# ERGODIC THEORY: NONSINGULAR TRANSFORMATIONS

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## Article Outline



## Glossary

**Nonsingular dynamical system:** Let $(X, \mathcal{B}, \mu)$ be a standard Borel space equipped with a $\sigma$-finite measure. A Borel map $T : X \to X$ is a *nonsingular transformation* of $X$ if for any $N \in \mathcal{B}$, $\mu(T^{-1}N) = 0$ if and only if $\mu(N) = 0$. In this case the measure $\mu$ is called *quasi-invariant* for $T$; and the quadruple $(X, \mathcal{B}, \mu, T)$ is called a *nonsingular dynamical system*. If $\mu(A) = \mu(T^{-1}A)$ for all $A \in \mathcal{B}$ then $\mu$ is said to be *invariant* under $T$ or, equivalently, $T$ is *measure-preserving*.

**Conservativeness:** $T$ is *conservative* if for all sets $A$ of positive measure there exists an integer $n > 0$ such that $\mu(A \cap T^{-n}A) > 0$.

**Ergodicity:** $T$ is *ergodic* if every measurable subset $A$ of $X$ that is invariant under $T$ (i.e., $T^{-1}A = A$) is either $\mu$-null or $\mu$-conull. Equivalently, every Borel function $f : X \to \mathbb{R}$ such that $f \circ T = f$ is constant a.e.

**Types II, II$_1$, II$_\infty$ and III:** Suppose that $\mu$ is non-atomic and $T$ ergodic (and hence conservative). If there exists a $\sigma$-finite measure $\nu$ on $\mathcal{B}$ which is equivalent to $\mu$ and invariant under $T$ then $T$ is said *to be of type II*. It is easy to see that $\nu$ is unique up to scaling. If $\nu$ is finite then $T$ is *of type $II_1$*. If $\nu$ is infinite then $T$ is of type II$_\infty$. If $T$ is not of type II then $T$ is said *to be of type III*.

## 1. Definition of the subject and its importance

An abstract measurable dynamical system consists of a set $X$ (phase space) with a transformation $T : X \to X$ (evolution law or time) and a finite or $\sigma$-finite measure $\mu$ on $X$ that specifies a class of negligible subsets. Nonsingular ergodic theory studies systems where $T$ respects $\mu$ in a weak sense: the transformation preserves only the class of negligible subsets but it may not preserve $\mu$. This survey is about dynamics and invariants of nonsingular systems. Such systems model 'non-equilibrium' situations in which events that are impossible at some time remain impossible at any other time. Of course, the first question that arises is whether it is possible to find an equivalent invariant measure, i.e. pass to a hidden equilibrium without changing the negligible subsets? It turns out that there exist systems which do not admit an equivalent invariant finite or even $\sigma$-finite measure. They are of our primary interest here. In a way (Baire category) most of systems are like that.

Nonsingular dynamical systems arise naturally in various fields of mathematics: topological and smooth dynamics, probability theory, random walks, theory of numbers, von Neumann algebras, unitary representations of groups, mathematical physics and so on. They also can appear in the study of probability preserving systems: some criteria of mild mixing and distality, a problem of Furstenberg on disjointness, etc. We briefly discuss this in § 11. Nonsingular ergodic theory studies all of them from a general point of view:

— What is the qualitative nature of the dynamics?
— What are the orbits?
— Which properties are typical withing a class of systems?
— How do we find computable invariants to compare or distinguish various systems?

Typically there are two kinds of results: some are extensions to nonsingular systems of theorems for finite measure-preserving transformations (for instance, the entire § 2) and the other are about new properly 'nonsingular' phenomena (see § 4 or § 6). Philosophically speaking, the dynamics of nonsingular systems is more diverse comparatively with their finite measure-preserving counterparts. That is why it is usually easier to construct counterexamples than to develop a general theory. Because of shortage of space we concentrate only on invertible transformations, and we have not included as many references as we had wished. Nonsingular endomorphisms and general group or semigroup actions are practically not considered here (with some exceptions in § 11 devoted to applications). A number of open problems are scattered through the entire text.

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## 2. Basic Results

This section includes the basic results involving conservativeness and ergodicity as well as some direct nonsingular counterparts of the basic machinery from classic ergodic theory: mean and pointwise ergodic theorems, Rokhlin lemma, ergodic decomposition, generators, Glimm-Effros theorem and special representation of nonsingular flows. The historically first example of a transformation of type III (due to Ornstein) is also given here with full proof.

2.1. **Nonsingular transformations.** In this paper we will consider only *invertible* nonsingular transformations, i.e. those which are bijections when restricted to an invariant Borel subset of full measure. Thus when we refer to a nonsingular dynamical system $(X, \mathcal{B}, \mu, T)$ we shall assume that $T$ is an invertible nonsingular transformation. Of course, each measure $\nu$ on $\mathcal{B}$ which is *equivalent* to $\mu$, i.e. $\mu$ and $\nu$ have the same null sets, is also quasi-invariant under $T$. In particular, since $\mu$ is $\sigma$-finite, $T$ admits an equivalent quasi-invariant probability measure. For each $i \in \mathbb{Z}$, we denote by $\omega_i^\mu$ or $\omega_i$ the Radon-Nikodym derivative $d(\mu \circ T^i)/d\mu \in L^1(X, \mu)$. The derivatives satisfy the cocycle equation $\omega_{i+j}(x) = \omega_i(x)\omega_j(T^i x)$ for a.e. $x$ and all $i, j \in \mathbb{Z}$.

2.2. **Basic properties of conservativeness and ergodicity.** A measurable set $W$ is said to be *wandering* if for all $i, j \geq 0$ with $i \neq j$, $T^{-i}W \cap T^{-j}W = \emptyset$. Clearly, if $T$ has a wandering set of positive measure then it cannot be conservative. A nonsingular transformation $T$ is *incompressible* if whenever $T^{-1}C \subset C$, then $\mu(C \setminus T^{-1}C) = 0$. A set $W$ of positive measure is said to be *weakly wandering* if there is a sequence $n_i \to \infty$ such that $T^{n_i}W \cap T^{n_j}W = \emptyset$ for all $i \neq j$. Clearly, a finite measure-preserving transformation cannot have a weakly wandering set. Hajian and Kakutani [79] showed that a nonsingular transformation $T$ admits an equivalent finite invariant measure if and only if $T$ does not have a weakly wandering set.

**Proposition 2.1.** (see e.g. [123]) *Let $(X, \mathcal{B}, \mu, T)$ be a nonsingular dynamical system. The following are equivalent:*

(i) *$T$ is conservative.*
(ii) *For every measurable set $A$, $\mu(A \setminus \bigcup_{n=1}^\infty T^{-n}A) = 0$.*
(iii) *$T$ is incompressible.*
(iv) *Every wandering set for $T$ is null.*

Since any finite measure-preserving transformation is incompressible, we deduce that it is conservative. This is the statement of the classical Poincaré recurrence lemma. If $T$ is a conservative nonsingular transformation of $(X, \mathcal{B}, \mu)$ and $A \in \mathcal{B}$ a subset of positive measure, we can define an *induced transformation* $T_A$ of the space $(A, \mathcal{B} \cap A, \mu \restriction A)$ by setting $T_A x := T^n x$ if $n = n(x)$ is the smallest natural number such that $T^n x \in A$. $T_A$ is also conservative. As shown in [179, 5.2], if $\mu(X) = 1$ and $T$ is conservative and ergodic, $\int_A \sum_{i=0}^{n(x)-1} \omega(x)\, d\mu(x) = 1$, which is a nonsingular version of the well-known Kaçs formula.

**Theorem 2.2** (Hopf Decomposition, see e.g. [3])**.** *Let $T$ be a nonsingular transformation. Then there exist disjoint invariant sets $C, D \in \mathcal{B}$ such that $X = C \sqcup D$, $T$ restricted to $C$ is conservative, and $D = \bigsqcup_{n=-\infty}^{\infty} T^n W$, where $W$ is a wandering set. If $f \in L^1(X, \mu)$, $f > 0$, then $C = \{x : \sum_{i=0}^{n-1} f(T^i x)\omega_i(x) = \infty$ a.e.$\}$ and $D = \{x : \sum_{i=0}^{n-1} f(T^i x)\omega_i(x) < \infty$ a.e.$\}$.*

The set $C$ is called the *conservative part* of $T$ and $D$ is called the *dissipative part* of $T$.

If $T$ is ergodic and $\mu$ is non-atomic then $T$ is automatically conservative. The translation by 1 on the group $\mathbb{Z}$ furnished with the counting measure is an example of an ergodic non-conservative (infinite measure-preserving) transformation.

**Proposition 2.3.** *Let $(X, \mathcal{B}, \mu, T)$ be a nonsingular dynamical system. The following are equivalent:*

(i) *$T$ is conservative and ergodic.*
(ii) *For every set $A$ of positive measure, $\mu(X \setminus \bigcup_{n=1}^{\infty} T^{-n}A) = 0$. (In this case we will say $A$ sweeps out.)*
(iii) *For every measurable set $A$ of positive measure and for a.e. $x \in X$ there exists an integer $n > 0$ such that $T^n x \in A$.*
(iv) *For all sets $A$ and $B$ of positive measure there exists an integer $n > 0$ such that $\mu(T^{-n}A \cap B) > 0$.*
(v) *If $A$ is such that $T^{-1}A \subset A$, then $\mu(A) = 0$ or $\mu(A^c) = 0$.*

This survey is mainly about systems of type III. For some time it was not quite obvious whether such systems exist at all. The historically first example was constructed by Ornstein in 1960.

**Example 2.4.** (Ornstein [149]) Let $A_n = \{0, 1, \dots, n\}$, $\nu_n(0) = 0.5$ and $\nu_n(i) = 1/(2n)$ for $0 < i \le n$ and all $n \in \mathbb{N}$. Denote by $(X, \mu)$ the infinite product probability space $\bigotimes_{n=1}^{\infty}(A_n, \nu_n)$. Of course, $\mu$ is non-atomic. A point of $X$ is an infinite sequence $x = (x_n)_{n=1}^{\infty}$ with $x_n \in A_n$ for all $n$. Given $a_1 \in A_1, \dots, a_n \in A_n$, we denote the cylinder $\{x = (x_i)_{i=1}^{\infty} \in X : x_1 = a_1, \dots, x_n = a_n\}$ by $[a_1, \dots, a_n]$. Define a Borel map $T : X \to X$ by setting

$$(1) \qquad (Tx)_i = \begin{cases} 0, & \text{if } i < l(x) \\ x_i + 1, & \text{if } i = l(x) \\ x_i, & \text{if } i > l(x), \end{cases}$$

where $l(x)$ is the smallest number $l$ such that $x_l \neq l$. It is easy to verify that $T$ is a nonsingular transformation of $(X, \mu)$ and

$$\frac{d\mu \circ T}{d\mu}(x) = \prod_{n=1}^{\infty} \frac{\nu_n((Tx)_n)}{\nu_n(x_n)} = \begin{cases} (l(x) - 1)!/l(x), & \text{if } x_{l(x)} = 0 \\ (l(x) - 1)!, & \text{if } x_{l(x)} \neq 0. \end{cases}$$

We prove that $T$ is of type III by contradiction. Suppose that there exists a $T$-invariant $\sigma$-finite measure $\nu$ equivalent to $\mu$. Let $\varphi := d\mu/d\nu$. Then

$$(2) \qquad \omega_i^{\mu}(x) = \varphi(x)\varphi(T^i x)^{-1} \quad \text{for a.a. } x \in X \text{ and all } i \in \mathbb{Z}.$$

Fix a real $C > 1$ such that the set $E_C := \varphi^{-1}([C^{-1}, C]) \subset X$ is of positive measure. By a standard approximation argument, for each sufficiently large $n$, there is a cylinder $[a_1, \dots, a_n]$ such that $\mu(E_C \cap [a_1, \dots, a_n]) > 0.9\mu([a_1, \dots, a_n])$. Since $\nu_{n+1}(0) = 0.5$, it follows that $\mu(E_C \cap [a_1, \dots, a_n, 0]) > 0.8\mu([a_1, \dots, a_n, 0])$. Moreover, by the pigeon hole principle there is $0 < i \le n + 1$ with $\mu(E_C \cap [a_1, \dots, a_n, i]) > 0.8\mu([a_1, \dots, a_n, i])$. Find $N_n > 0$ such that $T^{N_n}[a_1, \dots, a_n, 0] = [a_1, \dots, a_n, i]$. Since $\omega_{N_n}^{\mu}$ is constant on $[a_1, \dots, a_n, 0]$, there is a subset $E_0 \subset E_C \cap [a_1, \dots, a_n, 0]$ of positive measure such that $T^{N_n}E_0 \subset E_C \cap [a_1, \dots, a_n, i]$. Moreover, $\omega_{N_n}^{\mu}(x) = \nu_{n+1}(i)/\nu_{n+1}(0) = (n + 1)^{-1}$ for a.a. $x \in [a_1, \dots, a_n, 0]$. On the other hand, we deduce from (2) that $\omega_{N_n}^{\mu}(x) \ge C^{-2}$ for all $x \in E_0$, a contradiction.

2.3. **Mean and pointwise ergodic theorems. Rokhlin lemma.** Let $(X, \mathcal{B}, \mu, T)$ be a nonsingular dynamical system. Define a unitary operator $U_T$ of $L^2(X, \mu)$ by setting

$$(3) \qquad U_T f := \sqrt{(d(\mu \circ T)/d\mu} \cdot f \circ T.$$

We note that $U_T$ preserves the cone of positive functions $L^2_+(X,\mu)$. Conversely, every positive unitary operator in $L^2(X,\mu)$ that preserves $L^2_+(X,\mu)$ equals $U_T$ for a $\mu$-nonsingular transformation $T$.

**Theorem 2.5** (von Neumann mean Ergodic Theorem, see e.g. [3])**.** *If $T$ has no $\mu$-absolutely continuous $T$-invariant probability, then $n^{-1}\sum_{i=0}^{n-1} U_T^i \to 0$ in the strong operator topology.*

Denote by $\mathcal{I}$ the sub-$\sigma$-algebra of $T$-invariant sets. Let $\mathbb{E}_\mu[.|\mathcal{I}]$ stand for the conditional expectation with respect to $\mathcal{I}$. Note that if $T$ is ergodic, then $\mathbb{E}_\mu[f|\mathcal{I}] = \int f\,d\mu$. Now we state a nonsingular analogue of Birkhoff's pointwise ergodic theorem, due to Hurewicz [101] and in the form stated by Halmos [80].

**Theorem 2.6** (Hurewicz pointwise Ergodic Theorem)**.** *If $T$ is conservative, $\mu(X) = 1$, $f, g \in L^1(X,\mu)$ and $g > 0$, then*

$$\frac{\sum_{i=0}^{n-1} f(T^i x)\omega_i(x)}{\sum_{i=0}^{n-1} g(T^i x)\omega_i(x)} \to \frac{\mathbb{E}_\mu[f|\mathcal{I}]}{\mathbb{E}_\mu[g|\mathcal{I}]} \quad \text{as } n\to\infty \text{ for a.e. } x.$$

A transformation $T$ is *aperiodic* if the $T$-orbit of a.e. point from $X$ is infinite. The following classical statement can be deduced easily from Proposition 2.1.

**Lemma 2.7** (Rokhlin's lemma [161])**.** *Let $T$ be an aperiodic nonsingular transformation. For each $\varepsilon > 0$ and integer $N > 1$ there exists a measurable set $A$ such that the sets $A, TA, \dots, T^{N-1}A$ are disjoint and $\mu(A \cup TA \cup \cdots \cup T^{N-1}A) > 1-\varepsilon$.*

This lemma was refined later (for ergodic transformations) by Lehrer and Weiss as follows.

**Theorem 2.8** ($\epsilon$-free Rokhlin lemma [132])**.** *Let $T$ be ergodic and $\mu$ non-atomic. Then for a subset $B \subset X$ and any $N$ for which $\bigcup_{k=0}^\infty T^{-kN}(X\setminus B) = X$, there is a set $A$ such that the sets $A, TA, \dots, T^{N-1}A$ are disjoint and $A \cup TA \cup \cdots \cup T^{N-1}A \supset B$.*

The condition $\bigcup_{k=0}^\infty T^{-kN}(X\setminus B) = X$ holds of course for each $B \neq X$ if $T$ is *totally ergodic*, i.e. $T^p$ is ergodic for any $p$, or if $N$ is prime.

2.4. **Ergodic decomposition.** A proof of the following theorem may be found in [3, 2.2.8].

**Theorem 2.9** (Ergodic Decomposition Theorem)**.** *Let $T$ be a conservative nonsingular transformation on a standard probability space $(X,\mathcal{B},\mu)$. There there exists a standard probability space $(Y,\nu,\mathcal{A})$ and a family of probability measures $\mu_y$ on $(X,\mathcal{B})$, for $y\in Y$, such that*

(i) *For each $A\in\mathcal{B}$ the map $y\mapsto\mu_y(A)$ is Borel and for each $A\in\mathcal{B}$*

$$\mu(A) = \int \mu_y(A)d\nu(y).$$

(ii) *For $y, y'\in Y$ the measures $\mu_y$ and $\mu_{y'}$ are mutually singular.*

(iii) *For each $y\in Y$ the transformation $T$ is nonsingular and conservative, ergodic on $(X,\mathcal{B},\mu_y)$.*

(iv) *For each $y\in Y$*

$$\frac{d\mu\circ T}{d\mu} = \frac{d\mu_y\circ T}{d\mu_y} \quad \mu_y\text{-a.e.}$$

(v) *(Uniqueness) If there exists another probability space* $(Y', \nu', \mathcal{A}')$ *and a family of probability measures* $\mu'_{y'}$ *on* $(X, \mathcal{B})$, *for* $y' \in Y'$, *satisfying* (i)-(iv), then there exists a measure-preserving isomorphism $\theta : Y \to Y'$ such that $\mu_y = \mu'_{\theta y}$ for $\nu$-a.e. $y$.

It follows that if $T$ preserves an equivalent $\sigma$-finite measure then the system $(X, \mathcal{B}, \mu_y, T)$ is of type II for a.a. $y$. The space $(Y, \nu, \mathcal{A})$ is called *the space of $T$-ergodic components*.

2.5. **Generators.** It was shown in [162], [157] that a nonsingular transformation $T$ on a standard probability space $(X, \mathcal{B}, \mu)$ has a *countable generator*, i.e. a countable partition $\mathcal{P}$ so that $\bigvee_{n=-\infty}^{\infty} T^n \mathcal{P}$ generates the measurable sets. It was refined by Krengel [126]: if $T$ is of type $\mathrm{II}_\infty$ or III then there exists a generator $P$ consisting of two sets only. Moreover, given a sub-$\sigma$-algebra $\mathcal{F} \subset \mathcal{B}$ such that $\mathcal{F} \subset T\mathcal{F}$ and $\bigcup_{k>0} T^k \mathcal{F} = \mathcal{B}$, the set $\{A \in \mathcal{F} \mid (A, X \setminus A)$ is a generator of $T\}$ is dense in $\mathcal{F}$. It follows, in particular, that $T$ is isomorphic to the shift on $\{0,1\}^{\mathbb{Z}}$ equipped with a quasi-invariant probability measure.

2.6. **The Glimm-Effros Theorem.** The classical Bogoliouboff-Krylov theorem states that each homeomorphism of a compact space admits an ergodic invariant probability measure [33]. The following statement by Glimm [72] and Effros [57] is a "nonsingular" analogue of that theorem. (We consider here only a particular case of $\mathbb{Z}$-actions.)

**Theorem 2.10.** *Let $X$ be a Polish space and $T : X \to X$ an aperiodic homeomorphism. Then the following are equivalent:*

(i) *$T$ has a recurrent point $x$, i.e. $x = \lim_{n\to\infty} T^{n_i} x$ for a sequence $n_1 < n_2 < \cdots$.*
(ii) *There is an orbit of $T$ which is not locally closed.*
(iii) *There is no a Borel set which intersects each orbit of $T$ exactly once.*
(iv) *There is a continuous probability Borel measure $\mu$ on $X$ such that $(X, \mu, T)$ is an ergodic nonsingular system.*

A natural question arises: under the conditions of the theorem how many such $\mu$ can exists? It turns out that there is a wealth of such measures. To state a corresponding result we first write an important definition.

**Definition 2.11.** Two nonsingular systems $(X, \mathcal{B}, \mu, T)$ and $(X, \mathcal{B}', \mu', T')$ are called *orbit equivalent* if there is a one-to-one bi-measurable map $\varphi : X \to X$ with $\mu' \circ \varphi \sim \mu$ and such that $\varphi$ maps the $T$-orbit of $x$ onto the $T'$-orbit of $\varphi(x)$ for a.a. $x \in X$.

The following theorem was proved in [116], [174] and [128].

**Theorem 2.12.** *Let $(X, T)$ be as in Theorem 2.10. Then for each ergodic dynamical system $(Y, \mathcal{C}, \nu, S)$ of type $II_\infty$ or III, there exist uncountably many mutually disjoint Borel measures $\mu$ on $X$ such that $(X, T, \mathcal{B}, \mu)$ is orbit equivalent to $(Y, \mathcal{C}, \nu, S)$.*

On the other hand, $T$ may not have any finite invariant measure. Indeed, let $T$ be an irrational rotation on the circle $\mathbb{T}$ and $X$ a non-empty $T$-invariant $G_\delta$ subset of $\mathbb{T}$ of full Lebesgue measure. Let $(X, T)$ contain a recurrent point. Then the unique ergodicity of $(\mathbb{T}, T)$ implies that $(X, T)$ has no finite invariant measures.

Let $T$ be an aperiodic Borel transformation of a standard Borel space $X$. Denote by $\mathcal{M}(T)$ the set of all ergodic $T$-nonsingular continuous measures on $X$. Given $\mu \in \mathcal{M}(T)$, let $N(\mu)$ denote the family of all Borel $\mu$-null subsets. Shelah and Weiss showed [178] that $\bigcap_{\mu\in\mathcal{M}(T)} N(\mu)$ coincides with the collection of all Borel $T$-wandering sets.

2.7. **Special representations of ergodic flows.** Nonsingular flows (=$\mathbb{R}$-actions) appear naturally in the study of orbit equivalence for systems of type III (see Section 6). Here we record some basic notions related to nonsingular flows. Let $(X, \mathcal{B}, \mu)$ be a standard Borel space with a $\sigma$-finite measure $\mu$ on $\mathcal{B}$. A nonsingular *flow* on $(X, \mu)$ is a Borel map $S : X \times \mathbb{R} \ni (x, t) \mapsto S_t x \in X$ such that $S_t S_s = S_{t+s}$ for all $s, t \in \mathbb{R}$ and each $S_t$ is a nonsingular transformation of $(X, \mu)$. Conservativeness and ergodicity for flows are defined in a similar way as for transformations.

A very useful example of a flow is a flow built under a function. Let $(X, \mathcal{B}, \mu, T)$ be a nonsingular dynamical system and $f$ a positive Borel function on $X$ such that $\sum_{i=0}^{\infty} f(T^i x) = \sum_{i=0}^{\infty} f(T^{-i} x) = \infty$ for all $x \in X$. Set $X^f := \{(x, s) : x \in X, 0 \le s < f(x)\}$. Define $\mu^f$ to be the restriction of the product measure $\mu \times \text{Leb}$ on $X \times \mathbb{R}$ to $X^f$ and define, for $t \ge 0$,

$$S_t^f(x, s) := (T^n x, s + t - \sum_{i=0}^{n-1} f(T^i x),$$

where $n$ is the unique integer that satisfies

$$\sum_{i=0}^{n-1} f(T^i x) < s + t \le \sum_{i=0}^{n} f(T^i x).$$

A similar definition applies when $t < 0$. In particular, when $0 < s + t < \varphi(x)$, $S_t^f(x, s) = (x, s + t)$, so that the flow moves the point $(x, s)$ up $t$ units, and when it reaches $(x, \varphi(x))$ it is sent to $(Tx, 0)$. It can be shown that $S^f = (S_t^f)_{t\in\mathbb{R}}$ is a free $\mu^f$-nonsingular flow and that it preserves $\mu^f$ if and only if $T$ preserves $\mu$ [148]. It is called the *flow built under the function* $\varphi$ *with the base transformation* $T$. Of course, $S^f$ is conservative or ergodic if and only if so is $T$.

Two flows $S = (S_t)_{t\in\mathbb{R}}$ on $(X, \mathcal{B}, \mu)$ and $V = (V_t)_{t\in\mathbb{R}}$ on $(Y, \mathcal{C}, \nu)$ are said to be *isomorphic* if there exist invariant co-null sets $X' \subset X$ and $Y' \subset Y$ and an invertible nonsingular map $\rho : X' \to Y'$ that interwines the actions of the flows: $\rho \circ S_t = V_t \circ \rho$ on $X'$ for all $t$. The following nonsingular version of Ambrose–Kakutani representation theorem was proved by Krengel [120] and Kubo [130].

**Theorem 2.13.** *Let $S$ be a free nonsingular flow. Then it is isomorphic to a flow built under a function.*

Rudolph showed that in the Ambrose-Kakutani theorem one can choose the function $\varphi$ to take two values. Krengel [122] showed that this can also be assumed in the nonsingular case.

## 3. Panorama of Examples

This section is devoted entirely to examples of nonsingular systems. We describe here the most popular (and simple) constructions of nonsingular systems: odometers, nonsingular Markov odometers, tower transformations, rank-one and finite rank systems and nonsingular Bernoulli shifts.

3.1. **Nonsingular odometers.** Given a sequence $m_n$ of natural numbers, we let $A_n := \{0, 1, \dots, m_n - 1\}$. Let $\nu_n$ be a probability on $A_n$ and $\nu_n(a) > 0$ for all $a \in A_n$. Consider now the infinite product probability space $(X, \mu) := \bigotimes_{n=1}^{\infty}(A_n, \nu_n)$. Assume that $\prod_{n=1}^{\infty} \max\{\nu_n(a) \mid a \in A_n\} = 0$. Then $\mu$ is non-atomic. Given $a_1 \in A_1, \dots, a_n \in A_n$, we denote by $[a_1, \dots, a_n]$ the cylinder $x = (x_i)_{i>0} \mid x_1 = a_1, \dots, x_n = a_n$. If $x \neq (0, 0, \dots)$, we let $l(x)$ be the smallest number $l$ such that the $l$-th coordinate of $x$ is not $m_l - 1$. We define a Borel map $T : X \to X$ by (1) if $x \neq (m_1, m_2, \dots)$ and put $Tx := (0, 0, \dots)$ if $x = (m_1, m_2, \dots)$. Of course, $T$ is isomorphic to a rotation on a compact monothetic totally disconnected Abelian group. It is easy to check that $T$ is $\mu$-nonsingular and

$$\frac{d\mu \circ T}{d\mu}(x) = \prod_{n=1}^{\infty} \frac{\nu_n((Tx)_n)}{\nu_n(x_n)} = \frac{\nu_{l(x)}(x_{l(x)} + 1)}{\nu_{l(x)}(x_{l(x)})} \prod_{n=1}^{l(x)-1} \frac{\nu_n(0)}{\nu_n(m_n - 1)}$$

for a.a. $x = (x_n)_{n>0} \in X$. It is also easy to verify that $T$ is ergodic. It is called the *nonsingular odometer* associated to $(m_n, \nu_n)_{n=1}^{\infty}$. We note that Ornstein's transformation (Example 2.4) is a nonsingular odometer.

3.2. **Markov odometers.** We define Markov odometers as in [51]. An ordered Bratteli diagram $B$ [98] consists of

(i) a vertex set $V$ which is a disjoint union of finite sets $V^{(n)}$, $n \geq 0$, $V_0$ is a singleton;
(ii) an edge set $E$ which is a disjoint union of finite sets $E^{(n)}$, $n > 0$;
(iii) source mappings $s_n : E^{(n)} \to V^{(n-1)}$ and range mappings $r_n : E^{(n)} \to V^{(n)}$ such that $s_n^{-1}(v) \neq \emptyset$ for all $v \in V^{(n-1)}$ and $r_n^{-1}(v) \neq \emptyset$ for all $v \in V^{(n)}$, $n > 0$;
(iv) a partial order on $E$ so that $e, e' \in E$ are comparable if and only if $e, e' \in E^{(n)}$ for some $n$ and $r_n(e) = r_n(e')$.

A *Bratteli compactum* $X_B$ of the diagram $B$ is the space of infinite paths

$$\{x = (x_n)_{n>0} \mid x_n \in E^{(n)} \text{ and } r(x_n) = s(x_{n+1})\}$$

on $B$. $X_B$ is equipped with the natural topology induced by the product topology on $\prod_{n>0} E^{(n)}$. We will assume always that the diagram is *essentially simple*, i.e. there is only one infinite path $x_{\max} = (x_n)_{n>0}$ with $x_n$ maximal for all $n$ and only one $x_{\min} = (x_n)_{n>0}$ with $x_n$ minimal for all $n$. The *Bratteli-Vershik* map $T_B : X_B \to X_B$ is defined as follows: $Tx_{\max} = x_{\min}$. If $x = (x_n)_{n>0} \neq x_{\max}$ then let $k$ be the smallest number such that $x_k$ is not maximal. Let $y_k$ be a successor of $x_k$. Let $(y_1, \dots, y_k)$ be the unique path such that $y_1, \dots, y_{k-1}$ are all minimal. Then we let $T_B x := (y_1, \dots, y_k, x_{k+1}, x_{k+2}, \dots)$. It is easy to see that $T_B$ is a homeomorphism of $X_B$. Suppose that we are given a sequence $P^{(n)} = (P^{(n)}_{(v,e) \in V^{n-1} \times E^{(n)}})$ of stochastic matrices, i.e.

(i) $P^{(n)}_{v,e} > 0$ if and only if $v = s_n(e)$ and
(ii) $\sum_{\{e \in E^{(n)} \mid s_n(e) = v\}} P^{(n)}_{v,e} = 1$ for each $v \in V^{(n-1)}$.

For $e_1 \in E^{(1)}, \dots, e_n \in E^{(n)}$, let $[e_1, \dots, e_n]$ denote the cylinder $\{x = (x_j)_{j>0} \mid x_1 = e_1, \dots, x_n = e_n\}$. Then we define a *Markov measure* on $X_B$ by setting

$$\mu_P([e_1, \dots, e_n]) = P^1_{s_1(e_1), e_1} P^2_{s_2(e_2), e_2} \cdots P^n_{s_n(e_n), e_n}$$

for each cylinder $[e_1, \dots, e_n]$. The dynamical system $(X_B, \mu_P, T_B)$ is called a *Markov odometer*. It is easy to see that every nonsingular odometer is a Markov odometer where the corresponding $V^{(n)}$ are all singletons.

3.3. **Tower transformations.** This construction is a discrete analogue of flow under a function. Given a nonsingular dynamical system $(X, \mu, T)$ and a measurable map $f : X \to \mathbb{N}$, we define a new dynamical system $(X^f, \mu^f, T^f)$ by setting

$$\begin{aligned} X^f &:= \{(x,i) \in X \times \mathbb{Z}_+ \mid 0 \le i < f(x)\}, \\ d\mu^f(x,i) &:= d\mu(x) \text{ and} \\ T^f(x,i) &:= \begin{cases} (x, i+1), & \text{if } i+1 < f(x) \\ (Tx, 0), & \text{otherwise.} \end{cases} \end{aligned}$$

Then $T^f$ is $\mu^f$-nonsingular and $(d\mu^f \circ T^f / d\mu^f)(x,i) = (d\mu \circ T / d\mu)(x)$ for a.a. $(x,i) \in X^f$. This transformation is called the (Kakutani) *tower over $T$ with height function $f$*. It is easy to check that $T^f$ is conservative if and only if $T$ is conservative; $T^f$ is ergodic if and only if $T$ is ergodic; $T^f$ is of type III if and only if $T$ is of type III. Moreover, the induced transformation $(T^f)_{X \times \{0\}}$ is isomorphic to $T$. Given a subset $A \subset X$ of positive measure, $T$ is the tower over the induced transformation $T_A$ with the first return time to $A$ as the height function.

3.4. **Rank-one transformations. Chacón maps. Finite rank.** The definition uses the process of "cutting and stacking." We construct by induction a sequence of columns $C_n$. A *column* $C_n$ consists of a finite sequence of bounded intervals (left-closed, right-open) $C_n = \{I_{n,0}, \dots, I_{n,h_n-1}\}$ of *height* $h_n$. A column $C_n$ determines a *column map* $T_{C_n}$ that sends each interval $I_{n,i}$ to the interval above it $I_{n,i+1}$ by the unique orientation-preserving affine map between the intervals. $T_{C_n}$ remains undefined on the top interval $I_{n,h_n-1}$. Set $C_0 = \{[0,1)\}$ and let $\{r_n > 2\}$ be a sequence of positive integers, let $\{s_n\}$ be a sequence of functions $s_n : \{0, \dots, r_n - 1\} \to \mathbb{N}_0$, and let $\{w_n\}$ be a sequence of probability vectors on $\{0, \dots, r_n - 1\}$. If $C_n$ has been defined, column $C_{n+1}$ is defined as follows. First "cut" (i.e., subdivide) each interval $I_{n,i}$ in $C_n$ into $r_n$ subintervals $I_{n,i}[j], j = 0, \dots, r_n - 1$, whose lengths are in the proportions $w_n(0) : w_n(1) : \dots : w_n(r_n - 1)$. Next place, for each $j = 0, \dots r_n - 1$, $s_n(j)$ new subintervals above $I_{n,h_n-1}[j]$, all of the same length as $I_{n,h_n-1}[j]$. Denote these intervals, called *spacers*, by $S_{n,0}[j], \dots S_{n,s_n(j)-1}[j]$. This yields, for each $j \in \{0, \dots, r_n - 1\}$, $r_n$ subcolums each consisting of the subintervals

$$I_{n,0}[j], \dots I_{n,h_n-1}[j] \text{ followed by the spacers } S_{n,0}[j], \dots S_{n,s_n(j)-1}[j].$$

Finally each subcolumn is stacked from left to right so that the top subinterval in subcolumn $j$ is sent to the bottom subinterval in subcolumn $j+1$, for $j = 0, \dots, r_n - 2$ (by the unique orientation-preserving affine map between the intervals). For example, $S_{n,s_n(0)-1}[0]$ is sent to $I_{n,0}[1]$. This defines a new column $C_{n+1}$ and new column map $T_{C_{n+1}}$, which remains undefined on its top subinterval. Let $X$ be the union of all intervals in all columns and let $\mu$ be Lebesgue measure restricted to $X$. We assume that as $n \to \infty$ the maximal length of the intervals in $C_n$ converges to 0, so we may define a transformation $T$ of $(X, \mu)$ by $Tx := \lim_{n \to \infty} T_{C_n} x$. One can verify that $T$ is well-defined a.e. and that it is nonsingular and ergodic. $T$ is said to be the *rank-one* transformation associated with $(r_n, w_n, s_n)_{n=1}^{\infty}$. If all the probability vectors $w_n$ are uniform the resulting transformation is measure-preserving. The measure is infinite ($\sigma$-finite) if and only if the total mass of the spacers is infinite. In the case $r_n = 3$ and $s_n(0) = s_n(2) = 0$, $s_n(1) = 1$ for all $n \geq 0$, the associated rank-one transformation is called a *nonsingular Chacón map*.

It is easy to see that every nonsingular odometer is of rank-one (the corresponding maps $s_n$ are all trivial). Each rank-one map $T$ is a tower over a nonsingular odometer (to obtain such an odometer reduce $T$ to a column $C_n$).

A rank $N$ transformation is defined in a similar way. A nonsingular transformation $T$ is said to be of *rank $N$ or less* if at each stage of its construction there exits $N$ disjoint columns, the levels of the columns generate the $\sigma$-algebra and the Radon-Nikodym derivative of $T$ is constant on each non-top level of every column. $T$ is said to be of *rank $N$* if it is of rank $N$ or less and not of rank $N-1$ or less. A rank $N$ transformation, $N \geq 2$, need not be ergodic.

3.5. **Nonsingular Bernoulli transformations – Hamachi's example.** A *nonsingular Bernoulli* transformation is a transformation $T$ such that there exists a countable generator $\mathcal{P}$ (see § 2.5) such that the partitions $T^n\mathcal{P}$, $n \in \mathbb{Z}$, are mutually independent and such that the Radon-Nikodym derivative $\omega_1$ is measurable with respect to the sub-$\sigma$-algebra $\bigvee_{n=-\infty}^{0} T^n\mathcal{P}$.

In [83], Hamachi constructed examples of conservative nonsingular Bernoulli transformations, hence ergodic (see §4.1), with a 2-set generating partition that are of type III. Krengel [121] asked if there are of type $\mathrm{II}_\infty$ examples of nonsingular Bernoulli automorphisms and the question remains open. Hamachi's construction is the left-shift on the space $X = \prod_{n=-\infty}^{\infty}\{0, 1\}$. The measure is a product $\mu = \prod_{n=-\infty}^{\infty} \mu_n$ where $\mu_n = (1/2, 1/2)$ for $n \geq 0$ and for $n < 0$ $\mu_n$ is chosen carefully alternating on large blocks between the uniform measure and different non-uniform measures. Kakutani's criterion for equivalence of infinite product measures is used to verify that $\mu$ is nonsingular.

## 4. Mixing notions and multiple recurrence

The study of mixing and multiple recurrence are central topics in classical ergodic theory [33], [66]. Unfortunately, these notions are considerably less 'smooth' in the world of nonsingular systems. The very concepts of any kind of mixing and multiple recurrence are not well understood in view of their ambiguity. Below we discuss nonsingular systems possessing a surprising diversity of such properties that seem equivalent but are different indeed.

4.1. **Weak mixing, mixing, $K$-property.** Let $T$ be an ergodic conservative nonsingular transformation. A number $\lambda \in \mathbb{C}$ is an *$L^\infty$-eigenvalue* for $T$ if there exists a nonzero $f \in L^\infty$ so that $f \circ T = \lambda f$ a.e. It follows that $|\lambda| = 1$ and $f$ has constant modulus, which we assume to be 1. Denote by $e(T)$ the set of all $L^\infty$-eigenvalues of $T$. $T$ is said to be *weakly mixing* if $e(T) = \{1\}$. We refer to [3, Theorem 2.7.1] for proof of the following Keane's ergodic multiplier theorem: given an ergodic probability preserving transformation $S$, the product transformation $T \times S$ is ergodic if and only if $\sigma_S(e(T)) = 0$, where $\sigma_S$ denotes the measure of (reduced) maximal spectral type of the unitary $U_S$ (see (3)). It follows that $T$ is weakly mixing if and only $T \times S$ is ergodic for every ergodic probability preserving $S$. While in the finite measure-preserving case this implies that $T \times T$ is ergodic, it was shown in [5] that there exits a weakly mixing nonsingular $T$ with $T \times T$ not conservative, hence not ergodic. In [11], a weakly mixing $T$ was constructed with $T \times T$ conservative but not ergodic. A nonsingular transformation $T$ is said to be *doubly ergodic* if for all sets of positive measure $A$ and $B$ there exists an integer $n > 0$ such that $\mu(A \cap T^{-n}A) > 0$ and $\mu(A \cap T^{-n}B) > 0$. Furstenberg [66] showed that for finite measure-preserving transformations double ergodicity is equivalent to weak mixing. In [20] it is shown that for nonsingular transformations weak mixing does not imply double ergodicity and double ergodicity does not imply that $T \times T$ is ergodic.

$T$ is said to have *ergodic index* $k$ if the Cartesian product of $k$ copies of $T$ is ergodic but the product of $k+1$ copies of $T$ is not ergodic. If all finite Cartesian products of $T$ are ergodic then $T$ is said to have *infinite ergodic index*. Parry and Kakutani [113] constructed for each $k \in \mathbb{N} \cup \{\infty\}$, an infinite Markov shift of ergodic index $k$. A stronger property is *power weak mixing*, which requires that for all nonzero integers $k_1, \ldots, k_r$ the product $T^{k_1} \times \cdots \times T^{k_r}$ is ergodic [47]. The following examples were constructed in [12], [36], [38]:

(i) power weakly mixing rank-one transformations,
(ii) non-power weakly mixing rank-one transformations with infinite ergodic index,
(iii) non-power weakly mixing rank-one transformations with infinite ergodic index and such that $T^{k_1} \times \cdots \times T^{k_r}$ are all conservative, $k_1, \ldots, k_r \in \mathbb{Z}$,

of types $\mathrm{II}_\infty$ and III (and various subtypes of III, see Section 6). Thus we have the following scale of properties (equivalent to weak mixing in the probability preserving case), where every next property is strictly stronger than the previous ones:

$$T \text{ is weakly mixing} \Leftarrow T \text{ is doubly ergodic} \Leftarrow T \times T \text{ is ergodic} \Leftarrow T \times T \times T \text{ is ergodic}$$
$$\Leftarrow \cdots \Leftarrow T \text{ has infinite ergodic index} \Leftarrow T \text{ is power weakly mixing.}$$

We also mention a recent example of a power weakly mixing transformation of type $\mathrm{II}_\infty$ which embeds into a flow [46].

We now consider several attempts to generalize the notion of (strong) mixing. Given a sequence of measurable sets $\{A_n\}$ let $\sigma_k(\{A_n\})$ denote the $\sigma$-algebra generated by $A_k, A_{k+1}, \ldots$. A sequence $\{A_n\}$ is said to be *remotely trivial* if $\bigcap_{k=0}^\infty \sigma_k(\{A_n\}) = \{\emptyset, X\} \mod \mu$, and it is *semi-remotely trivial* if every subsequence contains a subsequence that is remotely trivial. Krengel and Sucheston [124] define a nonsingular transformation $T$ of a $\sigma$-finite measure space to be *mixing* if for every set $A$ of finite measure the sequence $\{T^{-n}A\}$ is semi-remotely trivial, and *completely mixing* if $\{T^{-n}A\}$ is semi-remotely trivial for all measurable sets $A$. They show that $T$ is completely mixing if and only if it is type $\mathrm{II}_1$ and mixing for the equivalent finite invariant measure. Thus there are no type III and $\mathrm{II}_\infty$ completely mixing nonsingular transformations on probability spaces. We note that this definition of mixing in infinite measure spaces depends on the choice of measure inside the equivalence class (but it is independent if we replace the measure by an equivalent measure with the same collection of sets of finite measure).

Hajian and Kakutani showed [79] that an ergodic infinite measure-preserving transformation $T$ is either of *zero type*: $\lim_{n\to\infty} \mu(T^{-n}A \cap A) = 0$ for all sets $A$ of finite measure, or of *positive type*: $\limsup_{n\to\infty} \mu(T^{-n}A \cap A) > 0$ for all sets $A$ of finite positive measure. $T$ is mixing if and only if it is of zero type [124]. For $0 \le \alpha \le 1$ Kakutani suggested a related definition of $\alpha$-type: an infinite measure preserving transformation is *of $\alpha$-type* if

$\limsup_{n\to\infty}\mu(A\cap T^nA)=\alpha\mu(A)$ for every subset $A$ of finite measure. In [153] examples of ergodic transformations of any $\alpha$-type and a transformation of not any type were constructed.

It may seem that mixing is stronger than any kind of nonsingular weak mixing considered above. However, it is not the case: if $T$ is a weakly mixing infinite measure preserving transformation of zero type and $S$ is an ergodic probability preserving transformation then $T\times S$ is ergodic and of zero type. On the other hand, the $L^\infty$-spectrum $e(T\times S)$ is nontrivial, i.e. $T\times S$ is not weakly mixing, whenever $S$ is not weakly mixing. We also note that there exist rank-one infinite measure-preserving transformations $T$ of zero type such that $T\times T$ is not conservative (hence not ergodic) [11]. In contrast to that, if $T$ is of positive type all of its finite Cartesian products are conservative [7]. Another result that suggests that there is no good definition of mixing in the nonsingular case was proved recently in [106]. It is shown there that while the mixing finite measure-preserving transformations are measurably sensitive, there exists no infinite measure-preserving system that is measurably sensitive. (Measurable sensitivity is a measurable version of the strong sensitive dependence on initial conditions—a concept from topological theory of chaos.)

A nonsingular transformation $T$ of $(X,\mathcal{B},\mu)$ is called *$K$-automorphism* [180] if there exists a sub-$\sigma$-algebra $\mathcal{F}\subset\mathcal{B}$ such that $T^{-1}\mathcal{F}\subset\mathcal{F}$, $\bigcap_{k\geq 0}T^{-k}\mathcal{F}=\{\emptyset,X\}$, $\bigvee_{k=0}^{+\infty}T^k\mathcal{F}=\mathcal{B}$ and the Radon-Nikodym derivative $d\mu\circ T/d\mu$ is $\mathcal{F}$-measurable (see also [156] for the case when $T$ is of type $\mathrm{II}_\infty$; the authors in [180] required $T$ to be conservative). Evidently, a nonsingular Bernoulli transformation (see § 3.5) is a $K$-automorphism. Parry [156] showed that a type $\mathrm{II}_\infty$ $K$-automorphism is either dissipative or ergodic. Krengel [121] proved the same for a class of Bernoulli nonsingular transformations, and finally Silva and Thieullen extended this result to nonsingular $K$-automorphisms [180]. It is also shown in [180] that if $T$ is a nonsingular $K$-automorphism, for any ergodic nonsingular transformation $S$, if $S\times T$ is conservative, then it is ergodic. It follows that a conservative nonsingular $K$-automorphism is weakly mixing. However, it does not necessarily have infinite ergodic index [113]. Krengel and Sucheston [124] showed that an infinite measure-preserving conservative $K$-automorphism is mixing.

4.2. **Multiple and polynomial recurrence.** Let $p$ be a positive integer. A nonsingular transformation $T$ is called *$p$-recurrent* if for every subset $B$ of positive measure there exists a positive integer $k$ such that

$$\mu(B\cap T^{-k}B\cap\cdots\cap T^{-kp}B)>0.$$

If $T$ is $p$-recurrent for any $p>0$, then it is called *multiply recurrent*. It is easy to see that $T$ is 1-recurrent if and only if it is conservative. $T$ is called *rigid* if $T^{n_k}\to\mathrm{Id}$ for a sequence $n_k\to\infty$. Clearly, if $T$ is rigid then it is multiply recurrent. Furnstenberg showed [66] that every finite measure-preserving transformation is multiply recurrent. In contrast to that Eigen, Hajian and Halverson [60] constructed for any $p\in\mathbb{N}\cup\{\infty\}$, a nonsingular odometer of type $\mathrm{II}_\infty$ which is $p$-recurrent but not $(p+1)$-recurrent. Aaronson and Nakada showed in [7] that an infinite measure preserving Markov shift $T$ is $p$-recurrent if and only if the product $T\times\cdots\times T$ ($p$ times) is conservative. It follows from this and [5] that in the class of ergodic Markov shifts infinite ergodic index implies multiple recurrence. However, in general this is not true. It was shown in [12], [78] and [45] that for each $p\in\mathbb{N}\cup\{\infty\}$ there exist

(i) power weakly mixing rank-one transformations and

(ii) non-power weakly mixing rank-one transformations with infinite ergodic index which are $p$-recurrent but not $(p+1)$-recurrent (the latter holds when $p \neq \infty$, of course).

A subset $A$ is called *$p$-wandering* if $\mu(A \cap T^k A \cap \cdots \cap T^{pk} A) = 0$ for each $k$. Aaronson and Nakada established in [7] a $p$-analogue of Hopf decomposition (see Theorem 2.2).

**Proposition 4.1.** *If $(X, \mathcal{B}, \mu, T)$ is conservative aperiodic nonsingular dynamical system and $p \in \mathbb{N}$ then $X = C_p \cup D_p$, where $C_p$ and $D_p$ are $T$-invariant disjoint subsets, $D_p$ is a countable union of $p$-wandering sets, $T \restriction C_p$ is $p$-recurrent and $\sum_{k=1}^{\infty} \mu(B \cap T^{-k} B \cap \cdots \cap T^{-dk} B) = \infty$ for every $B \subset C_p$.*

Let $T$ be an infinite measure-preserving transformation and let $\mathcal{F}$ be a $\sigma$-finite factor (i.e., invariant subalgebra) of $T$. Inoue [102] showed that for each $p > 0$, if $T \restriction \mathcal{F}$ is $p$-recurrent then so is $T$ provided that the extension $T \to T \restriction \mathcal{F}$ is isometric. It is unknown yet whether the latter assumption can be dropped. However, partial progress was recently achieved in [140]: if $T \restriction \mathcal{F}$ is multiply recurrent then so is $T$.

Let $\mathcal{P} := \{q \in \mathbb{Q}[t] \mid q(\mathbb{Z}) \subset \mathbb{Z} \text{ and } q(0) = 0\}$. An ergodic conservative nonsingular transformation $T$ is called *$p$-polynomially recurrent* if for every $q_1, \dots, q_p \in \mathcal{P}$ and every subset $B$ of positive measure there exists $k \in \mathbb{N}$ with

$$\mu(B \cap T^{q_1(k)} B \cap \cdots \cap T^{q_p(k)} B) > 0.$$

If $T$ is $p$-polynomially recurrent for every $p \in \mathbb{N}$ then it is called *polynomially recurrent*. Furstenberg's theorem on multiple recurrence was significantly strengthened in [17], where it was shown that every finite measure-preserving transformation is polynomially recurrent. However, Danilenko and Silva [45] constructed

(i) nonsingular transformations $T$ which are $p$-polynomially recurrent but not $(p+1)$-polynomially recurrent (for each fixed $p \in \mathbb{N}$),
(ii) polynomially recurrent transformations $T$ of type $\mathrm{II}_\infty$,
(iii) rigid (and hence multiply recurrent) transformations $T$ which are not polynomially recurrent.

Moreover, such $T$ can be chosen inside the class of rank-one transformations with infinite ergodic index.

## 5. Topological group $\mathrm{Aut}(X, \mu)$

Let $(X, \mathcal{B}, \mu)$ be a standard probability space and let $\mathrm{Aut}(X, \mu)$ denote the group of all nonsingular transformations of $X$. Let $\nu$ be a finite or $\sigma$-finite measure equivalent to $\mu$; the subgroup of the $\nu$-preserving transformations is denoted by $\mathrm{Aut}_0(X, \nu)$. Then $\mathrm{Aut}(X, \mu)$ is a simple group [58] and it has no outer automorphisms [59]. Ryzhikov showed [169] that every element of this group is a product of three involutions (i.e. transformations of order 2). Moreover, a nonsingular transformation is a product of two involutions if and only if it is conjugate to its inverse by an involution.

Inspired by [81], Ionescu Tulcea [103] and Chacon and Friedman [21] introduced the *weak* and the *uniform* topologies respectively on $\mathrm{Aut}(X, \mu)$. The weak one—we denote it by $d_w$—is induced from the weak operator topology on the group of unitary operators in $L^2(X, \mu)$ by the embedding $T \mapsto U_T$ (see § 2.3). Then $(\mathrm{Aut}(X, \mu), d_w)$ is a Polish topological group and $\mathrm{Aut}_0(X,\nu)$ is a closed subgroup of $\mathrm{Aut}(X,\mu)$. This topology will not be affected if we replace $\mu$ with any equivalent measure. We note that $T_n$ weakly converges to $T$ if and only if $\mu(T_n^{-1}A\triangle T^{-1}A)\to 0$ for each $A\in\mathcal{B}$ and $d(\mu\circ T_n)/d\mu\to d(\mu\circ T)/d\mu$ in $L^1(X,\mu)$. Danilenko showed in [34] that $(\mathrm{Aut}(X,\mu),d_w)$ is contractible. It follows easily from the Rokhlin lemma that periodic transformations are dense in $\mathrm{Aut}(X,\mu)$.

For each $p\geq 1$, one can also embed $\mathrm{Aut}(X,\mu)$ into the isometry group of $L^p(X,\mu)$ via a formula similar to (3) but with another power of the Radon-Nikodym derivative in it. The strong operator topology on the isometry group induces the very same weak topology on $\mathrm{Aut}(X,\mu)$ for all $p\geq 1$ [24].

It is natural to ask which properties of nonsingular transformations are typical in the sense of Baire category. The following technical lemma (see see [64], [24]) is an indispensable tool when considering such problems.

**Lemma 5.1.** *The conjugacy class of each aperiodic transformation $T$ is dense in* $\mathrm{Aut}(X,\mu)$ *endowed with the weak topology.*

Using this lemma and the Hurewicz ergodic theorem Choksi and Kakutani [24] proved that the ergodic transformations form a dense $G_\delta$ in $\mathrm{Aut}(X,\mu)$. The same holds for the subgroup $\mathrm{Aut}_0(X,\nu)$ ([170] and [24]). Combined with [103] the above implies that the ergodic transformations of type III is a dense $G_\delta$ in $\mathrm{Aut}(X,\mu)$. For further refinement of this statement we refer to Section 6.

Since the map $T\mapsto T\times\cdots\times T$ ($p$ times) from $\mathrm{Aut}(X,\mu)$ to $\mathrm{Aut}(X^p,\mu^{\otimes p})$ is continuous for each $p>0$, we deduce that the set $\mathcal{E}_\infty$ of transformations with infinite ergodic index is a $G_\delta$ in $\mathrm{Aut}(X,\mu)$. It is non-empty by [113]. Since this $\mathcal{E}_\infty$ is invariant under conjugacy, it is dense in $\mathrm{Aut}(X,\mu)$ by Lemma 5.1. Thus we obtain that $\mathcal{E}_\infty$ is a dense $G_\delta$. In a similar way one can show that $\mathcal{E}_\infty\cap\mathrm{Aut}_0(X,\nu)$ is a dense $G_\delta$ in $\mathrm{Aut}_0(X,\nu)$ (see also [170], [24], [26] for original proofs of these claims).

The rigid transformations form a dense $G_\delta$ in $\mathrm{Aut}(X,\mu)$. It follows that the set of multiply recurrent nonsingular transformations is residual [13]. A finer result was established in [45]: the set of polynomially recurrent transformations is also residual.

Given $T\in\mathrm{Aut}(X,\mu)$, we denote the *centralizer* $\{S\in\mathrm{Aut}(X,\mu)\mid ST=TS\}$ of $T$ by $C(T)$. Of course, $C(T)$ is a closed subgroup of $\mathrm{Aut}(X,\mu)$ and $C(T)\supset\{T^n\mid n\in\mathbb{Z}\}$. The following problems solved recently (by the efforts of many authors) for probability preserving systems are still open for the nonsingular case. Are the properties:

(i) $T$ has square root;
(ii) $T$ embeds into a flow;
(iii) $T$ has non-trivial invariant sub-$\sigma$-algebra;
(iv) $C(T)$ contains a torus of arbitrary dimension

typical (residual) in $\mathrm{Aut}(X,\mu)$?

The *uniform* topology on $\mathrm{Aut}(X,\mu)$, finer than $d_w$, is defined by the metric

$$d_u(T,S)=\mu(\{x:Tx\neq Sx\})+\mu(\{x:T^{-1}x\neq S^{-1}x\}).$$

This topology is also complete metric. It depends only on the measure class of $\mu$. However the uniform topology is not separable and that is why it is of less importance in ergodic theory. We refer to [21], [64], [24] and [27] for the properties of $d_u$.

## 6. Orbit theory

Orbit theory is, in a sense, the most complete part of nonsingular ergodic theory. We present here the seminal Krieger's theorem on orbit classification of ergodic nonsingular transformations in terms of ratio sets and associated flows. Examples of transformations of various types $\mathrm{III}_\lambda$, $0 \le \lambda \le 1$ are also given here. Next, we consider the outer conjugacy problem for automorphisms of the orbit equivalence relations. This problem is solved in terms of a simple complete system of invariants. We discuss also a general theory of cocycles (of nonsingular systems) taking values in locally compact Polish groups and present an important orbit classification theorem for cocycles. This theorem is an analogue of the aforementioned result of Krieger. We complete the section by considering ITPFI-systems and their relation to AT-flows.

6.1. **Full groups. Ratio set and types $\mathrm{III}_\lambda$, $0 \le \lambda \le 1$.** Let $T$ be a nonsingular transformation of a standard probability space $(X, \mathcal{B}, \mu)$. Denote by $\mathrm{Orb}_T(x)$ the $T$-orbit of $x$, i.e. $\mathrm{Orb}_T(x) = \{T^n x \mid n \in \mathbb{Z}\}$. The *full group* $[T]$ of $T$ consists of all transformations $S \in \mathrm{Aut}(X, \mu)$ such that $Sx \in \mathrm{Orb}_T(x)$ for a.a. $x$. If $T$ is ergodic then $[T]$ is topologically simple (or even algebraically simple if $T$ is not of type $\mathrm{II}_\infty$) [58]. It is easy to see that $[T]$ endowed with the uniform topology $d_u$ is a Polish group. If $T$ is ergodic then $([T], d_u)$ is contractible [34].

The *ratio set* $r(T)$ of $T$ was defined by Krieger [Kr70] and as we shall see below it is the key concept in the orbit classification (see Definition 2.11). The ratio set is a subset of $[0, +\infty)$ defined as follows: $t \in r(T)$ if and only if for every $A \in \mathcal{B}$ of positive measure and each $\epsilon > 0$ there is a subset $B \subset A$ of positive measure and an integer $k \neq 0$ such that $T^k B \subset A$ and $|\omega_k^\mu(x) - t| < \epsilon$ for all $x \in B$. It is easy to verify that $r(T)$ depends only on the equivalence class of $\mu$ and not on $\mu$ itself. A basic fact is that $1 \in r(T)$ if and only if $T$ is conservative. Assume now $T$ to be conservative and ergodic. Then $r(T) \cap (0, +\infty)$ is a closed subgroup of the multiplicative group $(0, +\infty)$. Hence $r(T)$ is one of the following sets:

(i) $\{1\}$;
(ii) $\{0, 1\}$; in this case we say that $T$ is of *type* $\mathit{III}_0$,
(iii) $\{\lambda^n \mid n \in \mathbb{Z}\} \cup \{0\}$ for $0 < \lambda < 1$; then we say that $T$ is of *type* $\mathit{III}_\lambda$,
(iv) $[0, +\infty)$; then we say that $T$ is of *type* $\mathit{III}_1$.

Krieger showed that $r(T) = \{1\}$ if and only if $T$ is of type II. Hence we obtain a further subdivision of type III into subtypes $\mathrm{III}_0$, $\mathrm{III}_\lambda$, or $\mathrm{III}_1$.

**Example 6.1.** (i) Fix $\lambda \in (0, 1)$. Let $\nu_n(0) := 1/(1 + \lambda)$ and $\nu_n(1) := \lambda/(1 + \lambda)$ for all $n = 1, 2, \ldots$. Let $T$ be the nonsingular odometer associated with the sequence $(2, \nu_n)_{n=1}^\infty$ (see § 3.1). We claim that $T$ is of type $\mathrm{III}_\lambda$. Indeed, the group $\Sigma$ of finite permutations of $\mathbb{N}$ acts on $X$ by $(\sigma x)_n = x_{\sigma^{-1}(n)}$, for all $n \in \mathbb{N}$, $\sigma \in \Sigma$ and $x = (x_n)_{n=1}^\infty \in X$. This action preserves $\mu$. Moreover, it is ergodic by the Hewitt-Savage 0-1 law. It remains to notice that $(d\mu \circ T/d\mu)(x) = \lambda$ on the cylinder $[0]$ which is of positive measure.

(ii) Fix positive reals $\rho_1$ and $\rho_2$ such that $\log \rho_1$ and $\log \rho_2$ are rationally independent. Let $\nu_n(0) := 1/(1 + \rho_1 + \rho_2)$, $\nu_n(1) := \rho_1/(1 + \rho_1 + \rho_2)$ and $\nu_n(2) := \rho_2/(1 + \rho_1 + \rho_2)$ for all

$n = 1, 2, \ldots$. Then the nonsingular odometer associated with the sequence $(3, \nu_n)_{n=1}^{\infty}$ is of type $\mathrm{III}_1$. This can be shown in a similar way as (i).

Non-singular odometer of type $\mathrm{III}_0$ will be constructed in Example 6.3 below.

6.2. **Maharam extension, associated flow and orbit classification of type III systems.** On $X \times \mathbb{R}$ with the $\sigma$-finite measure $\mu \times \kappa$, where $d\kappa(y) = \exp(y)dy$, consider the transformation

$$\widetilde{T}(x, y) := (Tx, y - \log \frac{d\mu \circ T}{d\mu}(x)).$$

We call it the *Maharam extension* of $T$ (see [136], where these transformations were introduced). It is measure-preserving and it commutes with the flow $S_t(x, y) := (x, y + t)$, $t \in \mathbb{R}$. It is conservative if and only if $T$ is conservative [136]. However $\widetilde{T}$ is not necessarily ergodic. Let $(Z, \nu)$ denote the space of $\widetilde{T}$-ergodic components. Then $(S_t)_{t\in\mathbb{R}}$ acts nonsingularly on this space. The restriction of $(S_t)_{t\in\mathbb{R}}$ to $(Z, \nu)$ is called the *associated flow* of $T$. The associated flow is ergodic whenever $T$ is ergodic. It is easy to verify that the isomorphism class of the associated flow is an invariant of the orbit equivalence of the underlying system.

**Proposition 6.2** ([86]). (i) *$T$ is of type II if and only if its associated flow is the translation on $\mathbb{R}$, i.e. $x \mapsto x + t$, $x, t \in \mathbb{R}$,*

(ii) *$T$ is of type $III_\lambda$, $0 \le \lambda < 1$ if and only if its associated flow is the periodic flow on the interval $[0, -\log \lambda)$, i.e. $x \mapsto x + t \mod (-\log \lambda)$,*

(iii) *$T$ is of type $III_1$ if and only if its associated flow is the trivial flow on a singleton or, equivalently, $\widetilde{T}$ is ergodic,*

(iv) *$T$ is of type $III_0$ if and only if its associated flow is nontransitive.*

**Example 6.3.** Let $A_n = \{0, 1, \ldots, 2^{2^n}\}$ and $\nu_n(0) = 0.5$ and $\nu_n(i) = 0.5 \cdot 2^{-2^n}$ for all $0 < i \le 2^n$. Let $T$ be the nonsingular odometer associated with $(2^{2^n} + 1, \nu_n)_{n=0}^{\infty}$. It is straightforward that the associated flow of $T$ is the flow built under the constant function 1 with the probability preserving 2-adic odometer (associated with $(2, \kappa_n)_{n=1}^{\infty}$, $\kappa_n(0) = \kappa_n(1) = 0.5$) as the base transformation. In particular, $T$ is of type $\mathrm{III}_0$.

A natural problem arises: to compute Krieger's type (or the ratio set) for the nonsingular odometers—the simplest class of nonsingular systems. Some partial progress was achieved in [141], [152], [50], etc. However in the general setting this problem remains open.

The map $\Psi : \mathrm{Aut}(X, \mu) \ni T \mapsto \widetilde{T} \in \mathrm{Aut}(X \times \mathbb{R}, \mu \times \kappa)$ is a continuous group homomorphism. Since the set $\mathcal{E}$ of ergodic transformations on $X \times \mathbb{R}$ is a $G_\delta$ in $\mathrm{Aut}(X \times \mathbb{R}, \mu \times \kappa)$ (See § 5), the subset $\Psi^{-1}(\mathcal{E})$ of type $\mathrm{III}_1$ ergodic transformations on $X$ is also $G_\delta$. The latter subset is non-empty in view of Example 6.1(ii). Since it is invariant under conjugacy, we deduce from Lemma 5.1 that the set of ergodic transformations of type $\mathrm{III}_1$ is a dense $G_\delta$ in $(\mathrm{Aut}(X, \mu), d_w)$ ([159], [23]).

Now we state the main result of this section—Krieger's theorem on orbit classification for ergodic transformations of type III. It is a far reaching generalization of the basic result by H. Dye: any two ergodic probability preserving transformations are orbit equivalent [56].

**Theorem 6.4** (Orbit equivalence for type III systems [125]—[129]). *Two ergodic transformations of type III are orbit equivalent if and only if their associated flows are isomorphic.*

*In particular, for a fixed* $0 < \lambda \le 1$, *any two ergodic transformations of type* $III_\lambda$ *are orbit equivalent.*

The original proof of this theorem is rather complicated. Simpler treatment of it can be found in [86] and [117].

We also note that every free ergodic flow can be realized as the associated flow of a type $\mathrm{III}_0$ transformation. However it is somewhat easier to construct a $\mathbb{Z}^2$-action of type $\mathrm{III}_0$ whose associated flow is the given one. For this, we take an ergodic nonsingular transformation $Q$ on a probability space $(Z, \mathcal{B}, \lambda)$ and a measure-preserving transformation $R$ of an infinite $\sigma$-finite measure space $(Y, \mathcal{F}, \nu)$ such that there is a continuous homomorphism $\pi : \mathbb{R} \to C(R)$ with $(d\nu \circ \pi(t)/d\nu)(y) = \exp(t)$ for a.a. $y$ (for instance, take a type $\mathrm{III}_1$ transformation $T$ and put $R := \widetilde{T}$ and $\pi(t) := S_t$). Let $\varphi : Z \to \mathbb{R}$ be a Borel map with $\inf_Z \varphi > 0$. Define two transformations $R_0$ and $Q_0$ of $(Z \times Y, \lambda \times \nu)$ by setting:

$$R_0(x, y) := (x, Ry), \quad Q_0(x, y) = (Qx, U_x y),$$

where $U_x = \pi(\varphi(x) - \log(d\mu \circ Q/d\mu)(x))$. Notice that $R_0$ and $Q_0$ commute. The corresponding $\mathbb{Z}^2$-action generated by these transformations is ergodic. Take any transformation $V \in \mathrm{Aut}(Z \times Y, \lambda \times \nu)$ whose orbits coincide with the orbits of the $\mathbb{Z}^2$-action. (According to [29], any ergodic nonsingular action of any countable amenable group is orbit equivalent to a single transformation.) Then $V$ is of type $\mathrm{III}_0$. It is now easy to verify that the associated flow of $V$ is the special flow built under $\varphi \circ Q^{-1}$ with the base transformation $Q^{-1}$. Since $Q$ and $\varphi$ are arbitrary, we deduce the following from Theorem 2.13.

**Theorem 6.5.** *Every nontransitive ergodic flow is an associated flow of an ergodic transformation of type* $III_0$.

In [129] Krieger introduced a map $\Phi$ as follows. Let $T$ be an ergodic transformation of type $\mathrm{III}_0$. Then the associated flow of $T$ is a flow built under function with a base transformation $\Phi(T)$. We note that the orbit equivalence class of $\Phi(T)$ is well defined by the orbit equivalent class of $T$. If $\Phi^n(T)$ fails to be of type $\mathrm{III}_0$ for some $1 \le n < \infty$ then $T$ is said to *belong to Krieger's hierarchy.* For instance, the transformation constructed in Example 6.3 belongs to Krieger's hierarchy. Connes gave in [28] an example of $T$ such that $\Phi(T)$ is orbit equivalent to $T$ (see also [86] and [69]). Hence $T$ is not in Krieger's hierarchy.

6.3. **Normalizer of the full group. Outer conjugacy problem.** Let

$$N[T] = \{R \in \mathrm{Aut}(X, \mu) \mid R[T]R^{-1} = [T]\},$$

i.e. $N[T]$ is the *normalizer* of the full group $[T]$ in $\mathrm{Aut}_(X, \mu)$. We note that a transformation $R$ belongs to $N[T]$ if and only if $R(\mathrm{Orb}_T(x)) = \mathrm{Orb}_T(Rx)$ for a.a. $x$. To define a topology on $N[T]$ consider the $T$-orbit equivalence relation $\mathcal{R}_T \subset X \times X$ and a $\sigma$-finite measure $\mu_{\mathcal{R}}$ on $\mathcal{R}_T$ given by $\mu_{\mathcal{R}_T} = \int_X \sum_{y \in \mathrm{Orb}_T(x)} \delta_{(x,y)} d\mu(x)$. For $R \in N[T]$, we define a transformation $i(R) \in \mathrm{Aut}(\mathcal{R}_T, \mu_{\mathcal{R}_T})$ by setting $i(R)(x, y) := (Rx, Ry)$. Then the map $R \mapsto i(R)$ is an embedding of $N[T]$ into $\mathrm{Aut}(\mathcal{R}_T, \mu_{\mathcal{R}_T})$. Denote by $\tau$ the topology on $N[T]$ induced by the weak topology on $\mathrm{Aut}(\mathcal{R}_T, \mu_{\mathcal{R}_T})$ via $i$ [34]. Then $(N[T], \tau)$ is a Polish group. A sequence $R_n$ converges to $R$ in $N[T]$ if $R_n \to R$ weakly (in $\mathrm{Aut}(X, \mu)$) and $R_n T R_n^{-1} \to RTR^{-1}$ uniformly (in $[T]$).

Given $R \in N[T]$, denote by $\widetilde{R}$ the Maharam extension of $R$. Then $\widetilde{R} \in N[\widetilde{T}]$ and it commutes with $(S_t)_{t\in\mathbb{R}}$. Hence it defines a nonsingular transformation $\bmod R$ on the space $(Z,\nu)$ of the associated flow $W = (W_t)_{t\in\mathbb{R}}$ of $T$. Moreover, $\bmod R$ belongs to the centralizer $C(W)$ of $W$ in $\mathrm{Aut}(Z,\nu)$. Note that $C(W)$ is a closed subgroup of $(\mathrm{Aut}(Z,\nu), d_w)$.

Let $T$ be of type $\mathrm{II}_\infty$ and let $\mu'$ be the invariant $\sigma$-finite measure equivalent to $\mu$. If $R \in N[T]$ then it is easy to see that the Radon-Nikodym derivative $d\mu' \circ R/d\mu'$ is invariant under $T$. Hence it is constant, say $c$. Then $\bmod R = \log c$.

**Theorem 6.6** ([86], [82]). *If $T$ is of type III then the map $\bmod : N[T] \to C(W)$ is a continuous onto homomorphism. The kernel of this homomorphism is the $\tau$-closure of $[T]$. Hence the quotient group $N[T]/\overline{[T]}^\tau$ is (topologically) isomorphic to $C(W)$. In particular, $\overline{[T]}^\tau$ is co-compact in $N[T]$ if and only if $W$ is a finite measure-preserving flow with a pure point spectrum.*

The following theorem describes the homotopical structure of normalizers.

**Theorem 6.7** ([34]). *Let $T$ be of type II or $III_\lambda$, $0 \le \lambda < 1$. The group $\overline{[T]}^\tau$ is contractible. $N[T]$ is homotopically equivalent to $C(W)$. In particular, $N[T]$ is contractible if $T$ is of type II. If $T$ is of type $III_\lambda$ with $0 < \lambda < 1$ then $\pi_1(N[T]) = \mathbb{Z}$.*

The *outer period* $p(R)$ of $R \in N[T]$ is the smallest positive integer $n$ such that $R^n \in [T]$. We write $p(R) = 0$ if no such $n$ exists.

Two transformations $R$ and $R'$ in $N[T]$ are called *outer conjugate* if there are transformations $V \in N[T]$ and $S \in [T]$ such that $VRV^{-1} = R'S$. The following theorem provides convenient (for verification) necessary and sufficient conditions for the outer conjugacy.

**Theorem 6.8** ([30] for type II and [18] for type III). *Transformations $R, R' \in N[T]$ are outer conjugate if and only if $p(R) = p(R')$ and $\bmod R$ is conjugate to $\bmod R'$ in the centralizer of the associated flow for $T$.*

We note that in the case $T$ is of type II, the second condition in the theorem is just $\bmod R = \bmod R'$. It is always satisfied when $T$ is of type $II_1$.

6.4. **Cocycles of dynamical systems. Weak equivalence of cocycles.** Let $G$ be a locally compact Polish group and $\lambda_G$ a left Haar measure on $G$. A Borel map $\varphi : X \to G$ is called a *cocycle* of $T$. Two cocycles $\varphi$ and $\varphi'$ are *cohomologous* if there is a Borel map $b : X \to G$ such that

$$\varphi'(x) = b(Tx)^{-1}\varphi(x)b(x)$$

for a.a. $x \in X$. A cocycle cohomologous to the trivial one is called a *coboundary*. Given a dense subgroup $G' \subset G$, then every cocycle is cohomologous to a cocycle with values in $G'$ [77]. Each cocycle $\varphi$ extends to a (unique) map $\alpha_\varphi : \mathcal{R}_T \to G$ such that $\alpha_\varphi(Tx, x) = \varphi(x)$ for a.a. $x$ and $\alpha_\varphi(x,y)\alpha_\varphi(y,z) = \alpha_\varphi(x,z)$ for a.a. $(x,y),(y,z) \in \mathcal{R}_T$. $\alpha_\varphi$ is called the *cocycle of $\mathcal{R}_T$ generated by $\varphi$*. Moreover, $\varphi$ and $\varphi'$ are cohomologous via $b$ as above if and only if $\alpha_\varphi$ and $\alpha_{\varphi'}$ are *cohomologous* via $b$, i.e. $\alpha_\varphi(x,y) = b(x)^{-1}\alpha_{\varphi'}(x,y)b(y)$ for $\mu_{\mathcal{R}_T}$-a.a. $(x,y) \in \mathcal{R}_T$. The following notion was introduced by Golodets and Sinelshchikov [74], [77]: two cocycles $\varphi$ and $\varphi'$ are *weakly equivalent* if there is a transformation $R \in N[T]$ such that the cocycles $\alpha_\varphi$ and $\alpha'_\varphi \circ (R \times R)$ of $\mathcal{R}_T$ are cohomologous. Let $\mathcal{M}(X,G)$ denote the set of Borel maps from

$X$ to $G$. It is a Polish group when endowed with the topology of convergence in measure. Since $T$ is ergodic, it is easy to deduce from Rokhlin's lemma that the cohomology class of any cocycle is dense in $\mathcal{M}(X, G)$. Given $\varphi \in \mathcal{M}(X, G)$, we define the *$\varphi$-skew product extension* $T_\varphi$ of $T$ acting on $(X \times G, \mu \times \lambda_G)$ by setting $T_\varphi(x, g) := (Tx, \varphi(x)g)$. Thus Maharam extension is (isomorphic to) the Radon-Nikodym cocycle-skew product extension. We now specify some basic classes of cocycles [173], [19], [77], [35]:

(i) $\varphi$ is called *transient* if $T_\varphi$ is of type $I$,
(ii) $\varphi$ is called *recurrent* if $T_\varphi$ is conservative (equivalently, $T_\varphi$ is not transient),
(iii) $\varphi$ *has dense range in* $G$ if $T_\varphi$ is ergodic.
(iv) $\varphi$ is called *regular* if $\varphi$ cobounds with dense range into a closed subgroup $H$ of $G$ (then $H$ is defined up to conjugacy).

These properties are invariant under the cohomology and the weak equivalence. The Radon-Nikodym cocycle $\omega_1$ is a coboundary if and only if $T$ is of type II. It is regular if and only if $T$ is of type II or III$_\lambda$, $0 < \lambda \leq 1$. It has dense range (in the multiplicative group $\mathbb{R}^*_+$) if and only if $T$ is of type III$_1$. Notice that $\omega_1$ is never transient (since $T$ is conservative).

Schmidt introduced in [176] an invariant $R(\varphi) := \{g \in G \mid \varphi - g \text{ is recurrent}\}$. He showed in particular that

(i) $R(\varphi)$ is a cohomology invariant,
(ii) $R(\varphi)$ is a Borel set in $G$,
(iii) $R(\log \omega_1) = \{0\}$ for each aperiodic conservative $T$,
(iv) there are cocycles $\varphi$ such that $R(\varphi)$ and $G \setminus R(\varphi)$ are dense in $G$,
(v) if $\mu(X) = 1$, $\mu \circ T = \mu$ and $\varphi : X \to \mathbb{R}$ is integrable then $R(\varphi) = \{\int \varphi \, d\mu\}$.

We note that (v) follows from Atkinson theorem [15]. A nonsingular version of this theorem was established in [183]: if $T$ is ergodic and $\mu$-nonsingular and $f \in L^1(\mu)$ then

$$\liminf_{n\to\infty} \left| \sum_{j=0}^{n-1} f(T^j x)\omega_j(x) \right| = 0 \quad \text{for a.a. } x$$

if and only if $\int f \, d\mu = 0$.

Since $T_\varphi$ commutes with the action of $G$ on $X \times G$ by inverted right translations along the second coordinate, this action induces an ergodic $G$-action $W_\varphi = (W_\varphi(g))_{g\in G}$ on the space $(Z, \nu)$ of $T_\varphi$-ergodic components. It is called the *Mackey range (or Poincaré flow)* of $\varphi$ [135], [62], [173], [188]. We note that $\varphi$ is regular (and cobounds with dense range into $H \subset G$) if and only if $W_\varphi$ is transitive (and $H$ is the stabilizer of a point $z \in Z$, i.e. $H = \{g \in G \mid W_\varphi(g)z = z\}$). Hence every cocycle taking values in a compact group is regular.

It is often useful to consider the *double cocycle* $\varphi_0 := \varphi \times \omega_1$ instead of $\varphi$. It takes values in the group $G \times \mathbb{R}^*_+$. Since $T_{\varphi_0}$ is exactly the Maharam extension of $T_\varphi$, it follows from [136] that $\varphi_0$ is transient or recurrent if and only if $\varphi$ is transient or recurrent respectively.

**Theorem 6.9** (Orbit classification of cocycles [77])**.** *Let $\varphi, \varphi' : X \to G$ be two recurrent cocycles of an ergodic transformation $T$. They are weakly equivalent if and only if their Mackey ranges $W_{\varphi_0}$ and $W_{\varphi'_0}$ are isomorphic.*

Another proof of this theorem was presented in [61].

**Theorem 6.10.** *Let $T$ be an ergodic nonsingular transformation. Then there is a cocycle of $T$ with dense range in $G$ if and only if $G$ is amenable.*

It follows that if $G$ is amenable then the subset of cocycles of $T$ with dense range in $G$ is a dense $G_\delta$ in $\mathcal{M}(X, G)$ (just adapt the argument following Example 6.3). The 'only if' part of Theorem 6.10 was established in [187]. The 'if' part was considered by many authors in particular cases: $G$ is compact [186], $G$ is solvable or amenable almost connected [75], $G$ is amenable unimodular [104], etc. The general case was proved in [74] and [96] (see also a recent treatment in [9]).

Theorem 6.5 is a particular case of the following result.

**Theorem 6.11** ([76], [61], [10])**.** *Let $G$ be amenable. Let $V$ be an ergodic nonsingular action of $G \times \mathbb{R}^*_+$. Then there is an ergodic nonsingular transformation $T$ and a recurrent cocycle $\varphi$ of $T$ with values in $G$ such that $V$ is isomorphic to the Mackey range of the double cocycle $\varphi_0$.*

Given a cocycle $\varphi \in \mathcal{M}(X, G)$ of $T$, we say that a transformation $R \in N[T]$ is *compatible with* $\varphi$ if the cocycles $\alpha_\varphi$ and $\alpha_\varphi \circ (R \times R)$ of $\mathcal{R}_T$ are cohomologous. Denote by $D(T, \varphi)$ the group of all such $R$. It has a natural Polish topology which is stronger than $\tau$ [41]. Since $[T]$ is a normal subgroup in $D(T, \varphi)$, one can consider the outer conjugacy equivalence relation inside $D(T, \varphi)$. It is called *$\varphi$-outer conjugacy.* Suppose that $G$ is Abelian. Then an analogue of Theorem 6.8 for the $\varphi$-outer conjugacy is established in [41]. Also, the cocycles $\varphi$ with $D(T, \varphi) = N[T]$ are described there.

6.5. **ITPFI transformations and AT-flows.** A nonsingular transformation $T$ is called *ITPFI*[1] if it is orbit equivalent to a nonsingular odometer (associated to a sequence $(m_n, \nu_n)_{n=1}^\infty$, see § 3.1). If the sequence $m_n$ can be chosen bounded then $T$ is called ITPFI of bounded type. If $m_n = 2$ for all $n$ then $T$ is called $\text{ITPFI}_2$. By [70], every ITPFI-transformation of bounded type is $\text{ITPFI}_2$. A remarkable characterization of ITPFI transformations in terms of their associated flows was obtained by Connes and Woods [31]. We first single out a class of ergodic flows. A nonsingular flow $V = (V_t)_{t \in \mathbb{R}}$ on a space $(\Omega, \nu)$ is called *approximate transitive (AT)* if given $\epsilon > 0$ and $f_1, \dots, f_n \in L^1_+(X, \mu)$, there exists $f \in L^1_+(X, \mu)$ and $\lambda_1, \dots, \lambda_n \in L^1_+(\mathbb{R}, dt)$ such that

$$\left|\left| f_j - \int_{\mathbb{R}} f \circ V_t \frac{d\nu \circ V_t}{d\nu} \lambda_j(t) dt \right|\right|_1 < \epsilon$$

for all $1 \le j \le n$. A flow built under a constant ceiling function with a funny rank-one [63] probability preserving base transformation is AT [31]. In particular, each ergodic finite measure-preserving flow with a pure point spectrum is AT.

**Theorem 6.12** ([31])**.** *An ergodic nonsingular transformation is ITPFI if and only if its associated flow is AT.*

The original proof of this theorem was given in the framework of von Neumann algebras theory. A simpler, purely measure theoretical proof was given later in [92] (the 'only if' part)

[1]This abbreviates 'infinite tensor product of factors of type $I$' (came from the theory of von Neumann algebras).

and [84] (the 'if' part). It follows from Theorem 6.12 that every ergodic flow with pure point spectrum is the associated flow of an ITPFI transformation. If the point spectrum of $V$ is $\theta\Gamma$, where $\Gamma$ is a subgroup of $\mathbb{Q}$ and $\theta \in \mathbb{R}$, then $V$ is the associated flow of an $\mathrm{ITPFT}_2$ transformation [87].

**Theorem 6.13** ([51])**.** *Each ergodic nonsingular transformation is orbit equivalent to a Markov odometer (see §3.2).*

The existence of non-ITPFI transformations and ITPFI transformations of unbounded type was shown in [127]. In [52], an explicit example of a non-ITPFI Markov odometer was constructed.

## 7. Smooth nonsingular transformations

Diffeomorphisms of smooth manifolds equipped with smooth measures are commonly considered as physically natural examples of dynamical systems. Therefore the construction of smooth models for various dynamical properties is a well established problem of the modern (probability preserving) ergodic theory. Unfortunately, the corresponding 'nonsingular' counterpart of this problem is almost unexplored. We survey here several interesting facts related to the topic.

For $r \in \mathbb{N}\cup\{\infty\}$, denote by $\mathrm{Diff}^r_+(\mathbb{T})$ the group of orientation preserving $C^r$-diffeomorphisms of the circle $\mathbb{T}$. Endow this set with the natural Polish topology. Fix $T \in \mathrm{Diff}^r_+(\mathbb{T})$. Since $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, there exists a $C^1$-function $f : \mathbb{R} \to \mathbb{R}$ such that $T(x+\mathbb{Z}) = f(x)+\mathbb{Z}$ for all $x \in \mathbb{R}$. The *rotation number* $\rho(T)$ of $T$ is the limit $\lim_{n\to\infty}(\underbrace{f \circ \cdots \circ f}_{n \text{ times}})(x) \pmod 1$. The limit exists and does not depend on the choice of $x$ and $f$. It is obvious that $T$ is nonsingular with respect to Lebesgue measure $\lambda_{\mathbb{T}}$. Moreover, if $T \in \mathrm{Diff}^r_+(\mathbb{T})$ and $\rho(T)$ is irrational then the dynamical system $(\mathbb{T}, \lambda_{\mathbb{T}}, T)$ is ergodic [33]. It is interesting to ask: which Krieger's type can such systems have?

Katznelson showed in [114] that the subset of type III $C^\infty$-diffeomorphisms and the subset of type $\mathrm{II}_\infty$ $C^\infty$-diffeomorphisms are dense in $\mathrm{Diff}^\infty_+(\mathbb{T})$. Hawkins and Schmidt refined the idea of Katznelson from [114] to construct, for every irrational number $\alpha \in [0,1)$ which is not of constant type (i.e. in whose continued fraction expansion the denominators are not bounded) a transformation $T \in \mathrm{Diff}^2_+(\mathbb{T})$ which is of type $\mathrm{III}_1$ and $\rho(T) = \alpha$ [93]. It should be mentioned that class $C^2$ in the construction is essential, since it follows from a remarkable result of Herman that if $T \in \mathrm{Diff}^3_+(\mathbb{T})$ then under some condition on $\alpha$ (which determines a set of full Lebesgue measure), $T$ is measure theoretically (and topologically) conjugate to a rotation by $\rho(T)$ [97]. Hence $T$ is of type $II_1$.

In [90], Hawkins shows that every smooth paracompact manifold of dimension $\geq 3$ admits a type $\mathrm{III}_\lambda$ diffeomorphism for every $\lambda \in [0,1]$. This extends a result of Herman [96] on the existence of type $\mathrm{III}_1$ diffeomorphisms in the same circumstances.

It is also of interest to ask: which free ergodic flows are associated with smooth dynamical systems of type $\mathrm{III}_0$? Hawkins proved that any free ergodic $C^\infty$-flow on a smooth, connected, paracompact manifold is the associated flow for a $C^\infty$-diffeomorphism on another manifold (of higher dimension) [91].

A nice result was obtained in [115]: if $T \in \text{Diff}^2_+(\mathbb{T})$ and the rotation number of $T$ has unbounded continued fraction coefficients then $(\mathbb{T}, \lambda_\mathbb{T}, T)$ is ITPFI. Moreover, a converse also holds: given a nonsingular odometer $R$, the set of orientation-preserving $C^\infty$-diffeomorphisms of the circle which are orbit equivalent to $R$ is $C^\infty$-dense in the Polish set of all $C^\infty$-orientation-preserving diffeomorphisms with irrational rotation numbers. In contrast to that, Hawkins constructs in [89] a type $\text{III}_0$ $C^\infty$-diffeomorphism of the 4-dimensional torus which is not ITPFI.

## 8. Spectral theory for nonsingular systems

While the spectral theory for probability preserving systems is developed in depth, the spectral theory of nonsingular systems is still in its infancy. We discuss below some problems related to $L^\infty$-spectrum which may be regarded as an analogue of the discrete spectrum. We also include results on computation of the maximal spectral type of the 'nonsingular' Koopman operator for rank-one nonsingular transformations.

8.1. **$L^\infty$-spectrum and groups of quasi-invariance.** Let $T$ be an ergodic nonsingular transformation of $(X, \mathcal{B}, \mu)$. A number $\lambda \in \mathbb{T}$ belongs to the $L^\infty$-spectrum $e(T)$ of $T$ if there is a function $f \in L^\infty(X, \mu)$ with $f \circ T = \lambda f$. $f$ is called an $L^\infty$-*eigenfunction* of $T$ corresponding to $\lambda$. Denote by $\mathcal{E}(T)$ the group of all $L^\infty$-eigenfunctions of absolute value 1. It is a Polish group when endowed with the topology of converges in measure. If $T$ is of type $II_1$ then the $L^\infty$-eigenfunctions are $L^2(\mu')$-eigenfuctions of $T$, where $\mu'$ is an equivalent invariant probability measure. Hence $e(T)$ is countable. Osikawa constructed in [151] the first examples of ergodic nonsingular transformations with uncountable $e(T)$.

We state now a nonsingular version of the von Neumann-Halmos discrete spectrum theorem. Let $Q \subset \mathbb{T}$ be a countable infinite subgroup. Let $K$ be a compact dual of $Q_d$, where $Q_d$ denotes $Q$ with the discrete topology. Let $k_0 \in K$ be the element defined by $k_0(q) = q$ for all $q \in Q$. Let $R : K \to K$ be defined by $Rk = k + k_0$. The system $(K, R)$ is called a *compact group rotation.* The following theorem was proved in [6].

**Theorem 8.1.** *Assume that the $L^\infty$-eigenfunctions of $T$ generate the entire $\sigma$-algebra $\mathcal{B}$. Then $T$ is isomorphic to a compact group rotation equipped with an ergodic quasi-invariant measure.*

A natural question arises: which subgroups of $\mathbb{T}$ can appear as $e(T)$ for an ergodic $T$?

**Theorem 8.2** ([143], [1])**.** *$e(T)$ is a Borel subset of $\mathbb{T}$ and carries a unique Polish topology which is stronger than the usual topology on $\mathbb{T}$. The Borel structure of $e(T)$ under this topology agrees with the Borel structure inherited from $\mathbb{T}$. There is a Borel map $\psi : e(T) \ni \lambda \mapsto \psi_\lambda \in \mathcal{E}(T)$ such that $\psi_\lambda \circ T = \lambda\psi_\lambda$ for each $\lambda$. Moreover, $e(T)$ is of Lebesgue measure* 0 *and it can have an arbitrary Hausdorff dimension.*

A proper Borel subgroup $E$ of $\mathbb{T}$ is called

(i) *weak Dirichlet* if $\limsup_{n\to\infty} \widehat{\lambda}(n) = 1$ for each finite complex measure $\lambda$ supported on $E$;

(ii) *saturated* if $\limsup_{n\to\infty} |\widehat{\lambda}(n)| \ge |\lambda(E)|$ for each finite complex measure $\lambda$ on $\mathbb{T}$, where $\widehat{\lambda}(n)$ denote the $n$-th Fourier coefficient of $\lambda$.

Every countable subgroup of $\mathbb{T}$ is saturated.

**Theorem 8.3.** *$e(T)$ is $\sigma$-compact in the usual topology on $\mathbb{T}$ [100] and saturated ([139], [100]).*

It follows that $e(T)$ is weak Dirichlet (this fact was established earlier in [175]).

It is not known if every Polish group continuously embedded in $\mathbb{T}$ as a $\sigma$-compact saturated group is the eigenvalue group of some ergodic nonsingular transformation. This is the case for the so-called $H_2$-groups and the groups of quasi-invariance of measures on $\mathbb{T}$ (see below). Given a sequence $n_j$ of positive integers and a sequence $a_j \geq 0$, the set of all $z \in \mathbb{T}$ such that $\sum_{j=1}^{\infty} a_j |1 - z^{n_j}|^2 < \infty$ is a group. It is called an *$H_2$-group*. Every $H_2$-group is Polish in an intrinsic topology stronger than the usual circle topology.

**Theorem 8.4** ([100])**.** (i) *Every $H_2$-group is a saturated (and hence weak Dirichlet) $\sigma$-compact subset of $\mathbb{T}$.*

(ii) *If $\sum_{j=0}^{\infty} a_j = +\infty$ then the corresponding $H_2$-group is a proper subgroup of $\mathbb{T}$.*

(iii) *If $\sum_{j=0}^{\infty} a_j (n_j/n_{j+1})^2 < \infty$ then the corresponding $H_2$-group is uncountable.*

(iv) *Any $H_2$-group is $e(T)$ for an ergodic nonsingular compact group rotation $T$.*

It is an open problem whether every eigenvalue group $e(T)$ is an $H_2$-group. It is known however that $e(T)$ is close 'to be an $H_2$-group': if a compact subset $L \subset \mathbb{T}$ is disjoint from $e(T)$ then there is an $H_2$-group containing $e(T)$ and disjoint from $L$.

**Example 8.5** ([6], see also [151])**.** Let $(X, \mu, T)$ be the nonsingular odometer associated to a sequence $(2, \nu_j)_{j=1}^{\infty}$ Let $n_j$ be a sequence of positive integers such that $n_j > \sum_{i<j} n_i$ for all $j$. For $x \in X$, we put $h(x) := n_{l(x)} - \sum_{j<l(x)} n_j$. Then $h$ is a Borel map from $X$ to the positive integers. Let $S$ be the tower over $T$ with height function $h$ (see § 3.3). Then $e(S)$ is the $H_2$-group of all $z \in \mathbb{T}$ with $\sum_{j=1}^{\infty} \nu_j(0)\nu_j(1)|1 - z^{n_j}|^2 < \infty$.

It was later shown in [100] that if $\sum_{j=1}^{\infty} \nu_j(0)\nu_j(1)(n_j/n_{j+1})^2 < \infty$ then the $L^{\infty}$-eigenfunctions of $S$ generate the entire $\sigma$-algebra, i.e. $S$ is isomorphic (measure theoretically) to a nonsingular compact group rotation.

Let $\mu$ be a finite measure on $\mathbb{T}$. Let $H(\mu) := \{z \in \mathbb{Z} \mid \delta_z * \mu \sim \mu\}$, where $*$ means the convolution of measures. Then $H_\mu$ is a group called the *group of quasi-invariance of* $\mu$. It has a Polish topology whose Borel sets agree with the Borel sets which $H(\mu)$ inherits from $\mathbb{T}$ and the injection map of $H(\mu)$ into $\mathbb{T}$ is continuous. This topology is induced by the weak operator topology on the unitary group in the Hilbert space $L^2(\mathbb{T}, \mu)$ via the map $H(\mu) \ni z \mapsto U_z$, $(U_z f)(x) = \sqrt{(d(\delta_z * \mu)/d\mu)(x)} f(xz)$ for $f \in L^2(\mathbb{T}, \mu)$. Moreover, $H(\mu)$ is saturated [100]. If $\mu(H(\mu)) > 0$ then either $H(\mu)$ is countable or $\mu$ is equivalent to $\lambda_{\mathbb{T}}$ [137].

**Theorem 8.6** ( [6] )**.** *Let $\mu$ be an ergodic with respect to the $H(\mu)$-action by translations on $\mathbb{T}$. Then there is a compact group rotation $(K, R)$ and a finite measure on $K$ quasi-invariant and ergodic under $R$ such that $e(R) = H(\mu)$. Moreover, there is a continuous one-to-one homomorphism $\psi : e(R) \to E(R)$ such that $\psi_\lambda \circ R = \lambda \psi_\lambda$ for all $\lambda \in e(R)$.*

It was shown by Aaronson and Nadkarni [6] that if $n_1 = 1$ and $n_j = a_j a_{j-1} \cdots a_1$ for positive integers $a_j \geq 2$ with $\sum_{j=1}^{\infty} a_j^{-1} < \infty$ then the transformation $S$ from Example 8.5

does not admit a continuous homomorphism $\psi : e(S) \to E(S)$ with $\psi_\lambda \circ T = \lambda\psi_\lambda$ for all $\lambda \in e(S)$. Hence $e(S) \neq H(\mu)$ for any measure $\mu$ satisfying the conditions of Theorem 8.6.

Assume that $T$ is an ergodic nonsingular compact group rotation. Let $\mathcal{B}_0$ be the $\sigma$-algebra generated by a sub-collection of eigenfunctions. Then $\mathcal{B}_0$ is invariant under $T$ and hence a factor (see §10) of $T$. It is not known if every factor of $T$ is of this form. It is not even known whether every factor of $T$ must have non-trivial eigenvalues.

8.2. **Unitary operator associated with a nonsingular system.** Let $(X, \mathcal{B}, \mu, T)$ be a nonsingular dynamical system. In this subsection we consider spectral properties of the unitary operator $U_T$ defined by (3). First, we note that the spectrum of $T$ is the entire circle $\mathbb{T}$ [147]. Next, if $U_T$ has an eigenvector then $T$ is of type $II_1$. Indeed, if there are $\lambda \in \mathbb{T}$ and $0 \neq f \in L^2(X, \mu)$ with $U_T f = \lambda f$ then the measure $\nu$, $d\nu(x) := |f(x)|^2 d\mu(x)$, is finite, $T$-invariant and equivalent to $\mu$. Hence if $T$ is of type III or $\mathrm{II}_\infty$ then the maximal spectral type $\sigma_T$ of $U_T$ is continuous. Another 'restriction' on $\sigma_T$ was recently found in [166]: no Foïaş-Strătilă measure is absolutely continuous with respect to $\sigma_T$ if $T$ is of type $\mathrm{II}_\infty$. We recall that a symmetric measure on $\mathbb{T}$ possesses *Foïaş-Strătilă property* if for each ergodic probability preserving system $(Y, \nu, S)$ and $f \in L^2(Y, \nu)$, if $\sigma$ is the spectral measure of $f$ then $f$ is a Gaussian random variable [134]. For instance, measures supported on Kronecker sets possess this property.

Mixing is an $L^2$-spectral property for type $\mathrm{II}_\infty$ transformations: $T$ is mixing if and only if $\sigma_T$ is a Rajchman measure, i.e. $\widehat{\sigma}_T(n) := \int z^n \, d\sigma_T(z) \to 0$ as $|n| \to \infty$. Also, $T$ is mixing if and only if $n^{-1}\sum_{i=0}^{n-1} U_T^{k_i} \to 0$ in the strong operator topology for each strictly increasing sequence $k_1 < k_2 < \cdots$ [124]. This generalizes a well known theorem of Blum and Hanson for probability preserving maps. For comparison, we note that ergodicity is not an $L^2$-spectral property of infinite measure preserving systems.

Now let $T$ be a rank-one nonsingular transformation associated with a sequence $(r_n, w_n, s_n)_{n=1}^\infty$ as in §3.4.

**Theorem 8.7** ([100], [25])**.** *The spectral multiplicity of $U_T$ is 1 and the maximal spectral type $\sigma_T$ of $U_T$ (up to a discrete measure in the case $T$ is of type $II_1$) is the weak limit of the measures $\rho_k$ defined as follows:*

$$d\rho_k(z) = \prod_{j=1}^{k} w_j(0)|P_j(z)|^2 \, dz,$$

*where $P_j(z) := 1 + \sqrt{w_j(1)/w_j(0)}z^{-R_{1,j}} + \cdots + \sqrt{w_j(m_j-1)/w_j(0)}z^{-R_{r_j-1,j}}$, $z \in \mathbb{T}$, $R_{i,j} := ih_{j-1} + s_j(0) + \cdots + s_j(i)$, $1 \le i \le r_k - 1$ and $h_j$ is the hight of the $j$-th column.*

Thus the maximal spectral type of $U_T$ is given by a so-called *generalized Riesz product.* We refer the reader to [100], [99], [25], [148] for a detailed study of Riesz products: their convergence, mutual singularity, singularity to $\lambda_\mathbb{T}$, etc.

It was shown in [6] that $H(\sigma_T) \supset e(T)$ for any ergodic nonsingular transformation $T$. Moreover, $\sigma_T$ is ergodic under the action of $e(T)$ by translations if $T$ is isomorphic to an ergodic nonsingular compact group rotation. However it is not known:

(i) Whether $H(\sigma_T) = e(T)$ for all ergodic $T$.

(ii) Whether ergodicity of $\sigma_T$ under $e(T)$ implies that $T$ is an ergodic compact group rotation.

The first claim of Theorem 8.7 extends to the rank $N$ nonsingular systems as follows: if $T$ is an ergodic nonsingular transformation of rank $N$ then the spectral multiplicity of $U_T$ is bounded by $N$ (as in the finite measure-preserving case). It is not known whether this claim is true for a more general class of transformations which are defined as rank $N$ but without the assumption that the Radon-Nikodym cocycle is constant on the tower levels.

## 9. Entropy and other invariants

Let $T$ be an ergodic conservative nonsingular transformation of a standard probability space $(X, \mathcal{B}, \mu)$. If $\mathcal{P}$ is a finite partition of $X$, we define the entropy $H(\mathcal{P})$ of $\mathcal{P}$ as $H(\mathcal{P}) = -\sum_{P\in\mathcal{P}} \mu(P)\log\mu(P)$. In the study of measure-preserving systems the classical (Kolmogorov-Sinai) entropy proved to be a very useful invariant for isomorphism [33]. The key fact of the theory is that if $\mu\circ T = \mu$ then the limit $\lim_{n\to\infty} n^{-1}H(\bigvee_{i=1}^n T^{-i}\mathcal{P})$ exists for every $\mathcal{P}$. However if $T$ does not preserve $\mu$, the limit may no longer exist. Some efforts have been made to extend the use of entropy and similar invariants to the nonsingular domain. These include Krengel's entropy of conservative measure-preserving maps and its extension to nonsingular maps, Parry's entropy and Parry's nonsingular version of Shannon-McMillan-Breiman theorem, critical dimension by Mortiss and Dooley, etc. Unfortunately, these invariants are less informative than their classical counterparts and they are more difficult to compute.

9.1. **Krengel's and Parry's entropies.** Let $S$ be a conservative measure-preserving transformation of a $\sigma$-finite measure space $(Y, \mathcal{E}, \nu)$. The *Krengel entropy* [119] of $S$ is defined by

$$h_{\mathrm{Kr}}(S) = \sup\{\nu(E)h(S_E) \mid 0 < \nu(E) < +\infty\},$$

where $h(S_E)$ is the finite measure-preserving entropy of $S_E$. It follows from Abramov's formula for the entropy of induced transformation that $h_{\mathrm{Kr}}(S) = \mu(E)h(S_E)$ whenever $E$ *sweeps out*, i.e. $\bigcup_{i\geq 0} S^{-i}E = X$. A generic transformation from $\mathrm{Aut}_0(X,\mu)$ has entropy 0. Krengel raised a question in [119]: does there exist a zero entropy infinite measure-preserving $S$ and a zero entropy finite measure-preserving $R$ such that $h_{\mathrm{Kr}}(S\times R) > 0$? This problem was recently solved in [44] (a special case was announced by Silva and Thieullen in an October 1995 AMS conference (unpublished)):

(i) if $h_{\mathrm{Kr}}(S) = 0$ and $R$ is distal then $h_{\mathrm{Kr}}(S\times R) = 0$;
(ii) if $R$ is not distal then there is a rank-one transformation $S$ with $h_{\mathrm{Kr}}(S\times R) = \infty$.

We also note that if a conservative $S \in \mathrm{Aut}_0(X,\mu)$ commutes with another transformation $R$ such that $\nu\circ R = c\nu$ for a constant $c \neq 1$ then $h_{\mathrm{Kr}}(S)$ is either 0 or $\infty$ [180].

Now let $T$ be a type III ergodic transformation of $(X, \mathcal{B}, \mu)$. Silva and Thieullen define an entropy $h^*(T)$ of $T$ by setting $h^*(T) := h_{\mathrm{Kr}}(\widetilde{T})$, where $\widetilde{T}$ is the Maharam extension of $T$ (see § 6.2). Since $\widetilde{T}$ commutes with transformations which 'multiply' $\widetilde{T}$-invariant measure, it follows that $h^*(T)$ is either 0 or $\infty$.

Let $T$ be the standard $\mathrm{III}_\lambda$-odometer from Example 6.1(i). Then $h^*(T) = 0$. The same is true for a so-called ternary odometer associated with the sequence $(3, \nu_n)_{n=1}^\infty$, where $\nu_n(0) =$

$\nu_n(2) = \lambda/(1+2\lambda)$ and $\nu_n(1) = \lambda/(1+\lambda)$ [180]. It is not known however whether every ergodic nonsingular odometer has zero entropy. On the other hand, it was shown in [180] that $h^*(T) = \infty$ for every $K$-automorphism.

The Parry entropy [158] of $S$ is defined by

$$h_{\mathrm{Pa}}(S) := \{H(S^{-1}\mathfrak{F}|\mathfrak{F}) \mid \mathfrak{F} \text{ is a } \sigma\text{-finite subalgebra of } \mathfrak{B} \text{ such that } \mathfrak{F} \subset S^{-1}\mathfrak{F}\}.$$

Parry showed [158] that $h_{\mathrm{Pa}}(S) \le h_{\mathrm{Kr}}(S)$. It is still an open question whether the two entropies coincide. This is the case when $S$ is of rank one (since $h_{\mathrm{Kr}}(S) = 0$) and when $S$ is quasi-finite [158]. The transformation $S$ is called *quasi-finite* if there exists a subset of finite measure $A \subset Y$ such that the first return time partition $(A_n)_{n>0}$ of $A$ has finite entropy. We recall that $x \in A_n \iff n$ is the smallest positive integer such that $T^n x \in A$. An example of non-quasi-finite ergodic infinite measure preserving transformation was constructed recently in [8].

9.2. **Parry's generalization of Shannon-MacMillan-Breiman theorem.** Let $T$ be an ergodic transformation of a standard non-atomic probability space $(X, \mathcal{B}, \mu)$. Suppose that $f \circ T \in L^1(X,\mu)$ if and only if $f \in L^1(X,\mu)$. This means that there is $K > 0$ such that $K^{-1} < \frac{d\mu\circ T}{d\mu}(x) < K$ for a.a. $x$. Let $\mathcal{P}$ be a finite partition of $X$. Denote by $C_n(x)$ the atom of $\bigvee_{i=0}^n T^{-i}\mathcal{P}$ which contains $x$. We put $\omega_{-1} = 0$. Parry shows in [155] that

$$\frac{\sum_{j=0}^n \log \mu(C_{n-j}(T^j x))(\omega_j(x) - \omega_{j-1}(x))}{\sum_{i=0}^n \omega_j(x)} \to$$
$$H\Big(P \mid \bigvee_{i=1}^{\infty} T^{-1}P\Big) - \int_X \log E\Big(\frac{d\mu\circ T}{d\mu} \mid \bigvee_{i=0}^{\infty} T^{-i}\mathcal{P}\Big)\, d\mu$$

for a.a. $x$. Parry also shows that under the aforementioned conditions on $T$,

$$\frac{1}{n}\Big(\sum_{j=0}^n H\Big(\bigvee_{i=0}^{j} T^{-j}\mathcal{P}\Big) - \sum_{j=0}^{n-1} H\Big(\bigvee_{i=1}^{j+1} T^{-j}\mathcal{P}\Big)\Big) \to H\Big(\mathcal{P} \mid \bigvee_{i=1}^{\infty} T^{-i}\mathcal{P}\Big).$$

9.3. **Critical dimension.** The critical dimension introduced by Mortiss [146] measures the order of growth for sums of Radon-Nikodym derivatives. Let $(X, \mathcal{B}, \mu, T)$ be an ergodic nonsingular dynamical system. Given $\delta > 0$, let

$$X_\delta := \{x \in X \mid \liminf_{n\to\infty} \frac{\sum_{i=0}^{n-1} \omega_i(x)}{n^\delta} > 0\} \text{ and} \tag{4}$$

$$X^\delta := \{x \in X \mid \liminf_{n\to\infty} \frac{\sum_{i=0}^{n-1} \omega_i(x)}{n^\delta} = 0\}. \tag{5}$$

Then $X_\delta$ and $X^\delta$ are $T$-invariant subsets.

**Definition 9.1** ([146], [53])**.** The *lower critical dimension* $\alpha(T)$ of $T$ is $\sup\{\delta \mid \mu(X_\delta) = 1\}$. The *upper critical dimension* $\beta(T)$ of $T$ is $\inf\{\delta \mid \mu(X^\delta) = 1\}$.

It was shown in [53] that the lower and upper critical dimensions are invariants for isomorphism of nonsingular systems. Notice also that

$$\alpha(T) = \liminf_{n\to\infty} \frac{\log(\sum_{i=1}^n \omega_i(x))}{\log n} \text{ and } \beta(T) = \limsup_{n\to\infty} \frac{\log(\sum_{i=1}^n \omega_i(x))}{\log n}.$$

Moreover, $0 \le \alpha(T) \le \beta(T) \le 1$. If $T$ is of type $\mathrm{II}_1$ then $\alpha(T) = \beta(T) = 1$. If $T$ is the standard $\mathrm{III}_\lambda$-odometer from Example 6.1 then $\alpha(T) = \beta(T) = \log(1+\lambda) - \frac{\lambda}{1+\lambda}\log\lambda$.

**Theorem 9.2.** (i) *For every $\lambda \in [0,1]$ and every $c \in [0,1]$ there exists a nonsingular odometer of type $III_\lambda$ with critical dimension equal to $c$* [145].
(ii) *For every $c \in [0,1]$ there exists a nonsingular odometer of type $II_\infty$ with critical dimension equal to $c$* [53].

Let $T$ be the nonsingular odometer associated with a sequence $(m_n, \nu_n)_{n=1}^\infty$. Let $s(n) = m_1 \cdots m_n$ and let $H(\mathcal{P}_n)$ denote the entropy of the partition of the first $n$ coordinates with respect to $\mu$. We now state a nonsingular version of Shannon-MacMillan-Breiman theorem for $T$ from [53].

**Theorem 9.3.** *Let $m_i$ be bounded from above. Then*
(i) $\alpha(T) = \liminf_{n\to\infty} \inf \frac{-\sum_{i=1}^n \log m_i(x_i)}{\log s(n)} = \liminf_{n\to\infty} \frac{H(\mathcal{P}_n)}{\log s(n)}$ *and*
(ii) $\beta(T) = \limsup_{n\to\infty} \inf \frac{-\sum_{i=1}^n \log m_i(x_i)}{\log s(n)} = \limsup_{n\to\infty} \frac{H(\mathcal{P}_n)}{\log s(n)}$
*for a.a. $x = (x_i)_{i\ge 1} \in X$.*

It follows that in the case when $\alpha(T) = \beta(T)$, the critical dimension coincides with $\lim_{n\to\infty} \frac{H(\mathcal{P}_n)}{\log s(n)}$. In [145] this expression (when it exists) was called *AC-entropy* (average coordinate). It also follows from Theorem 9.3 that if $T$ is an odometer of bounded type then $\alpha(T^{-1}) = \alpha(T)$ and $\beta(T^{-1}) = \beta(T)$. In [54], Theorem 9.3 was extended to a subclass of Markov odometers. The critical dimensions for Hamachi shifts (see § 3.5) were investigated in [55]:

**Theorem 9.4.** *For any $\epsilon > 0$, there exists a Hamachi shift $S$ with $\alpha(S) < \epsilon$ and $\beta(S) > 1-\epsilon$.*

9.4. **Nonsingular restricted orbit equivalence.** In [144] Mortiss initiated study of a nonsingular version of Rudolph's restricted orbit equivalence [167]. This work is still in its early stages and does not yet deal with any form of entropy. However she introduced nonsingular orderings of orbits, defined sizes and showed that much of the basic machinery still works in the nonsingular setting.

## 10. Nonsingular Joinings and Factors

The theory of joinings is a powerful tool to study probability preserving systems and to construct striking counterexamples. It is interesting to study what part of this machinery can be extended to the nonsingular case. However, the definition of nonsingular joining is far from being obvious. Some progress was achieved in understanding 2-fold joinings and constructing prime systems of any Krieger type. As far as we know the higher-fold nonsingular joinings have not been considered so far. It turned out however that an alternative coding technique, predating joinings in studying the centralizer and factors of the classical measure-preserving Chacón maps, can be used as well to classify factors of Cartesian products of some nonsingular Chacón maps.

10.1. **Joinings, nonsingular MSJ and simplicity.** In this section all measures are probability measures. A *nonsingular joining* of two nonsingular systems $(X_1, \mathcal{B}_1, \mu_1, T_1)$ and $(X_2, \mathcal{B}_2, \mu_2, T_2)$ is a measure $\hat{\mu}$ on the product $\mathcal{B}_1 \times \mathcal{B}_2$ that is nonsingular for $T_1 \times T_2$ and satisfies: $\hat{\mu}(A \times X_2) = \mu_1(A)$ and $\hat{\mu}(X_1 \times B) = \mu_2(B)$ for all $A \in \mathcal{B}_1$ and $B \in \mathcal{B}_2$. Clearly, the product $\mu_1 \times \mu_2$ is a nonsingular joining. Given a transformation $S \in C(T)$, the measure $\mu_S$ given by $\mu_S(A \times B) := \mu(A \cap S^{-1}B)$ is a nonsingular joining of $(X, \mu, T)$ and $(X, \mu \circ S^{-1}, T)$. It is called a *graph*-joining since it is supported on the graph of $S$. Another important kind of joinings that we are going to define now is related to factors of dynamical systems. Recall that given a nonsingular system $(X, \mathcal{B}, \mu, T)$, a sub-$\sigma$-algebra $\mathcal{A}$ of $\mathcal{B}$ such that $T^{-1}(\mathcal{A}) = \mathcal{A}$ mod $\mu$ is called a *factor* of $T$. There is another, equivalent, definition. A nonsingular dynamical system $(Y, \mathcal{C}, \nu, S)$ is called a factor of $T$ if there exists a measure-preserving map $\varphi : X \to Y$, called a *factor map*, with $\varphi T = S\varphi$ a.e. (If $\varphi$ is only nonsingular, $\nu$ may be replaced with the equivalent measure $\mu \circ \varphi^{-1}$, for which $\varphi$ is measure-preserving.) Indeed, the sub-$\sigma$-algebra $\varphi^{-1}(\mathcal{C}) \subset \mathcal{B}$ is $T$-invariant and, conversely, any $T$-invariant sub-$\sigma$-algebra of $\mathcal{B}$ defines a factor map by immanent properties of standard probability spaces, see e.g. [3]. If $\varphi$ is a factor map as above, then $\mu$ has a disintegration with respect to $\varphi$, i.e., $\mu = \int \mu_y d\nu(y)$ for a measurable map $y \mapsto \mu_y$ from $Y$ to the probability measures on $X$ so that $\mu_y(\varphi^{-1}(y)) = 1$, the measure $\mu_{S\varphi(x)} \circ T$ is equivalent to $\mu_{\varphi(x)}$ and

$$\frac{d\mu \circ T}{d\mu}(x) = \frac{d\nu \circ S}{d\nu}(\varphi(x)) \frac{d\mu_{S\varphi(x)} \circ T}{d\mu_{\varphi(x)}}(x) \tag{6}$$

for a.e. $x \in X$. Define now the *relative product* $\hat{\mu} = \mu \times_\varphi \mu$ on $X \times X$ by setting $\hat{\mu} = \int \mu_y \times \mu_y \, d\nu(y)$. Then it is easy to deduce from (6) that $\hat{\mu}$ is a nonsingular self-joining of $T$.

We note however that the above definition of joining is not satisfactory since it does not reduce to the classical definition when we consider probability preserving systems. Indeed, the following result was proved in [168].

**Theorem 10.1.** *Let $(X_1, \mathcal{B}_1, \mu_1, T_1)$ and $(X_2, \mathcal{B}_2, \mu_2, T_2)$ be two finite measure-preserving systems such that $T_1 \times T_2$ is ergodic. Then for every $\lambda, 0 < \lambda < 1$, there exists a nonsingular joining $\hat{\mu}$ of $\mu_1$ and $\mu_2$ such that $(T_1 \times T_2, \hat{\mu})$ is ergodic and of type III$_\lambda$.*

It is not known however if the nonsingular joining $\hat{\mu}$ can be chosen in every orbit equivalence class. In view of the above, Rudolph and Silva [168] isolate an important subclass of joining. It is used in the definition of a nonsingular version of minimal self-joinings.

**Definition 10.2.**
(i) A nonsingular joining $\hat{\mu}$ of $(X_1, \mu_1, T_1)$ and $(X_2, \mu_2, T_2)$ is *rational* if there exit measurable functions $c^1 : X_1 \to \mathbb{R}_+$ and $c^2 : X_2 \to \mathbb{R}_+$ such that

$$\hat{\omega}_1^{\hat{\mu}}(x_1, x_2) = \omega_1^{\mu_1}(x_1)\omega_1^{\mu_2}(x_2)c^1(x_1) = \omega_1^{\mu_1}(x_1)\omega_1^{\mu_2}(x_2)c^2(x_2) \quad \hat{\mu} \text{ a.e.}$$

(ii) A nonsingular dynamical system $(X, \mathcal{B}, \mu, T)$ has *minimal self-joinings (MSJ)* over a class $\mathcal{M}$ of probability measures equivalent to $\mu$, if for every $\mu_1, \mu_2 \in \mathcal{M}$, for every rational joining $\hat{\mu}$ of $\mu_1, \mu_2$, a.e. ergodic component of $\hat{\mu}$ is either the product of its marginals or is the graph-joining supported on $T^j$ for some $j \in \mathbb{Z}$.

Clearly, product measure, graph-joinings and the relative products are all rational joinings. Moreover, a rational joining of finite measure-preserving systems is measure-preserving and a rational joining of type $\mathrm{II}_1$'s is of type $\mathrm{II}_1$ [168]. Thus we obtain the finite measure-preserving theory as a special case. As for the definition of MSJ, it depends on a class $\mathcal{M}$ of equivalent measures. In the finite measure-preserving case $\mathcal{M} = \{\mu\}$. However, in the nonsingular case no particular measure is distinguished. We note also that Definition 10.2(ii) involves some restrictions on all rational joinings and not only ergodic ones as in the finite measure-preserving case. The reason is that an ergodic component of a nonsingular joining needs not be a joining of measures equivalent to the original ones [2]. For finite measure-preserving transformations, MSJ over $\{\mu\}$ is the same as the usual 2-fold MSJ [109].

A nonsingular transformation $T$ on $(X, \mathcal{B}, \mu)$ is called *prime* if its only factors are $\mathcal{B}$ and $\{X, \emptyset\} \bmod \mu$. A (nonempty) class $\mathcal{M}$ of probability measures equivalent to $\mu$ is said to be *centralizer stable* if for each $S \in C(T)$ and $\mu_1 \in \mathcal{M}$, the measure $\mu_1 \circ S$ is in $\mathcal{M}$.

**Theorem 10.3** ([168]). *Let $(X, \mathcal{B}, \mu, T)$ be a ergodic non-atomic dynamical system such that $T$ has MSJ over a class $\mathcal{M}$ that is centralizer stable. Then $T$ is prime and the centralizer of $T$ consists of the powers of $T$.*

A question that arises is whether if such nonsingular dynamical system (not of type $\mathrm{II}_1$) exist. Expanding on Ornstein's original construction from [150], Rudolph and Silva construct in [168], for each $0 \leq \lambda \leq 1$, a nonsingular rank-one transformation $T_\lambda$ that is of type $\mathrm{III}_\lambda$ and that has MSJ over a class $\mathcal{M}$ that is centralizer stable. Type $\mathrm{II}_\infty$ examples with analogues properties were also constructed there. In this connection it is worth to mention the example by Aaronson and Nadkarni [6] of $\mathrm{II}_\infty$ ergodic transformations that have no factor algebras on which the invariant measure is $\sigma$-finite (except for the trivial and the entire ones); however these transformations are not prime.

A more general notion than MSJ called *graph self-joinings (GSJ)*, was introduced [181]: just replace the the words "on $T^j$ for some $j \in \mathbb{Z}$" in Definition 10.2(ii) with "on $S$ for some element $S \in C(T)$". For finite measure-preserving transformations, GSJ over $\{\mu\}$ is the same as the usual 2-fold simplicity [109]. The famous Veech theorem on factors of 2-fold simple maps (see [109]) was extended to nonsingular systems in [181] as follows: if a system $(X, \mathcal{B}, \mu, T)$ has GSJ then for every non-trivial factor $\mathcal{A}$ of $T$ there exists a locally compact subgroup $H$ in $C(T)$ (equipped with the weak topology) which acts smoothly (i.e. the partition into $H$-orbits is measurable) and such that $\mathcal{A} = \{B \in \mathcal{B} \mid \mu(hB \triangle B) = 0 \text{ for all } h \in H\}$. It follows that there is a cocycle $\varphi$ from $(X, \mathcal{A}, \mu \restriction \mathcal{A})$ to $H$ such that $T$ is isomorphic to the $\varphi$-skew product extension $(T \restriction A)_\varphi$ (see § 6.4). Of course, the ergodic nonsingular odometers and, more generally, ergodic nonsingular compact group rotation (see § 8.1) have GSJ. However, except for this trivial case (the Cartesian square is non-ergodic) plus the systems with MSJ from [168], no examples of type III systems with GSJ are known. In particular, no smooth examples have been constructed so far. This is in sharp contrast with the finite measure preserving case where abundance of simple (or close to simple) systems are known (see [109], [182], [40], [39]).

10.2. **Nonsingular coding and factors of Cartesian products of nonsingular maps.** As we have already noticed above, the nonsingular MSJ theory was developed in [168] only for

2-fold self-joinings. The reasons for this were technical problems with extending the notion of rational joinings form 2-fold to $n$-fold self-joinings. However while the 2-fold nonsingular MSJ or GSJ properties of $T$ are sufficient to control the centralizer and the factors of $T$, it is not clear whether it implies anything about the factors or centralizer of $T \times T$. Indeed, to control them one needs to know the 4-fold joinings of $T$. However even in the finite measure-preserving case it is a long standing open question whether 2-fold MSJ implies $n$-fold MSJ. That is why del Junco and Silva [111] apply an alternative—nonsingular coding—techniques to classify the factors of Cartesian products of nonsingular Chacón maps. The techniques were originally used in [108] to show that the classical Chacón map is prime and has trivial centralizer. They were extended to nonsingular systems in [110].

For each $0 < \lambda < 1$ we denote by $T_\lambda$ the Chacón map (see § 3.4) corresponding the sequence of probability vectors $w_n = (\lambda/(1+2\lambda), 1/(1+2\lambda), \lambda/(1+2\lambda))$ for all $n > 0$. One can verify that the maps $T_\lambda$ are of type $\mathrm{III}_\lambda$. (The classical Chacón map corresponds to $\lambda = 1$.) All of these transformations are defined on the same standard Borel space $(X, \mathcal{B})$. These transformations were shown to be power weakly mixing in [12]. The centralizer of any finite Cartesian product of nonsingular Chacón maps is computed in the following theorem.

**Theorem 10.4** ([111]). *Let $0 < \lambda_1 < \ldots < \lambda_k \le 1$ and $n_1, \ldots, n_k$ be positive integers. Then the centralizer of the Cartesian product $T_{\lambda_1}^{\otimes n_1} \times \ldots \times T_{\lambda_k}^{\otimes n_k}$ is generated by maps of the form $U_1 \times \ldots \times U_k$, where each $U_i$, acting on the $n_i$-dimensional product space $X^{n_i}$, is a Cartesian product of powers of $T_{\lambda_i}$ or a co-ordinate permutation on $X^{n_i}$.*

Let $\pi$ denote the permutation on $X \times X$ defined by $\pi(x, y) = (y, x)$ and let $\mathcal{B}^{2\odot}$ denote the symmetric factor, i.e. $\mathcal{B}^{2\odot} = \{A \in \mathcal{B} \otimes \mathcal{B} \mid \pi(A) = A\}$. The following theorem classifies the factors of the Cartesian product of any two nonsingular type $\mathrm{III}_\lambda$, $0 < \lambda < 1$, or type $\mathrm{II}_1$ Chacón maps.

**Theorem 10.5** ([111]). *Let $T_{\lambda_1}$ and $T_{\lambda_2}$ be two nonsingular Chacón systems. Let $\mathcal{F}$ be a factor algebra of $T_{\lambda_1} \times T_{\lambda_2}$.*

(i) *If $\lambda_1 \ne \lambda_2$ then $\mathcal{F}$ is equal* mod 0 *to one of the four algebras $\mathcal{B} \otimes \mathcal{B}$, $\mathcal{B} \otimes \mathcal{N}$, $\mathcal{N} \otimes \mathcal{B}$, or $\mathcal{N} \otimes \mathcal{N}$, where $\mathcal{N} = \{\emptyset, X\}$.*
(ii) *If $\lambda_1 = \lambda_2$ then $\mathcal{F}$ is equal* mod 0 *to one of the following algebras $\mathcal{B} \otimes \mathcal{C}$, $\mathcal{B} \otimes \mathcal{N}$, $\mathcal{N} \otimes \mathcal{C}$, $\mathcal{N} \otimes \mathcal{N}$, or $(T^m \times Id)\mathcal{B}^{2\odot}$ for some integer $m$.*

It is not hard to obtain type $\mathrm{III}_1$ examples of Chacón maps for which the previous two theorems hold. However the construction of type $\mathrm{II}_\infty$ and type $\mathrm{III}_0$ nonsingular Chacón transformations is more subtle as it needs the choice of $\omega_n$ to vary with $n$. In [88], Hamachi and Silva construct type $\mathrm{III}_0$ and type $\mathrm{II}_\infty$ examples, however the only property proved for these maps is ergodicity of their Cartesian square. More recently, Danilenko [38] has shown that all of them (in fact, a wider class of nonsingular Chacón maps of all types) are power weakly mixing.

In [22], Choksi, Eigen and Prasad asked whether there exists a zero entropy, finite measure-preserving mixing automorphism $S$, and a nonsingular type III automorphism $T$, such that $T \times S$ has no Bernoulli factors. Theorem 10.5 provides a partial answer (with a mildly mixing only instead of mixing) to this question: if $S$ is the finite measure-preserving Chacón map and $T$ is a nonsingular Chacón map as above, the factors of $T \times S$ are only the trivial ones, so $T \times S$ has no Bernoulli factors.

## 11. Applications. Connections with other fields

In this—final—section we shed light on numerous mathematical sources of nonsingular systems. They come from the theory of stochastic processes, random walks, locally compact Cantor systems, horocycle flows on hyperbolic surfaces, von Neumann algebras, statistical mechanics, representation theory for groups and anticommutation relations, etc. We also note that such systems sometimes appear in the context of probability preserving dynamics (see also a criterium of distality in § 9.1).

11.1. **Mild mixing.** An ergodic finite measure-preserving dynamical system $(X, \mathcal{B}, \mu, T)$ is called *mildly mixing* if for each non-trivial factor algebra $\mathcal{A} \subset \mathcal{B}$, the restriction $T \restriction \mathcal{A}$ is not rigid. For equivalent definitions and extensions to actions of locally compact groups we refer to [3] and [177]. There is an interesting criterium of the mild mixing that involves nonsingular systems: $T$ is mildly mixing if and only if for each ergodic nonsingular transformation $S$, the product $T \times S$ is ergodic [67]. Furthermore, $T \times S$ is then orbit equivalent to $S$ [94]. Moreover, if $R$ is a nonsingular transformation such that $R \times S$ is ergodic for any ergodic nonsingular $S$ then $R$ is of type $\mathrm{II}_1$ (and mildly mixing) [177].

11.2. **Disjointness and Furstenberg's class $\mathcal{W}^{\perp}$.** Two probability preserving systems $(X, \mu, T)$ and $(Y, \nu, S)$ are called *disjoint* if $\mu \times \nu$ is the only $T \times S$-invariant probability measure on $X \times Y$ whose coordinate projections are $\mu$ and $\nu$ respectively. Furstenberg in [65] initiated studying the class $\mathcal{W}^{\perp}$ of transformations disjoint from all weakly mixing ones. Let $\mathcal{D}$ denote the class of distal transformations and $\mathcal{M}(\mathcal{W}^{\perp})$ the class of multipliers of $\mathcal{W}^{\perp}$ (for the definitions see [71]). Then $\mathcal{D} \subset \mathcal{M}(\mathcal{W}^{\perp}) \subset \mathcal{W}^{\perp}$. In [133] and [43] it was shown by constructing explicit examples that these inclusions are strict. We record this fact here because nonsingular ergodic theory was the key ingredient of the arguments in the two papers pertaining to the theory of probability preserving systems. The examples are of the form $T_{\varphi,S}(x, y) = (Tx, S_{\varphi(x)} y)$, where $T$ is an ergodic rotation on $(X, \mu)$, $(S_g)_{g \in G}$ a mildly mixing action of a locally compact group $G$ on $Y$ and $\varphi : X \to G$ a measurable map. Let $W_\varphi$ denote the Mackey action of $G$ associated with $\varphi$ and let $(Z, \kappa)$ be the space of this action. The key observation is that there exists an affine isomorphism of the simplex of $T_{\varphi,S}$-invariant probability measures whose pullback on $X$ is $\mu$ and the simplex of $W_\varphi \times S$ quasi-invariant probability measures whose pullback on $Z$ is $\kappa$ and whose Radon-Nikodym cocycle is measurable with respect to $Z$. This is a far reaching generalization of Furstenberg theorem on relative unique ergodicity of ergodic compact group extensions.

11.3. **Symmetric stable and infinitely divisible stationary processes.** Rosinsky in [163] established a remarkable connection between structural studies of stationary stochastic processes and ergodic theory of nonsingular transformations (and flows). For simplicity we consider only real processes in discrete time. Let $X = (X_n)_{n \in \mathbb{Z}}$ be a measurable stationary symmetric $\alpha$-stable (S$\alpha$S) process, $0 < \alpha < 2$. This means that any linear combination

$\sum_{k=1}^{n} a_k X_{j_k}$, $j_k \in \mathbb{Z}$, $a_k \in \mathbb{R}$ has an S$\alpha$S-distribution. (The case $\alpha = 2$ corresponds to Gaussian processes.) Then the process admits a spectral representation

$$X_n = \int_Y f_n(y)\, M(dy),\ n \in \mathbb{Z}, \tag{7}$$

where $f_n \in L^\alpha(Y, \mu)$ for a standard $\sigma$-finite measure space $(Y, \mathcal{B}, \mu)$ and $M$ is an independently scattered random measure on $\mathcal{B}$ such that $E \exp(iuM(A)) = \exp(-|u|^\alpha \mu(A))$ for every $A \in \mathcal{B}$ of finite measure. By [163], one can choose the kernel $(f_n)_{n\in\mathbb{Z}}$ in a special way: there are a $\mu$-nonsingular transformation $T$ and measurable maps $\varphi : X \to \{-1, 1\}$ and $f \in L^\alpha(Y, \mu)$ such that $f_n = U^n f$, $n \in \mathbb{Z}$, where $U$ is the isometry of $L^\alpha(X, \mu)$ given by $Ug = \varphi \cdot (d\mu \circ T/d\mu)^{1/\alpha} \cdot g \circ T$. If, in addition, the smallest $T$-invariant $\sigma$-algebra containing $f^{-1}(\mathcal{B}_{\mathbb{R}})$ coincides with $\mathcal{B}$ and $\mathrm{Supp}\{f \circ T^n : n \in \mathbb{Z}\} = Y$ then the pair $(T, \varphi)$ is called minimal. It turns out that minimal pairs always exist. Moreover, two minimal pairs $(T, \varphi)$ and $(T', \varphi')$ representing the same S$\alpha$S process are equivalent in some natural sense [163]. Then one can relate ergodic-theoretical properties of $(T, \varphi)$ to probabilistic properties of $(X_n)_{n\in\mathbb{Z}}$. For instance, let $Y = C \sqcup D$ be the Hopf decomposition of $Y$ (see Theorem 2.2). We let $X_n^D := \int_D f_n(y)\, M(dy)$ and $X_n^C := \int_C f_n(y)\, M(dy)$. Then we obtain a unique (in distribution) decomposition of $X$ into the sum $X^D + X^C$ of two independent stationary S$\alpha$S-processes.

Another kind of decomposition was considered in [172]. Let $P$ be the largest invariant subset of $Y$ such that $T \restriction P$ has a finite invariant measure. Partitioning $Y$ into $P$ and $N := Y \setminus N$ and restricting the integration in (7) to $P$ and $N$ we obtain a unique (in distribution) decomposition of $X$ into the sum $X^P + X^N$ of two independent stationary S$\alpha$S-processes. Notice that the process $X$ is ergodic if and only if $\mu(P) = 0$.

Recently, Roy considered a more general class of *infinitely divisible (ID)* stationary processes [165]. Using Maruyama's representation of the characteristic function of an ID process $X$ without Gaussian part he singled out the Lévy measure $Q$ of $X$. Then $Q$ is a shift invariant $\sigma$-finite measure on $\mathbb{R}^{\mathbb{Z}}$. Decomposing the dynamical system $(\mathbb{R}^{\mathbb{Z}}, \tau, Q)$ in various natural ways (Hopf decomposition, 0-type and positive type, so-called 'rigidity free' part and its complement) he obtains corresponding decompositions for the process $X$. Here $\tau$ stands for the shift on $\mathbb{R}^{\mathbb{Z}}$.

11.4. **Poisson suspensions.** Poisson suspensions are widely used in statistical mechanics to model ideal gas, Lorentz gas, etc (see [33]). Let $(X, \mathcal{B}, \mu)$ be a standard $\sigma$-finite non-atomic measure space and $\mu(X) = \infty$. Denote by $\widetilde{X}$ the space of unordered countable subsets of $X$. It is called the space of *configurations.* Fix $t > 0$. Let $A \in \mathcal{B}$ have positive finite measure and let $j \in \mathbb{Z}_+$. Denote by $[A, j]$ the subset of all configurations $\widetilde{x} \in \widetilde{X}$ such that $\#(\widetilde{x} \cap A) = j$. Let $\widetilde{\mathcal{B}}$ be the $\sigma$-algebra generated by all $[A, j]$. We define a probability measure $\widetilde{\mu}_t$ on $\widetilde{\mathcal{B}}$ by two conditions:

(i) $\widetilde{\mu}_t([A, j]) = \frac{(t\mu(A))^j}{j!} \exp(-t\mu(A))$;
(ii) if $A_1, \dots, A_p$ are pairwise disjoint then $\widetilde{\mu}_t(\bigcap_{k=1}^p [A_k, j_k]) = \prod_{k=1}^p \widetilde{\mu}_t([A_k, j_k])$.

If $T$ is a $\mu$-preserving transformation of $X$ and $\widetilde{x} = (x_1, x_2, \dots)$ is a configuration then we set $\widetilde{T}\omega := (Tx_1, Tx_2, \dots)$. It is easy to verify that $\widetilde{T}$ is a $\widetilde{\mu}$-preserving transformation of $\widetilde{X}$.

The dynamical system $(\widetilde{X}, \widetilde{\mathcal{B}}, \widetilde{\mu}, \widetilde{T})$ is called the *Poisson suspension* above $(X, \mathcal{B}, \mu, T)$. It is ergodic if and only if $T$ has no invariant sets of finite positive measure. There is a canonical representation of $L^2(\widetilde{X}, \widetilde{\mu})$ as the Fock space over $L^2(X, \mu)$ such that the unitary operator $U_{\widetilde{T}}$ is the 'exponent' of $U_T$. Thus, the maximal spectral type of $U_{\widetilde{T}}$ is $\sum_{n\geq 0}(n!)^{-1}\sigma^{*n}$, where $\sigma$ is a measure of the maximal spectral type of $U_T$. It is easy to see that a $\sigma$-finite factor of $T$ corresponds to a factor (called Poissonian) of $\widetilde{T}$. Moreover, a $\sigma$-finite measure-preserving joining (with $\sigma$-finite projections) of two infinite measure-preserving transformations $T_1$ and $T_2$ generates a joining (called Poissonian) of $\widetilde{T}_1$ and $\widetilde{T}_2$ [164], [48]. Thus we see a similarity with the well studied theory of Gaussian dynamical systems [134]. However, the Poissonian case is less understood. There was a recent progress in this field. Parreau and Roy constructed Poisson suspensions whose ergodic self-joinings are all Poissonian [154]. In [107] partial solutions of the following (still open) problems are found:

(i) whether the Pinsker factor of $\widetilde{T}$ is Poissoninan,
(ii) what is the relationship between Krengel's entropy of $T$, Parry's entropy of $T$ and Kolmogorov-Sinai entropy of $\widetilde{T}$.

11.5. **Recurrence of random walks with non-stationary increments.** Using nonsingular ergodic theory one can introduce the notion of recurrence for random walks obtained from certain non-stationary processes. Let $T$ be an ergodic nonsingular transformation of a standard probability space $(X, \mathcal{B}, \mu)$ and let $f : X \to \mathbb{R}^n$ a measurable function. Define for $m \geq 1$, $Y_m : X \to \mathbb{R}^n$ by $Y_m := \sum_{n=0}^{m-1} f \circ T^n$. In other words, $(Y_m)_{m\geq 1}$ is the random walk associated with the (non-stationary) process $(f \circ T^n)_{n\geq 0}$. Let us call this random walk *recurrent* if the cocycle $f$ of $T$ is recurrent (see § 6.4). It was shown in [176] that in the case $\mu \circ T = \mu$, i.e. the process is stationary, this definition is equivalent to the standard one.

11.6. **Boundaries of random walks.** Boundaries of random walks on groups retain valuable information on the underlying groups (amenability, entropy, etc.) and enable one to obtain integral representation for harmonic functions of the random walk [187], [186], [112]. Let $G$ be a locally compact group and $\nu$ a probability measure on $G$. Let $T$ denote the (one-sided) shift on the probability space $(X, \mathcal{B}_X, \mu) := (G, \mathcal{B}_G, \nu)^{\mathbb{Z}_+}$ and $\varphi : X \to G$ a measurable map defined by $(y_0, y_1, \dots) \mapsto y_0$. Let $T_\varphi$ be the $\varphi$-skew product extension of $T$ acting on the space $(X \times G, \mu \times \lambda_G)$ (for non-invertible transformations the skew product extension is defined in the very same way as for invertible ones, see § 6.4). Then $T_\varphi$ is isomorphic to the *homogeneous random walk* on $G$ with jump probability $\nu$. Let $\mathcal{I}(T_\varphi)$ denote the sub-$\sigma$-algebra of $T_\varphi$-invariant sets and let $\mathcal{F}(T_\varphi) := \bigcap_{n>0} T_\varphi^{-n}(\mathcal{B}_X \otimes \mathcal{B}_G)$. The former is called the *Poisson boundary* of $T_\varphi$ and the latter one is called the *tail boundary* of $T_\varphi$. Notice that a nonsingular action of $G$ by inverted right translations along the second coordinate is well defined on each of the two boundaries. The two boundaries (or, more precisely, the $G$-actions on them) are ergodic. The Poisson boundary is the Mackey range of $\varphi$ (as a cocycle of $T$). Hence the Poisson boundary is amenable [187]. If the support of $\nu$ generates a dense subgroup of $G$ then the corresponding Poisson boundary is weakly mixing [4]. As for the tail boundary, we first note that it can be defined for a wider family of *non-homogeneous* random walks. This means that the jump probability $\nu$ is no longer fixed and a sequence $(\nu_n)_{n>0}$ of probability measures on $G$ is considered instead. Now let $(X, \mathcal{B}_X, \mu) := \prod_{n>0}(G, \mathcal{B}_G, \nu)$.

The one-sided shift on $X$ may not be nonsingular now. Instead of it, we consider the tail equivalence relation $\mathcal{R}$ on $X$ and a cocycle $\alpha : \mathcal{R} \to G$ given by $\alpha(x,y) = x_1 \cdots x_n y_n^{-1} \cdots y_1$, where $x = (x_i)_{i>0}$ and $y = (y_i)_{i>0}$ are $\mathcal{R}$-equivalent and $n$ in the smallest integer such that $x_i = y_i$ for all $i > n$. The tail boundary of the random walk on $G$ with time dependent jump probabilities $(\nu_n)_{n>0}$ is the Mackey $G$-action associated with $\alpha$. In the case of homogeneous random walks this definition is equivalent to the initial one. Connes and Woods showed [32] that the tail boundary is always amenable and AT. It is unknown whether the converse holds for general $G$. However it is true for $G = \mathbb{R}$ and $G = \mathbb{Z}$: the class of AT-flows coincides with the class of tail boundaries of the random walks on $\mathbb{R}$ and a similar statement holds for $\mathbb{Z}$ [32]. Jaworsky showed [105] that if $G$ is countable and a random walk is homogeneous then the tail boundary of the random walk possesses a so-called SAT-property (which is stronger than AT).

11.7. **Classifying $\sigma$-finite ergodic invariant measures.** The description of ergodic finite invariant measures for topological (or, more generally, standard Borel) systems is a well established problem in the classical ergodic theory [33]. On the other hand, it seems impossible to obtain any useful information about the system by analyzing the set of all ergodic quasi-invariant (or just $\sigma$-finite invariant) measures because this set is wildly huge (see § 2.6). The situation changes if we impose some restrictions on the measures. For instance, if the system under question is a homeomorphism (or a topological flow) defined on a locally compact Polish space then it is natural to consider the class of ($\sigma$-finite) invariant Radon measures, i.e. measures taking finite values on the compact subsets. We give two examples.

First, the seminal results of Giordano, Putnam and Skau on the topological orbit equivalence of compact Cantor minimal systems were extended to locally compact Cantor minimal (l.c.c.m.) systems in [37] and [138]. Given a l.c.c.m. system $X$, we denote by $\mathcal{M}(X)$ and $\mathcal{M}_1(X)$ the set of invariant Radon measures and the set of invariant probability measures on $X$. Notice that $\mathcal{M}_1(X)$ may be empty [37]. It was shown in [138] that two systems $X$ and $X'$ are topologically orbit equivalent if and only if there is a homeomorphism of $X$ onto $X'$ which maps bijectively $\mathcal{M}(X)$ onto $\mathcal{M}(X')$ and $\mathcal{M}_1(X)$ onto $\mathcal{M}_1(X')$. Thus $\mathcal{M}(X)$ retains an important information on the system—it is 'responsible' for the topological orbit equivalence of the underlying systems. Uniquely ergodic l.c.c.m. systems (with unique up to scaling infinite invariant Radon measure) were constructed in [37].

The second example is related to study of the smooth horocycle flows on tangent bundles of hyperbolic surfaces. Let $\mathbb{D}$ be the open disk equipped with the hyperbolic metric $|dz|/(1-|z|^2)$ and let Möb($\mathbb{D}$) denote the group of Möbius transformations of $\mathbb{D}$. A hyperbolic surface can be written in the form $M := \Gamma\backslash\text{Möb}(\mathbb{D})$, where $\Gamma$ is a torsion free discrete subgroup of Möb($\mathbb{D}$). Suppose that $\Gamma$ is a nontrivial normal subgroup of a lattice $\Gamma_0$ in Möb($\mathbb{D}$). Then $M$ is a regular cover of the finite volume surface $M_0 := \Gamma_0\backslash\text{Möb}(\mathbb{D})$. The group of deck transformations $G = \Gamma_0/\Gamma$ is finitely generated. The horocycle flow $(h_t)_{t\in\mathbb{R}}$ and the geodesic flow $(g_t)_{t\in\mathbb{R}}$ defined on the unit tangent bundle $T^1(\mathbb{D})$ descend naturally to flows, say $h$ and $g$, on $T^1(M)$. We consider the problem of classification of the $h$-invariant Radon measures on $M$. According to Ratner, $h$ has no finite invariant measures on $M$ if $G$ is infinite (except for measures supported on closed orbits). However there are infinite invariant Radon measures, for instance the volume measure. In the case when $G$ is free Abelian and $\Gamma_0$ is co-compact, every homomorphism $\varphi : G \to \mathbb{R}$ determines a unique up to scaling ergodic invariant Radon measure (e.i.r.m.) $m$ on $T^1(M)$ such that $m \circ dD = \exp(\varphi(D))m$ for all $D \in G$ [16] and every e.i.r.m. arises this way [171]. Moreover all these measures are quasi-invariant under $g$. In the general case, an interesting bijection is established in [131] between the e.i.r.m. which are quasi-invariant under $g$ and the 'non-trivial minimal' positive eigenfunctions of the hyperbolic Laplacian on $M$.

11.8. **Von Neumann algebras.** There is a fascinating and productive interplay between nonsingular ergodic theory and von Neumann algebras. The two theories alternately influenced development of each other. Let $(X, \mathcal{B}, \mu, T)$ be a nonsingular dynamical system. Given $\varphi \in L^\infty(X, \mu)$ and $j \in \mathbb{Z}$, we define operators $A_\varphi$ and $U_j$ on the Hilbert space $L^2(Z \times \mathbb{Z}, \mu \times \nu)$ by setting

$$(A_\varphi f)(x, i) := \varphi(T^i x) f(x, i),\ (U_j f)(x, i) := f(x, i - j)$$

Then $U_j A_\varphi U_j^* = A_{\varphi \circ T^j}$. Denote by $\mathcal{M}$ the von Neumann algebra (i.e. the weak closure of the $*$-algebra) generated by $A_\varphi$, $\varphi \in L^\infty(X, \mu)$ and $U_j$, $j \in \mathbb{Z}$. If $T$ is ergodic and aperiodic then $\mathcal{M}$ is a factor, i.e. $\mathcal{M} \cap \mathcal{M}' = \mathbb{C}1$, where $\mathcal{M}'$ denotes the algebra of bounded operators commuting with $\mathcal{M}$. It is called a *Krieger's* factor. Murray-von Neumann-Connes' type of $\mathcal{M}$ is exactly the Krieger's type of $T$. The flow of weights of $\mathcal{M}$ is isomorphic to the associated flow of $T$. Two Krieger's factors are isomorphic if and only if the underlying dynamical systems are orbit equivalent [129]. Moreover, a number of important problems in the theory of von Newmann algebras such as classification of subfactors, computation of the flow of weights and Connes' invariants, outer conjugacy for automorphisms, etc. are intimately related to the corresponding problems in nonsingular orbit theory. We refer to [142], [62], [69], [70], [85], [42] for details.

11.9. **Representations of CAR.** Representations of canonical anticommutation relations (CAR) is one of the most elegant and useful chapters of mathematical physics, providing a natural language for many body quantum physics and quantum field theory. By a representation of CAR we mean a sequence of bounded linear operators $a_1, a_2, \dots$ in a separable Hilbert space $\mathcal{K}$ such that $a_j a_k + a_k a_j = 0$ and $a_j a_k^* + a_k^* a_j = \delta_{j,k}$.

Consider $\{0, 1\}$ as a group with addition mod 2. Then $X = \{0, 1\}^{\mathbb{N}}$ is a compact Abelian group. Let $\Gamma := \{x = (x_1, x_2, \dots) : \lim_{n\to\infty} x_n = 0\}$. Then $\Gamma$ is a dense countable subgroup of $X$. It is generated by elements $\gamma_k$ whose $k$-coordinate is 1 and the other ones are 0. $\Gamma$ acts on $X$ by translations. Let $\mu$ be an ergodic $\Gamma$-quasi-invariant measure on $X$. Let $(C_k)_{k\geq 1}$ be Borel maps from $X$ to the group of unitary operators in a Hilbert space $\mathcal{H}$ satisfying $C_k^*(x) = C_k(x + \delta_k)$, $C_k(x)C_l(x + \delta_l) = C_l(x)C_k(x + \delta_k)$, $k \neq l$ for a.a. $x$. In other words, $(C_k)_{k\geq 1}$ defines a cocycle of the $\Gamma$-action. We now put $\widetilde{\mathcal{H}} := L^2(X, \mu) \otimes \mathcal{H}$ and define operators $a_k$ in $\widetilde{\mathcal{H}}$ by setting

$$(a_k f)(x) = (-1)^{x_1 + \cdots + x_{k-1}} (1 - x_k) C_k(x) \sqrt{\frac{d\mu \circ \delta_k}{d\mu}(x)} f(x + \delta_k),$$

where $f : X \to \mathcal{H}$ is an element of $\widetilde{\mathcal{H}}$ and $x = (x_1, x_2, \dots) \in X$. It is easy to verify that $a_1, a_2, \dots$ is a representation of CAR. The converse was established in [68] and [73]: every factor-representation (this means that the von Neumann algebra generated by all $a_k$ is a factor) of CAR can be represented as above for some ergodic measure $\mu$, Hilbert space $\mathcal{H}$ and a $\Gamma$-cocycle $(C_k)_{k\geq 1}$. Moreover, using nonsingular ergodic theory Golodets [73] constructed for each $k = 2, 3, \dots, \infty$, an irreducible representation of CAR such that $\dim \mathcal{H} = k$. This answered a question of Gårding and Wightman [68] who considered only the case $k = 1$.

11.10. **Unitary representations of locally compact groups.** Nonsingular actions appear in a systematic way in the theory of unitary representations of groups. Let $G$ be a locally compact second countable group and $H$ a closed normal subgroup of $G$. Suppose that $H$ is commutative (or, more generally, of type I, see [49]). Then the natural action of $G$ by conjugation on $H$ induces a Borel $G$-action, say $\alpha$, on the dual space $\widehat{H}$—the set of unitarily equivalent classes of irreducible unitary representations of $H$. If now $U = (U_g)_{g\in G}$ is a unitary representation of $G$ in a separable Hilbert space then by applying Stone decomposition theorem to $U \restriction H$ one can deduce that $\alpha$ is nonsingular with respect to a measure $\mu$ of the 'maximal spectral type' for $U \restriction H$ on $\widehat{H}$. Moreover, if $U$ is irreducible then $\alpha$ is ergodic. Whenever $\mu$ is fixed, we obtain a one-to-one correspondence between the set of cohomology classes of irreducible cocycles for $\alpha$ with values in the unitary group on a Hilbert space $\mathcal{H}$ and the subset of $\widehat{G}$ consisting of classes of those unitary representations $V$ for which the measure associated to $V \restriction H$ is equivalent to $\mu$. This correspondence is used in both directions. From information about cocycles we can deduce facts about representations and vise versa [118], [49].

## 12. Concluding remarks

While some of the results that we have cited for nonsingular $\mathbb{Z}$-actions extend to actions of locally compact Polish groups (or subclasses of Abelian or amenable ones), many natural questions remain open in the general setting. For instance: what is Rokhlin lemma, or the pointwise ergodic theorem, or the definition of entropy for nonsingular actions of general countable amenable groups? The theory of abstract nonsingular equivalence relations [62] or, more generally, nonsingular groupoids [160] and polymorphisms [184] is also a beautiful part of nonsingular ergodic theory that has nice applications: description of semifinite traces of AF-algebras, classification of factor representations of the infinite symmetric group [185], path groups [14], etc. Nonsingular ergodic theory is getting even more sophisticated when we pass from $\mathbb{Z}$-actions to noninvertible endomorphisms or, more generally, semigroup actions (see [3] and references therein). However, due to restrictions of space we do not consider these issues in our survey.

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