

In this section we recall the definition of non-singular Chacon maps [JS95]. We first describe them geometrically as rank one cutting and stacking constructions, and then use a natural partition to code them to symbolic systems with alphabet $\{0, 1\}$. We fix a $\lambda \in (0, 1]$. When $\lambda = 1$ the construction yields the classical finite measure-preserving Chacon map. When $0 < \lambda < 1$ it yields a type III_λ non-singular map. We describe later how to obtain a type III_1 map.

To start the construction let

$$\alpha = \frac{1+\lambda}{1+2\lambda},$$

and let $I_0 = [0, \alpha)$. Partition I_0 into the contiguous intervals J_1^0, J_2^0 and J_4^0 of lengths $\frac{\lambda}{1+2\lambda}\alpha$, $\frac{1}{1+2\lambda}\alpha$, and $\frac{\lambda}{1+2\lambda}\alpha$; i.e., their lengths are in the proportion $\lambda, 1, \lambda$. Then let J_3^0 be an interval of length $\frac{1}{1+2\lambda}\alpha$ abutting on I_0. Define $T_\lambda : J_i^0 \to J_{i+1}^0$ for $i = 1, 2, 3$ to be the affine map that takes J_i^0 to J_{i+1}^0, and leave T_λ undefined on J_4. This produces a *column* or *tower* ξ_1 for T_λ of height 4 whose *levels* are intervals; T_λ has constant Radon-Nikodym derivatives on each level, except the top one, where T_λ is not yet defined. The union of the levels is a new interval I_1.

Assume by induction that at stage n of the construction we have a column ξ_n of height h_n whose levels are intervals with union I_n, also an interval, and T_λ is an affine map on each level to the one above, and as yet is undefined on the top level. We extend the definition of T_λ as follows. Partition the *base* (i.e., lowest level) $B(\xi_n)$ of column ξ_n into intervals J_1^n, J_2^n, J_4^n of lengths proportional to $\lambda, 1, \lambda$, and let J_3^n be an interval of the same length as J_2^n abutting on I_n. Extend T_λ affinely so that

$$\begin{aligned} T_\lambda &: T_\lambda^{h_n-1} J_1^n \to J_2^n \ , \\ T_\lambda &: T_\lambda^{h_n-1} J_2^n \to J_3^n \ , \\ T_\lambda &: \qquad\quad J_3^n \to J_4^n \ , \end{aligned}$$

to obtain a column ξ_{n+1} of height $h_{n+1} = 3h_n + 1$. We note that $\omega_{\mu,T_\lambda} = \lambda^{-1}$ on $T_\lambda^{h_n-1} J_1^n$, $\omega_{\mu,T_\lambda} = 1$ on $T_\lambda^{h_n-1} J_2^n$ and $\omega_{\mu,T_\lambda} = \lambda$ on $J_3^n = T_\lambda^{h_n} J_2^n$.

We have constructed a sequence of columns ξ_n on intervals I_n, and one can verify that $I_n \uparrow [0, 1)$, T_λ is defined a.e. on $X = [0, 1)$ and is a non-singular automorphism with respect to Lebesgue measure μ on X, with $\mathcal{B}$ the Borel σ-algebra of X.

To obtain a type III_1 example choose $0 < \lambda_1 < \lambda_2 < 1$ such that $\log(\lambda_1)/\log(\lambda_2)$ is irrational and in the construction of the columns ξ_n, for n even divide $B(\xi_n)$ in the ratio a, b, a with $a/b = \lambda_1$, and for n odd in the ratio a, b, a with $a/b = \lambda_2$; denote this transformation by T_{λ_1,λ_2}.

The construction of type II_∞ and type III_0 nonsingular Chacon transformations is more complex as it needs the choice of λ to vary with n in a controlled way. This has been done recently in [HS00] (written after the first version of this paper); however the only property proved for these maps in [HS00] is ergodicity of their 2-fold Cartesian product. In [HS00], at stage n in the construction, the levels of ξ_n are divided in the rations a, b, a with $a/b = \lambda_n$, and in the type III_0 and type II_∞ cases the choice of the λ_n is such the the bounds in Proposition 3.1 do not hold.

As explained later (cf. Remark 1) the bounds of Corollary 3.2, which follow from Proposition 3.1, are crucial to our arguments.

In this section we obtain some estimates about the Radon-Nikodym derivatives of joinings of products of non-singular Chacon maps. These estimates are possible along the column heights, and are based on the following proposition whose proof is contained in the proof of [JS95], Proposition 3.1. We include the proof for completeness.

Proposition 3.1. ([JS95]) *Let* $\mathbf{X} = (X, \mathcal{B}, \mu, T_\lambda)$ *be a non-singular Chacon system. Then*

1. $\lambda \leq \omega_{\mu,T_\lambda} \leq \lambda^{-1}\ \mu$ *a.e.*
2. $\forall n$, $\lambda^2 \leq \omega_{\mu,T_\lambda^{h_n}} \leq \lambda^{-2}\ \mu$ *a.e.*

Proof. Let $\omega = \omega_{\mu,T_\lambda}$. Clearly, $\omega(x) \in \{\lambda^{-1}, 1, \lambda\}$ a.e. Note that if ω_{h_n} is constant on an interval I, then for all $x \in I$, $\omega_{h_n}(x) = \frac{\mu(T^{h_n}(I))}{\mu(I)}$. Let J be any level of column ξ_n and let $J_{(1)}, J_{(2)}, J_{(3)}$ denote the three subintervals that J is divided into when constructing the levels of ξ_{n+1}. Since ω is constant on levels of ξ_n, by the cocycle relation, ω_{h_n} is constant on $J_{(1)}$ and on $J_{(2)}$. Also, it is constant on the first and second thirds of $J_{(3)}$, and so on. It follows that if $x \in \xi_n$, then $\omega_{h_n}(x) = \lambda^{-1}$ or $\omega_{h_n+1}(x) = \lambda$ (according to whether or not $T^i(x)$ is in ξ_n for $i = 0, \ldots, h_n$). If x is not in any level of ξ_n then $\omega_{h_n+1}(x) = \lambda$ or $\omega_{h_n+1}(x) = 1$. (The last possibility occurs only when x lies in J_3^n [the interval added at the n^{th} stage of the construction] and $T^{h_n+1}x \in B(\xi_n)$.) The cocycle relation implies that $\lambda^{+2} \leq \omega_{h_n}(x) \leq \lambda^{-2}$. □

The following is now a consequence of rationality and Proposition 3.1.

Corollary 3.2. *Let* $\mathbf{X} = (X, \mathcal{B}, \mu_{\lambda_1}, T)$ *and* $\mathbf{X} = (X, \mathcal{B}, \mu_{\lambda_2}, T)$ *be two non-singular Chacon systems. Let* $m_{1,2} = \mu_{\lambda_1} \times \mu_{\lambda_2}$ *and* $\hat{m}$ *be any rational joining of* $(T \times T, m_{1,2})$ *and* $(T \times T, m_{1,2})$*. Let* T_4 *denote the transformation* $(T \times T) \times (T \times T)$ *with measure* $m_{1,2} \times m_{1,2}$*. Let* $z \in (X \times X) \times (X \times X)$ *and write* Ω_z *for the measure* $\Omega_{\hat{m},z}$ *on* $\mathbb{Z}$ *as defined earlier. Let* Ω_x *denote the measure* $\Omega_{\lambda_1,x}$*. Then*

1. $\forall n$, $(\lambda_1\lambda_2)^6 \leq \omega_{\hat{m},T_4^{h_n}} \leq (\lambda_1\lambda_2)^{-6}\ \hat{m}$ *a.e., and so* $(T_4, \hat{m})$ *is conservative.*
2. $\forall k$, $(\lambda_1\lambda_2)^{3k} \leq \omega_{\hat{m},T_4^k} \leq (\lambda_1\lambda_2)^{-3k}\ \hat{m}$ *a.e.*
3. $\forall C > 0\ \exists C' > 0$ *such that for* $\hat{m}$*-a.a.* z *whenever* $I_1 \subset I_2$ *are intervals in* $\mathbb{Z}$ *with* $|I_2| < C|I_1|$ *then* $\Omega_z(I_2) < C'\Omega_z(I_1)$*. The analogous statement also holds for the measure* $\Omega_{\lambda_1,x}$*.*

4. *$\forall k \in \mathbb{Z}^+$, $\varepsilon > 0$ $\exists L > 0$ such that for $\hat{m}$-a.a. z whenever $I_1 \subset I_2$ are intervals in $\mathbb{Z}$ with $|I_1| < k$ and $|I_2| > L$ then $\Omega_z(I_1) < \varepsilon\Omega_z(I_2)$. The analogous statement also holds for the measure $\Omega_{\lambda_1,x}$.*

5. *$\forall n > 0$ and $i \in \mathbb{Z}$, $(\lambda_1\lambda_2)^6 < \Omega_z\{i\}/\Omega_z\{i + h_n\} < (\lambda_1\lambda_2)^{-6}$. The analogous statement also holds for the measure $\Omega_{\lambda_1,x}$.*

Proof. We first show part 1. Let T_2 denote $T \times T$ with measure $m_{1,2}$. Since

$$\omega_{m_{1,2},T_2^{h_n}}(x_1, x_2) = \omega_{\mu_{\lambda_1},T^{h_n}}(x_1)\ \omega_{\mu_{\lambda_2},T^{h_n}}(x_2),$$

using Proposition 3.1,

$$(\lambda_1\lambda_2)^2 \leq \omega_{m_{1,2},T_2^{h_n}} \leq (\lambda_1\lambda_2)^{-2}.$$

Now Lemma 2.1 completes the proof. Part 2 follows in a similar way (using the cocycle relation instead of Proposition 3.1). For part 3, since for all n, $h_{n+1}/h_n \leq 4$ there is an n such that $h_n \leq |I_1| \leq 4h_n$. Let $J \subset I_1$ be any interval of length h_n. Then I_2 can be covered by at most $4C + 1$ intervals $J + ih_n$ with $|i| < 4C + 1$. By part 1 we get

$$\Omega_{(x_1,x_2)}(I_2) \leq C'\Omega_{(x_1,x_2)}(J) \leq C'\Omega_{(x_1,x_2)}(I_1)$$

with $C' = (4C + 1)(\lambda_1\lambda_2)^{-6(4C+1)}$. The proof of the second part for the measure $\Omega_{\lambda_1,x}$ is similar.

To prove part 4 we may assume $k = 2$ (i.e., $|I_1| = 1$). Since $h_n \leq 4^n$, using part 1 we obtain

$$\sum_{i=0}^{4^n} \omega_{\hat{m},T_4^i} > n(\lambda_1\lambda_2)^6 \quad \text{and} \quad \sum_{i=-4^n}^{0} \omega_{\hat{m},T_4^i} > n(\lambda_1\lambda_2)^6 \quad \hat{\mu}\text{-a.e.}$$

This means that $\Omega_z\{j\}/\Omega_z[j, j + 4^n] < \frac{1}{n}(\lambda_1\lambda_2)^{-6}$ and $\Omega_{(x_1,x_2)}\{j\}/\Omega_z[j - 4^n, j] < \frac{1}{n}(\lambda_1\lambda_2)^{-6}$, so 4 follows. Part 5 is just a restatement of 1. □

Proposition 3.3. *For each $0 < \lambda < 1$, the map T_λ is a non-singular conservative, ergodic type III$_\lambda$ automorphism. For $\lambda = 1$, T_1 is the classical finite measure-preserving Chacon transformation. If $\log(\lambda_1)/log(\lambda_2)$ is irrational then T_{λ_1,λ_2} is a conservative, ergodic type III$_1$ automorphism.*

Proof. Let $T = T_\lambda$. As before, let $J_{(1)}, J_{(2)}, J_{(3)}$ denote the three subintervals that a level J in column ξ_n is divided into when constructing the levels of column ξ_{n+1}.

As the levels in $\{\xi_n\}_{n=0}^{\infty}$ generate, for any sets A, B of positive measure there exist levels J and K in some column ξ_n so that for each $k = 1, 2, 3$,

$$\mu(J_{(k)} \cap A) > \frac{1}{2}\mu(J_{(k)}) \text{ and } \mu(K_{(k)} \cap B) > \frac{1}{2}\mu(K_{(k)}).$$

It is clear then that there is an integer $\ell > 0$ such that $\mu(T^{\ell}(A) \cap B) > 0$, so T is conservative ergodic. Also, $T^{h_n}(J_{(1)}) = J_{(2)}$ and $\omega_{T^{h_n}} = 1/\lambda$ on $J_{(1)}$. Therefore, for every $\varepsilon > 0$, $\mu(A \cap T^{-h_n}(A) \cap \{x : \omega_{T^{h_n}}(x) \in N(1/\lambda, \varepsilon)\}) > 0$ and so $1/\lambda \in r(T)$. Since ω_{T^i} takes only values that are powers of λ it follows that T is type III_λ. In the case of T_{λ_1,λ_2} a similar argument shows that $\lambda_1, \lambda_2 \in r(T_{\lambda_1,\lambda_2})$, and since $r(T_{\lambda_1,\lambda_2}) \setminus \{0\}$ is a closed multiplicative subgroup the hypothesis implies that the ratio set is $[0, \infty)$. □

We now state two theorems about nonsingular Chacon maps that we use. Theorem 1 was proved in [JS95], however section 5 contains all the ideas needed for its proof.

Theorem 1. ([JS95]) *Let* $\mathbf{X} = (X, \mathcal{B}, \mu, T_\lambda)$ *a non-singular Chacon system. Let* $\mathcal{F}$ *be a factor algebra of* T. *Then* T *is prime, i.e.,* $\mathcal{F}$ *is equal* $\bmod \mu$ *to the full* σ*-algebra* $\mathcal{B}$ *or the trivial* σ*-algebra* $\mathcal{N}$.

Using the techniques of this paper the authors have shown that $T \times T$ is ergodic (unpublished). We will use the stronger property below (Theorem 2) that was shown in [AFS01] by different methods.

Theorem 2. ([AFS01]) *Let* $0 < \lambda_1 \leq \ldots \leq \lambda_k \leq 1$ *and* $n_1, \ldots, n_k$ *be nonzero integers. Then* $T_{\lambda_1}^{n_1} \times \ldots \times T_{\lambda_k}^{n_k}$ *is ergodic.*

3.2 Symbolic Chacon Maps

We now describe the symbolic version of T_λ. Let P be the two-set partition which, viewed as function into $\{0, 1\}$, is defined by

$$P(x) = \begin{cases} 0, & \text{if } x \in I_0, \\ 1, & \text{if } x \in X - I_0. \end{cases}$$

For $x \in X$ let $\theta(x) \in \{0, 1\}^{\mathbb{Z}}$ denote the " P, T_λ-name" of x:

$$\theta(x)(i) \ = \ P(T_\lambda^i x) \ .$$

Define blocks of 0's and 1's inductively as follows:

$$B_0 = 0, \quad B_1 = 0010, \quad B_{n+1} = B_n B_n 1 B_n \ .$$

Clearly B_n is the partial P, T_λ-name $\theta(x)[0, h_n - 1]$ of any point in the base of ξ_n.

One can show that the cutting and stacking construction we have described is isomorphic via θ to a symbolic system with alphabet $\{0, 1\}$ which we shall denote by $\mathbf{X} = (X, \mathcal{B}, \mu_\lambda, T)$; here T is always the shift, $\mu_\lambda = \mu \circ \theta^{-1}$, and for $\lambda_1 \neq \lambda_2$ the measures μ_{λ_1} and μ_{λ_2} are mutually singular. We end with the following proposition, whose proof can be found in [JS95], about the symbolic structure of the Chancon maps.

Proposition 3.4. (cf. [JS95]) *For each $\lambda \in (0, 1]$ there is a Borel subset $X_\lambda \subset X$ such that $\mu(X_\lambda) = 1$ with the following properties:*

(a) For each n, $x \in X_\lambda$ decomposes uniquely as a concatenation of n-blocks B_n, some of which are separated by a single 1. Let us call such 1's in x (n-block) spacers. Any appearance of B_n in x must be one of the B_n's in the unique decomposition.

(b) For all $x \in X_\lambda$ and for all sufficiently large $n = n(x)$ the 0^{th} co-ordinate in x lies inside an n-block, which we call the time 0 n-block. The union over n of the intervals on which the time 0 n-blocks occur is $\mathbb{Z}$.

(c) If $x, y \in X_\lambda$ and $x \notin \mathcal{O}(y) = \{T^n y : n \in \mathbb{Z}\}$, then for infinitely many n there are intervals I and J such that

(i) $x(I) = B_n = y(J)$

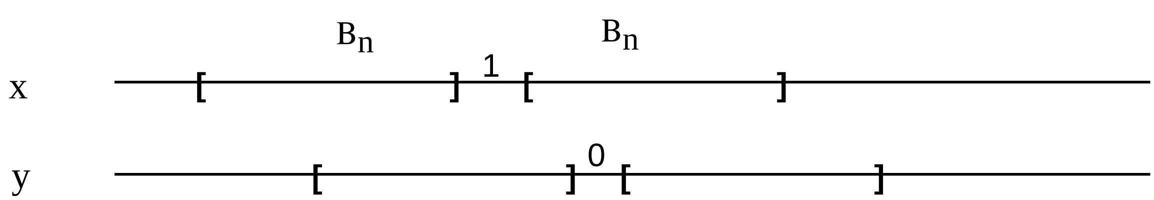
(ii) $|I \cap J| \geq \frac{1}{10} h_n = \frac{1}{10}|I| = \frac{1}{10}|J|$

(iii) $I \cup J \subset [-10h_n, 10h_n]$

(iv) $x(I)$ is followed by a spacer but $y(J)$ is not.

Roughly speaking, there are n-blocks in x and y, not too far from time 0 and overlapping substantially, one followed by a spacer and the other not. Let us call such a situation "broken n-blocks". Refer to Figure 1. By abuse of notation, in figures such as Figure 1, we use 0 between n-blocks to denote the absence of a spacer; of course 1 denotes the spacer.

Figure 1: Broken n-blocks



Remark 1. *We can think of* $(X, \mathcal{B}, \mu_\lambda, T)$ *as the topological Chacon map* T *with a nonsingular, non-atomic, ergodic Borel measure* μ_λ*;* μ_λ *is obtained from the isomorphism* θ *with the geometric construction* $(X, \mathcal{B}, \mu, T_\lambda)$*, where* μ *is Lebesgue measure on the interval. For our arguments in sections* 4 *and* 5 *we will use the symbolic version but properties of the measure that come from the geometric construction. However, on close inspection of the proofs we see that the only properties that we use, in addition to the name structure of the topological map, are the bounds on the Radon-Nikodym derivatives in Proposition 3.1; once they are established Corollary 3.2 follows. Thus we are able to prove our results for any non-singular, non-atomic ergodic Borel probability measure for which the Radon-Nikodym derivatives obey bounds such as in Proposition 3.1. In particular, while we discuss our proofs in terms of type* III_λ *measures,* $0 < \lambda < 1$*, they also apply to the type* III_1 *transformation* T_{λ_1,λ_2}*. (Theorem 2, as remarked in [AFS01], also holds for* III_1 *transformations.)*

4 The Centralizer of Products

Our main theorem in this section is the classification of the centralizer of Cartesian products of non-singular Chacon maps (Theorem 3).

Remark 2. *In this and the following sections* (T, μ_λ) *is always the symbolic representation of the type* III_λ *Chacon map, and* T *is the shift. To simplify notation sometimes we write* T_λ *instead of* (T, μ_λ)*, such as in the statement of Theorem 3. We let* $T_\lambda^{\otimes n}$ *denote the* n*-fold Cartesian product of* T_λ *and let* X^n *denote the Cartesian product of* n *copies of* X*.*

The following theorem classifies the centralizer of Cartesian products and is the main result of this section.

Theorem 3. *Let* $0 < \lambda_1 < \ldots < \lambda_k \leq 1$ *and* $n_1, \ldots, n_k$ *be integers. Then* $C(T_{\lambda_1}^{\otimes n_1} \times \ldots \times T_{\lambda_k}^{\otimes n_k})$ *is generated by maps of the form* $U_1 \times \ldots \times U_k$*, where each* U_i *acting on* X^{n_i} *is a Cartesian product of powers of* T_{λ_i}*, or a co-ordinate permutation on* X^{n_i}*.*

We start with a special case that will form the base case of the inductive proof of Theorem 3.

Theorem 4. *Any homomorphism from* (T, μ_λ) *to itself is a power of* T*. If* $\lambda_1 \neq \lambda_2$ *then there is no homomorphism from* (T, μ_{λ_1}) *to* (T, μ_{λ_2})*.*

Proof. Suppose ϕ is a homomorphism from T to itself. The proof that ϕ is a power of T is essentially the same as the by now classical argument [J78] that the centralizer of the measure-preserving Chacon map is trivial, with some technical modifications to deal with the weighted version of the $\bar{d}$-distance. We will present the proof informally.

Find a finite code ϕ' of length k such that $D[\phi, \phi']$ has measure less than ϵ. Since T is ergodic we have for a.a. x

$$\bar{d}_x(\phi(x), \phi'(x)) = \mu(D[\phi, \phi']) < \epsilon. \tag{4.1}$$

Moreover a.a. x will satisfy

$$\bar{d}_x(\phi(x), T^n(x)) = \mu(D[\phi, T^n]), \tag{4.2}$$

$$\bar{d}_x(\phi(x), \phi T(x)) = \mu(D[\phi, \phi \circ T]) \tag{4.3}$$

and

$$\text{both } x \text{ and } \phi(x) \text{ lie in the set } X_\lambda \text{ of Proposition 3.4.} \tag{4.4}$$

Fix x satisfying (4.1),(4.2), (4.3), (4.4) and suppose that $y = \phi(x)$ is not in the orbit of x. Now choose n such that h_n is much larger than $k = |\phi|$ and we see broken n-blocks in (x, y). So, there are n-blocks in x and y occurring on intervals $I, J \subset [-10h_n, 10h_n]$ with $|I \cap J| > h_n/10$ and these n-blocks are followed by spacers s_1, s_2 with $s_1 \neq s_2$. (As explained earlier, "followed by a spacer s" means followed or not followed by a spacer according as s is 1 or 0.) For concreteness we will suppose $s_1 = 1$ and $s_2 = 0$.

Our goal now is to see that

$$\bar{d}_x^{I\cap J}(y(I \cap J), y((I \cap J) + h_n + 1)) \text{ is small .} \tag{4.5}$$

To see this we will be comparing sequences defined on $I \cap J$, $I \cap J + h_n$ and $I \cap J + h_n + 1$. This will be justified by Corollary 3.2 (5) which guarantees that each of the metrics $\bar{d}_x^{I\cap J}, d_x^{(I\cap J)+h_n}, d_x^{(I\cap J)+h_n+1}$ is bounded by a constant times any other, independent of n. Thus, if α and β are any two sequences of length $|I \cap J|$ we will simply write $\alpha \sim \beta$ to mean that the sequences are close with respect to one, and hence, all of these metrics.

First observe that since $s_1 = 1$ we have $x(I \cap J) = x((I \cap J) + h_n + 1)$ so

$$\phi'(x)(I \cap J) \sim \phi'(x)((I \cap J) + h_n + 1), \tag{4.6}$$

since these two sequences will agree except on the leftmost and rightmost k places. When h_n is sufficiently large these places contribute only a small amount to $\bar{d}_x$ by Corollary 3.2 (4). Next, if h_n is sufficiently large $\phi'(x)[-10h_n, 10h_n] \sim \phi(x)[-10h_n, 10h_n]$ by (4.2) and since $|I \cap J| > h_n/10$ it follows from Corollary 3.2 (3) that

$$\phi'(x)(I \cap J) \sim \phi(x)(I \cap J). \tag{4.7}$$

Similarly

$$\phi'(x((I \cap J) + h_n + 1) \sim \phi(x((I \cap J) + h_n + 1). \tag{4.8}$$

Combining (4.6), (4.7) and (4.8) gives (4.5).

Now observe that $Ty(I \cap J) = y((I \cap J) + h_n + 1)$, since $s_2 = 0$. Thus (4.5) becomes

$$\bar{d}_x^{I \cap J}(y(I \cap J), Ty((I \cap J))) \text{ is small .}$$

Of course "is small" here means "$< \epsilon_1$" where ϵ_1 goes to zero with ϵ. $I = I_n$ and $J = J_n$ depend on n and $n \to \infty$ as $\epsilon \to 0$. Since $|I_n \cap J_n| > h_n/10$, applying (4.3), Lemma 2.2 and Corollary 3.2 (3) we can conclude that

$$\bar{d}_x(\phi(x), T(\phi(x)) = 0 = \mu(D[\phi, \phi \circ T]).$$

From this it follows that $\phi = \phi \circ T$, so, by ergodicity of T, ϕ is constant a.e., a contradiction. This contradiction means that in fact y is in the orbit of x, that is $\phi(x) = T^n(x)$ for some n. By (4.2) it follows that $\phi = T^n$ a.e.

Finally suppose that $\lambda_1 \neq \lambda_2$ and ϕ is a homomorphism from (T, μ_{λ_1}) to (T, μ_{λ_2}). Since the measures μ_{λ_1} and μ_{λ_2} are mutually singular they give full measure to disjoint T-invariant sets. This means that we may assume as above that $y = \phi(x)$ is not in the orbit of x and arrive at a contradiction as above. □

The following lemma forms the basis of the inductive step that we shall need.

Lemma 4.1. *If ϕ is any homomorphism from $T_{\lambda_1}^{\otimes n_1} \times \ldots \times T_{\lambda_k}^{\otimes n_k}$ to T_λ with $n = n_1 + \ldots + n_k > 1$ then ϕ is a function of some strict subset of the co-ordinates of $x \in X^{n_1} \times \ldots \times X^{n_k}$.*

Proof. Let μ^n denote the product measure on X^n, let δ denote the n-tuple $(-1, 0, \ldots, 0)$ and let $T^\delta = T^{-1} \times T^0 \times \ldots T^0$. Let ϕ' be a finite code of length k with $\mu^n(D[\phi, \phi']) < \epsilon$. μ^n-a.a. $x \in X^n$ will satisfy

$$\bar{d}_x(\phi x, \phi' x) = \mu^n(D[\phi, \phi']) < \epsilon, \tag{4.9}$$

$$\bar{d}_x(\phi T^\delta x, \phi x) = \mu^n D[\phi \circ T^\delta, \phi] \quad \text{and} \tag{4.10}$$

$$\bar{d}_x(\phi T^\delta x, T\phi x) = \mu^n D[\phi \circ T^\delta, T \circ \phi]. \tag{4.11}$$

Given $x \in X^n$ let us say $m > 0$ is favorable if the following happens: there are m-blocks in $x_1, \ldots, x_n$ occurring on intervals $I_1, \ldots I_n$ such that $I_j \subset [-10h_m, 10h_m]$, $|I_1 \cap \ldots \cap I_n| > h_m/10$ and I_1 is followed by a spacer but the remaining I_j are not followed by spacers. The Borel-Cantelli lemma ensures that for μ^n-a.a. x

$$\text{there are infinitely many favorable } m. \tag{4.12}$$

Now fix an x satisfying (4.9), (4.10), (4.11) and (4.12) and choose a favorable m with $h_m \gg |\,\phi'| = k$. Let $I_1, \ldots I_n$ be the intervals implied by the favorability of m and let $I = I_1 \cap \ldots \cap I_n$. The placement of spacers implies that $T^\delta x(I) = x(I + h_m)$ so it follows that

$$\phi'(T^\delta x)(I) = \phi'(x)(I + h_m), \tag{4.13}$$

except for possibly the left and rightmost k indices. Now the left and right hand sides of (4.13) are close to $\phi(T^\delta x)(I)$ and $\phi(x)(I + h_m)$ in the metrics $\bar{d}_x^I$ and $\bar{d}_{T^\delta x}^{I+h_m}$ respectively. As in the proof of Theorem 4 these two metrics are comparable to each other so we conclude from (4.13) that

$$\bar{d}_x(\phi(T^\delta x)(I), \phi(x)(I + h_m)) \text{ is small.} \tag{4.14}$$

So far we have said nothing about the m-blocks and spacers in $y = \phi(x)$. In fact all we need is the following simple observation: if y is any point in the set X_λ defined in Proposition 3.4 and I is any interval of length greater than $h_m/10$ then I has a subinterval of length greater than $h_m/30$ such that $y(I + h_m)(I')$ agrees with either $y(I')$ or $T^{-1}(y)(I')$. (If no spacer occurs in I we may take $I = I'$, otherwise we just take the part of I to the left or right of the spacer, whichever is largest.) Applying this to $y = \phi(x)$ we get an $I' \subset I$ such that $\phi x(I' + h_m) = T^\epsilon \phi x(I')$ where ϵ is either 0 or -1. If $\epsilon = 0$ we conclude that $\phi T^\delta x(I')$ and $\phi x(I')$ are close in $\bar{d}_x$ and then using (4.10) and arguing as in Theorem 4 we find that $\phi \circ T^\delta = \phi$ so ϕ is a function of all but the first co-ordinate of $x \in X$. If $\epsilon = -1$ we conclude that $\phi(T^\delta x)(I')$ and $T^{-1}\phi(x)(I')$ are close in $\bar{d}_x^{I'}$, whence $\phi \circ T^\delta = T^{-1} \circ \phi$ by (4.11). This means that

$$\begin{aligned}\phi = T \circ \phi \circ T^\delta &= \phi \circ (T \times \ldots \times T) \circ T^\delta \\ &= \phi \circ (Id \times T^{\otimes(n-1)}).\end{aligned}$$

Since $T^{\otimes(n-1)}$ is ergodic it follows that ϕ depends only on the first co-ordinate of x. $\square$

Now we are ready to prove the following theorem.

Theorem 5. *If ϕ is a homomorphism from $T_{\lambda_1}^{\otimes n_1} \times \ldots \times T_{\lambda_k}^{\otimes n_k}$ to T_λ then after a co-ordinate permutation we must have $\lambda = \lambda_1$ and ϕ is the projection onto the first co-ordinate followed by a power of T_λ.*

Proof. The proof is by induction using Theorem 4 and Lemma 4.1. □

Now an easy argument shows that Theorem 3 above is a consequence of Theorem 5. (In fact Theorem 5 can easily be used to characterize all homomorphisms from $T_{\lambda_1}^{\otimes n_1} \times \ldots \times T_{\lambda_k}^{\otimes n_k}$ to $T_{\lambda'_1}^{\otimes n'_1} \times \ldots \times T_{\lambda'_{k'}}^{\otimes n'_{k'}}$.)

5 Factors of Cartesian products

Let π denote the permutation on $X \times X$ defined by $\pi(x, y) = (y, x)$, and let $\mathcal{B}^{2\odot}$ denote the symmetric factor, i.e., $\mathcal{B}^{2\odot} = \{A \in \mathcal{B} \otimes \mathcal{B} : \pi(A) = A\}$. We now state our main theorem.

Theorem 6. *Let $\mathbf{X} = (X, \mathcal{B}, \mu_{\lambda_1}, T)$ and $\mathbf{X} = (X, \mathcal{B}, \mu_{\lambda_2}, T)$ be two non-singular Chacon systems. Let $\mathcal{F}$ be a factor algebra of $(T \times T, \mu_{\lambda_1} \times \mu_{\lambda_2})$.*

(a) If $\lambda_1 \neq \lambda_2$ then $\mathcal{F}$ is equal $\bmod \mu$ to one of the four algebras $\mathcal{B} \otimes \mathcal{C}$, $\mathcal{B} \otimes \mathcal{N}$, $\mathcal{N} \otimes \mathcal{C}$, or $\mathcal{N} \otimes \mathcal{N}$.

(b)If $\lambda_1 = \lambda_2$ then $\mathcal{F}$ is equal $\bmod \mu$ to one of the following algebras $\mathcal{B} \otimes \mathcal{C}$, $\mathcal{B} \otimes \mathcal{N}$, $\mathcal{N} \otimes \mathcal{C}$, $\mathcal{N} \otimes \mathcal{N}$, or $(T^m \times Id)\mathcal{B}^{2\odot}$ for some integer m.

The proof will proceed via several lemmas. Let us agree to refer to the factors specified in the statement of the theorem as standard.

Lemma 5.1. *It suffices to prove Theorem 6 in the case when $\mathcal{F}$ is generated by a homomorphism onto a non-singular symbolic system.*

Proof. Any finite $\mathcal{F}$-measurable partition of X^2 indexed by a finite set B generates a factor of $\mathcal{F}$ and a code ϕ onto $B^{\mathbb{Z}}$, which may be regarded as a homomorphism onto the non-singular symbolic system $(B^{\mathbb{Z}}, m_{1,2} \circ \phi^{-1})$. If we already know that each such factor of $\mathcal{F}$ is standard then we may take a sequence $\{P_i\}$ of finite partitions such that the corresponding factors $\mathcal{F}_i$ increase up to $\mathcal{F}$. Each $\mathcal{F}_i$ will be standard and since the family of standard factors has no strictly increasing chain of length greater than 3 we conclude that $\mathcal{F}_i$ is eventually equal to some standard $\mathcal{G}$, so $\mathcal{F} = \mathcal{G}$. □

Henceforth we assume that $\mathcal{F}$ is generated by a homomorphism ϕ onto a symbolic system with alphabet B.

Given $\mathbf{s} = (s_1, s_2, s_3, s_4) \in \{0,1\}^4$ and $z = (x, y, x', y') \in X^4$ we will say that $\mathbf{s}$ occurs in z at time n if there are n-blocks in x, y, x', y' occurring on intervals $I, J, I', J' \subset [-10h_n, 10h_n]$ such that $|I \cap J \cap I' \cap J'| > 10^{-2}h_n$ and these n-blocks are followed by spacers s_1, s_2, s_3, s_4 respectively. (As explained earlier, "followed by a spacer s" means followed or not followed by a spacer according as s is 1 or 0.) We will say that $\mathbf{s}$ occurs in z if it occurs at time n for infinitely many n. Finally if $E \subset X^4$ we will say that s occurs on E if it occurs in z for all $z \in E$.

Let $\hat{m}$ denote the relative product measure on X^4 corresponding to the factor algebra $\mathcal{F}$.

Lemma 5.2. *Suppose $\hat{m}(E) > 0$, $\mathbf{s}$ occurs on E, $s_1 = s_2$ and not all the s_i are equal. Then $\mathcal{F}$ is standard.*

Proof. For concreteness we will assume that $s_1 = s_2 = 0, s_3 = 1$ and $s_4 = 0$ and make a remark at the end of the proof about the argument in the other cases.

By Proposition 2.4 we may find a sequence $\{\phi_m\}$ of finite codes with the property that for $2^{-m} - \hat{m} - a.a.\ z = (x, y, x', y') \in X^4$

$$\bar{d}_z(\phi(x,y), \phi_m(x,y)) < 2^{-m} \text{ and } \bar{d}_z(\phi(x',y'), \phi_m(x',y')) < 2^{-m}. \tag{5.1}$$

(Here and later we use the convention that if (Ω, σ) is a probability space and $P(\omega)$ is a statement about $\omega \in \Omega$ then "for $\epsilon - \sigma - a.a.\ \omega\ P(\omega)$ holds" means that $\sigma\{\omega | P(\omega) \text{ holds}\} > 1 - \epsilon$.) In (5.1) (x, y) is to be viewed as a sequence in $A^{\mathbb{Z}}$ where $A = \{0,1\}^2$. At the same time we can ensure that for $2^{-m} - \hat{m} - a.a.\ z = (x, y, x', y') \in X^4$

$$\bar{d}_z(\phi \circ (T^{-1} \times id)(x', y'), \phi_m \circ (T^{-1} \times id)(x', y')) < 2^{-m}. \tag{5.2}$$

To satisfy (5.1) and (5.2) simultaneously we must make

$$D[\phi, \phi_m] \text{ and } D[\phi \circ (T^{-1} \times id), \phi_m \circ (T^{-1} \times id)]$$

small simultaneously. If $D[\phi, \phi_m]$ is sufficiently small then

$$D[\phi \circ (T^{-1} \times id), \phi_m \circ (T^{-1} \times id)] = (T \times id)(D[\phi, \phi_m])$$

will also be small by absolute continuity.

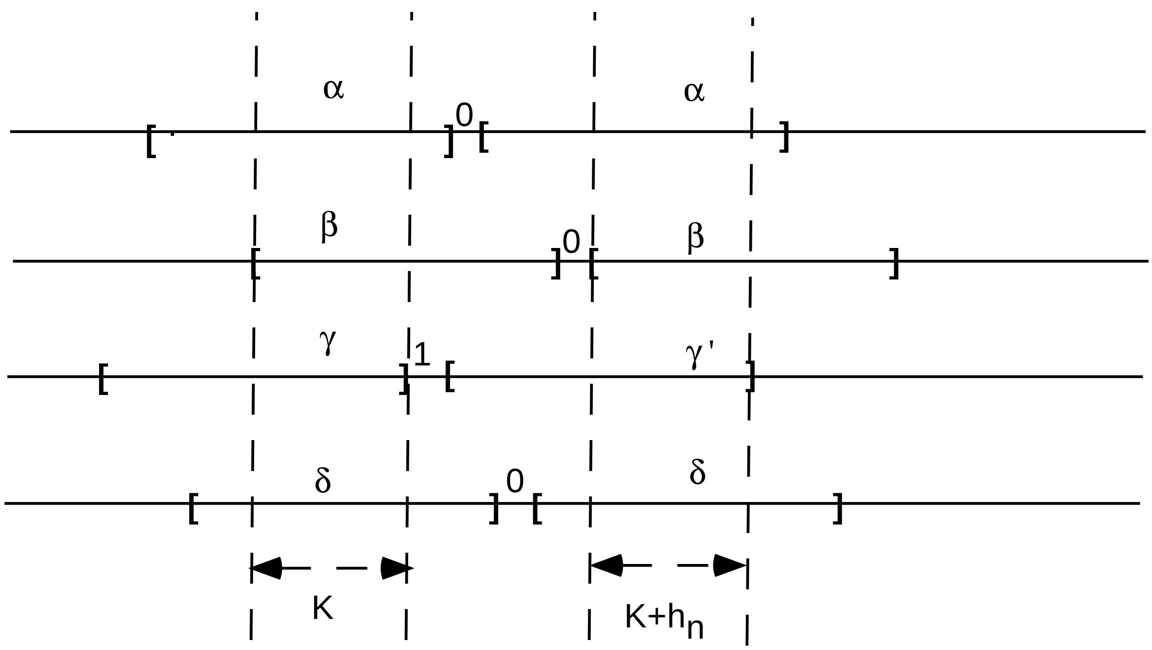
By the Borel-Cantelli lemma $\widehat{m} - a.a.\ z \in E$ will satisfy (5.1) and (5.2) for sufficiently large $m = m(z)$. In addition, by Lemma 2.3, for $\widehat{m} - a.a.\ z \in X^4$

$$\bar{d}_z(\phi(x', y'), \phi(T^{-1}x', y')) \text{ exists.} \tag{5.3}$$

Let us say a point $z \in E$ is **pleasant** if it satisfies (5.1) and (5.2) for sufficiently large m and also (5.3). So, *a.a.* $z \in E$ are pleasant.

Now fix a pleasant z and $m > m(z)$ and observe $\mathbf{s}$ occurring in z at time $n = n(m)$ for some large n such that $h_n \gg 2k + 1 = |\phi_m|$, see Figure 2. (We will specify how large n needs to be as the argument proceeds.) In Figure 2 $\alpha, \beta, \gamma, \delta$ denote the restrictions of x, y, x', y' to the intersection $K = K(m)$ of the intervals I, J, I', J' on which the relevant n-blocks in x, y, x', y' occur. On $K + h_n$ we see $\alpha, \beta, \gamma', \delta$ where γ' is essentially $T^{-1}\gamma$, more precisely γ' denotes the word γ with its last symbol deleted and either a 0 or 1 concatenated on the left.

Figure 2: n-blocks



Our next goal is to show that $\bar{d}_z^K(\phi_m(\gamma, \delta), \phi_m(\gamma', \delta))$ is small. Notice first that this statement is not quite meaningful because $\phi_m(\gamma, \delta)$ is not a word of length $|K|$ but only of length $|K| - 2k$ (recall that $2k + 1 = |\phi_m|$). However this is not a real problem because if n and hence $|K|$ is sufficiently large Corollary 3.2(4) ensures that the missing ends of these words can contribute at most η to the $\bar{d}$-errors, where η is as small as we please. Notice also that we are comparing the words (γ, δ) and (γ', δ) which occur on intervals K and $K + h_n$ but (arbitrarily) choosing to use the metric $\bar{d}_z^K$. However Corollary 3.2(5) implies that no matter how large n is

$$(\lambda_1\lambda_2)^{12}\bar{d}_z^K < \bar{d}_z^{K+h_n} < (\lambda_1\lambda_2)^{-12}\bar{d}_z^K.$$

If n is sufficiently large

$$\bar{d}_z^{[-10h_n,10h_n]}(\phi(x,y),\phi_m(x,y)) < 2^{-m}$$

and

$$\bar{d}_z^{[-10h_n,10h_n]}(\phi(x',y'),\phi_m(x',y')) < 2^{-m}$$

so

$$\bar{d}_z^{[-10h_n,10h_n]}(\phi_m(x,y),\phi_m(x',y')) < 2\cdot 2^{-m}.$$

Since everything in Figure 2 happens inside $[-10h_n, 10h_n]$ and $|K| > 10^{-2}h_n$, Corollary 3.2(3) now ensures the existence of a constant C' such that

$$\bar{d}_z^{K}(\phi_m(x,y),\phi_m(x',y')) < C'\cdot 2^{-m} \text{ and } \bar{d}_z^{K+h_n}(\phi_m(x,y),\phi_m(x',y')) < C'\cdot 2^{-m}.$$

In view of Corollary 3.2(4) if n is sufficiently large this may be rewritten as

$$\bar{d}_z^{K}(\phi_m(\alpha,\beta),\phi_m(\gamma,\delta)) < C'\cdot 2^{-m} \tag{5.4}$$

$$\bar{d}_z^{K+h_n}(\phi_m(\alpha,\beta),\phi_m(\gamma',\delta) < C'\cdot 2^{-m}. \tag{5.5}$$

Now Corollary 3.2(5) implies that no matter how large n is (5.5) may be replaced by

$$\bar{d}_z^{K}(\phi_m(\alpha,\beta),\phi_m(\gamma',\delta)) < (\lambda_1\lambda_2)^{-12}\cdot C' 2^{-m}. \tag{5.6}$$

Combining (5.4) and (5.6) we find that

$$\bar{d}_z^{K}(\phi_m(\gamma,\delta),\phi_m(\gamma',\delta)) < C'' 2^{-m},$$

where $C'' = C'(1 + (\lambda_1\lambda_2)^{-12})$, as we wanted.

Now we observe that (on all but the left endpoint of K) $(T^{-1}x')(K) = \gamma'$ so if n is sufficiently large Corollary 3.2(4) implies that the above inequality may be replaced by

$$\bar{d}_z^{K}(\phi_m(x,y),\phi_m(T^{-1}x,y)) < C'' 2^{-m}.$$

The next step is to replace ϕ_m above by ϕ at the cost of a somewhat larger error. This is done in the same way as before using inequality (5.2) and Corollary 3.2(3) to obtain

$$\bar{d}_z^{K}(\phi(x',y'),\phi(T^{-1}x',y')) < C''' 2^{-m} \tag{5.7}$$

where C''' is a new constant which there is no need to compute explicitly.

Now recall that m has been fixed in this discussion but $K = K(m)$ and $n = n(m)$ depend on m, $K(m) \subset [-10h_{n(m)}, 10h_{n(m)}]$, and $|K(m)| > 10^{-2}h_{n(m)}$. Thus by (5.3), Corollary 3.2(3) and Lemma 2.2 we can conclude that

$$\bar{d}_z(\phi(x',y'),\phi(T^{-1}x',y'))=0. \tag{5.8}$$

Recall that (5.8) holds for all pleasant z, hence for $\widehat{m}-a.a.\ z\in E$. By the ergodic theorem this means that $E(h|\mathcal{I})=0$ for $\widehat{m}-a.a.z\in E$ where h is the characteristic function of the set $\{z\in X^4|\phi(x',y')(0)\neq\phi(T^{-1}x',y')(0)\}$ and $\mathcal{I}$ is the algebra of $T^{\otimes 4}$- invariant sets in $(X^4,\widehat{m})$. Since $T\times T$ is ergodic it follows from Proposition 2.5 that $\phi(x',y')(0)=\phi(T^{-1}x',y')(0)\ \widehat{m}-a.e.$. Since this equation only depends on (x',y') and $\widehat{m}$ is a joining it also holds $m_{1,2}$-a.e. Then non-singularity of $T\times T$ and the fact that $T\times T$ and $T^{-1}\times T$ commute imply that $\phi(x',y')=\phi(T^{-1}x',y')\ m_{1,2}-a.e.$. By ergodicity of T it follows that ϕ depends only on y', that is $\mathcal{F}$ is contained in $\mathcal{N}\otimes\mathcal{B}$. Since T itself is prime, $\mathcal{F}$ is either $\mathcal{N}\otimes\mathcal{B}$ or $\mathcal{N}\otimes\mathcal{N}$.

We have just dealt with the case when $\mathbf{s}=(0,0,1,0)$. In a similar way all other cases will lead to invariance of ϕ under one of $T\times id, T\times T$ or $id\times T$ and the job is finished by ergodicity of either T or $T\times T$. (For example $\mathbf{s}=(0,0,1,1)$ leads to $T\times T$- invariance which implies $\mathcal{F}=\mathcal{N}\otimes\mathcal{N}$ by ergodicity of $T\times T$.) The key point is that the hypothesis that $s_1=s_2$ allows us to say that $\phi_m(x,y)$ is the same on K and $K+h_n+\delta$ where $\delta=0$ or 1. (For example if $\mathbf{s}=(1,1,0,0)$ then $\delta=1$.) In the argument we just carried out δ was 0. In general we need to compare $\bar{d}_z^K$ with $\bar{d}_z^{K+h_n+\delta}$. To do this when $\delta=1$ we use Corollary 3.2 (1) as well as (5). ☐

Lemma 5.3. *If $\widehat{m}\{z=(x,y,x',y')\in X^4|x\notin\mathcal{O}(x')\cup\mathcal{O}(y')\}>0$ then $\mathcal{F}$ is one of the standard factors.*

Proof. Let $z\in E=\{z\in X^4|x\notin\mathcal{O}(x')\cup\mathcal{O}(y')\}$. Since $x\notin\mathcal{O}(x')$ we see broken n-blocks in x and x' for infinitely many n. Fix such an n and let I and I' denote the intervals on which the broken n-blocks occur, so $|I\cap I'|>10^{-1}h_n$. Clearly we may then find n-blocks in y and y' occurring on intervals J and J' such that $|I\cap J\cap I'\cap J'|>10^{-2}h_n$. In other words, there is an $\mathbf{s}\in\{0,1\}^4$ which occurs in z with $(s_1,s_3)=(0,1)$ or $(1,0)$. Fixing such an $\mathbf{s}=\mathbf{s}(z)$ for each $z\in E$ we partition E according to the values of s_1 :

$$F_i=\{z|s_1=i\}.$$

Now consider any F_i of positive $\widehat{m}$-measure. Ruling out the values of $\mathbf{s}$ which are covered by Lemma 5.2 we may assume that $\mathbf{s}$ occurs on F_i, where

$$\begin{aligned}\mathbf{s}&=(0,1,1,0)\text{ if } i=0\\ \mathbf{s}&=(1,0,0,1)\text{ if } i=1.\end{aligned} \tag{5.9}$$

Since $\mathbf{s} = (0, 1, 1, 0)$ occurs on F_0 an argument similar to that in Lemma 5.2 tells us that

$$\bar{d}_z(\phi(x, T^{-1}y), \phi(T^{-1}x', y')) = 0 \text{ for } \widehat{m} - a.a.z \in F_0. \tag{5.10}$$

As in Lemma 5.2 this argument requires several assumptions about z which hold $\widehat{m}$ $a.s.$ We point out only one of them here, namely we need to know that

$$\bar{d}_z(\phi(x, T^{-1}y), \phi(T^{-1}x', y'))$$

exists. Notice that now the first argument of $\bar{d}_z$ involves x and y rather than x' and y' but the existence of the distance is still a consequence of Lemma 2.3, as the two arguments of the distance may be viewed as codes on X^4. Similarly

$$\phi(T^{-1}x, y) = \phi(x', T^{-1}y') \text{ for } \widehat{m} - a.a.z \in F_1. \tag{5.11}$$

Now use the fact that for $z \in E \quad x \notin \mathcal{O}(y')$ to find broken n-blocks in x and y' for infinitely many n. As in the last paragraph it follows that there is an $\mathbf{s} \in \{0, 1\}^4$ which occurs in z with $(s_1, s_4) = (0, 1)$ or $(1, 0)$. Fixing such an $\mathbf{s}$ for each $z \in E$ we partition E according to the values of s_1 :

$$G_j = \{z | s_1 = j\}.$$

As in the last paragraph if $\widehat{m}(G_j) > 0$ we may assume that $\mathbf{s}$ occurs on G_i, where

$$\begin{aligned} \mathbf{s} &= (0, 1, 0, 1) \text{ if } i = 0, \\ \mathbf{s} &= (1, 0, 1, 0) \text{ if } i = 1, \end{aligned} \tag{5.12}$$

and just as before we find that

$$\bar{d}_z(\phi(x, T^{-1}y), \phi(x', T^{-1}y')) = 0 \text{ for } \widehat{m} - a.a.\ z \in G_0, \tag{5.13}$$

and

$$\bar{d}_z(\phi(T^{-1}x, y), \phi(T^{-1}x', y')) = 0 \text{ for } \widehat{m} - a.a.z \in G_1. \tag{5.14}$$

Now choose i and j so that $\widehat{m}(F_i \cap G_j) > 0$. Suppose first that $i = 0, j = 0$. Then equations (5.10) and (5.13) give $\bar{d}_z(\phi((T^{-1}x', y'), \phi(x', T^{-1}y')) = 0$ $\widehat{m} - a.e.$ on $F_0 \cap G_0$. Since this equation now depends only on (x', y') we can conclude as in the proof of Lemma 5.2 that $\phi \circ (T^{-1} \times id)) = \phi \circ (id \times T^{-1})$ $m_{1,2} - a.e.$ It follows that $\phi \circ (T^{-1} \times T) = \phi$ and then ergodicity of $T^{-1} \times T$ tells us that ϕ is almost everywhere constant.

To finish the proof there are three other cases to look at corresponding to the values of (i, j) and it is easy to check that each one leads to invariance of ϕ under $T^{-1} \times T$ or $T \times T^{-1}$. □

We are now in a position to complete the proof of Theorem 3. Suppose first that $\lambda_1 = \lambda_2$. By Lemma 5.3 (applied twice, the second time with the roles of x and y reversed) we may assume that

$$\widehat{m}\{z|\{x,y\} \subset \mathcal{O}(x') \cup \mathcal{O}(y') = 1\}.$$

What this means is that for $\widehat{m}-a.a\ z \in X^4$ $(x,y) = R_z(x',y')$ where $R_z \in C(T \times T)$. Recall that this centralizer is the group generated by $T \times id$ and the co-ordinate interchange on X^2, which we denote by f. It is easy to see that in this group the only elements of finite order, other than the identity, are those of the form $R = f(T^m \times T^{-m})$ and these in fact have order 2. Thus if an element has infinite order its square is of the form $T^m \times T^n$, not both m and n equal to zero.

Suppose now R_z is a constant R for z in a set E of positive $\widehat{m}$-measure and that $R^2 = T^m \times T^n$, not both m and n equal to zero. On E we have $(x,y) = R(x',y')$ so $\phi(x',y') = \phi \circ R(x',y')$, since $\phi(x',y') = \phi(x,y)$. But this equation involves only (x',y') so as we have seen before it holds on a set of positive $m_{1,2}$ - measure and therefore it holds $m_{1,2} - a.e.$, that is ϕ is R-invariant. It follows that ϕ is also invariant under $R^2 = T^m \times T^n$. If, say, $n = 0$ then we conclude that ϕ is $\mathcal{N} \otimes \mathcal{B}$-measurable and we are done. If neither m nor n is 0 then ergodicity of $T^m \times T^n$ tells us that ϕ is almost everywhere constant.

So, finally we may assume that $\widehat{m}$-almost surely either $R_z = f(T^{m(z)} \times T^{-m(z)})$ or $R_z = id$. If $R_z = id$ almost surely then $\mathcal{F} = \mathcal{B} \otimes \mathcal{B}$. If $R_z = f(T^m \times T^{-m})$ with positive $\widehat{m}$-probability then as in the last paragraph we conclude that ϕ is R-invariant which means that $\mathcal{F} \subset (id \times T^m)\mathcal{B}^{2\odot}$. Since the algebras $(id \times T^m)\mathcal{B}^{2\odot}$ have pairwise trivial intersections this can only happen for one value of m. So now we have $\widehat{m}$-a.s. $R_z = f(T^m \times T^{-m})$ or $R_z = id$. This clearly means that $\mathcal{F} = (id \times T^m)\mathcal{B}^{2\odot}$ and concludes the proof in case $\lambda_1 = \lambda_2$.

In case $\lambda_1 \neq \lambda_2$ we observe that μ_{λ_1} and μ_{λ_2} are mutually singular. It follows that $\widehat{m} - a.s.\ x \notin \mathcal{O}(y')$ and $y \notin \mathcal{O}(x')$. So in this case we get that $\widehat{m} - a.s.$ $(x,y) = R_z(x',y')$ where now $R_z = T^{m(z)} \times T^{n(z)}$. The argument is then the same as before but the symmetric algebras do not arise.

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