ERGODIC PROPERTIES OF A CLASS OF DISCRETE ABELIAN GROUP EXTENSIONS OF RANK-ONE TRANSFORMATIONS

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Abstract. We define a class of discrete Abelian group extensions of rank-one transformations and establish necessary and sufficient conditions for these extensions to be power weakly mixing. We show that all members of this class are multiply recurrent. We then study conditions sufficient for showing that Cartesian products of transformations are conservative for a class of invertible infinite measure-preserving transformations and provide examples of these transformations.

1. Introduction

Group extensions of measure-preserving dynamical systems have received much attention in the literature. In most of the works the group has been assumed to be compact, and if the base transformation is finite measure-preserving then the extension is finite measure-preserving. A question that has been studied in this context is conditions under which dynamical properties of the base transformation (such as weak mixing or mixing) lift to the group extension; the reader may refer to e.g. [12], [11] and the references in these works. In this article we consider extensions of a class of rank-one transformations by countable discrete Abelian groups. While the base transformation is restricted to be a rank-one transformation we allow the group to possibly be infinite. We establish a simple condition that is equivalent to the ergodicity of the extensions, and another condition that is equivalent to power weak mixing of the extensions. Power weak mixing is equivalent to weak mixing for finite measure-preserving transformations, but it is a stronger property in the case of infinite measure-preserving transformations. We show that the extension is power weakly mixing if it is totally ergodic. We also show that our group extensions are multiply recurrent, and give several applications showing ergodicity or (power) weak mixing for certain extensions in both the finite and infinite measure-preserving cases. In the later sections we consider the question of the conservativity of products of powers of infinite measure-preserving transformations, and apply our results to staircase transformations.

Let $(X, \mathcal{B}, \mu)$ be a measure space isomorphic to a finite or infinite interval in $\mathbb{R}$ with Lebesgue measure μ (when the interval is finite we assume μ has been normalized to be a probability measure). Let $T : X \to X$ be an invertible measure-preserving transformation. The transformation T is **conservative** if for any set A of positive measure, there exists an integer $i > 0$ such that $\mu(T^{-i}A \bigcap A) > 0$.

1991 *Mathematics Subject Classification.* Primary 37A40; Secondary 37A25.

Key words and phrases. Infinite Measure-preserving, ergodic, group extensions, multiple recurrence.

T is **ergodic** if for any pair of sets set A and B of positive measure, there exists an integer $i \geq 0$ such that $\mu(T^{-i}A \bigcap B) > 0$. (As our transformations are invertible and defined on nonatomic spaces, ergodicity implies conservativity.) Let $T^{\otimes d}$ denote the Cartesian product of $d > 0$ copies of T. We define the **ergodic** (resp. **conservative**) **index** of a transformation T to be the largest integer d such that $T^{\otimes d}$ is ergodic (resp. conservative), (see [2]), and say that T has **infinite ergodic (resp. conservative) index** if this holds for all integers d. A transformation T is **power product conservative** if for all sequences of integers $k_1, k_2, \ldots k_d, T^{k_1} \times T^{k_2} \times \ldots \times T^{k_d} : X^{\otimes d} \to X^{\otimes d}$ is conservative; T is said to be **power weakly mixing** if for all nonzero $k_1, \ldots, k_d$, $T^{k_1} \times T^{k_2} \times \ldots \times T^{k_d}$ is ergodic.

Power weak mixing is clearly equivalent to weak mixing for finite measure-preserving transformations, but it is a stronger property in the case of infinite measure-preserving transformations [3]. In fact, there exists a transformation T_1 such that T_1 has infinite ergodic index but $T_1 \times T_1^2$ is not conservative, hence not ergodic [3]. These examples were extended in [4].

In Section 2 we define, for each countable discrete Abelian group G, a class of measure-preserving transformations. When G is an infinite group the transformation is infinite measure-preserving. As the last example in Section 6 shows, these contain group extensions of rank-one transformations. In Theorem 2.2 we give necessary and sufficient conditions for our construction to be power weakly mixing.

We also show that our group extensions are multiply recurrent. A transformation T is said to be **d-recurrent** if for all sets of positive measure A there exists an integer $n > 0$ such that $\mu(A \cap T^n(A) \cap \cdots \cap T^{nd}(A)) > 0$. T is said to be **multiply recurrent** if it is d-recurrent for all integers $d > 0$. As is well-known, Furstenberg showed that every finite measure-preserving transformation is multiply recurrent [7], but it is now known that infinite measure-preserving transformations need not be multiply recurrent [6], [2], even when they are power weakly mixing [8]. However, it was shown recently that compact group [9] and σ-finite [10] extensions of multiply recurrent infinite measure-preserving transformation are multiply recurrent. This need not be the case for extensions by non-compact groups as already observed in [9], but we obtain multiple recurrence for our class of (non-σ-finite) extensions. In particular, it follows that for each countable discrete Abelian group there is a multiply recurrent ergodic extension.

In Section 9 we introduce a condition for rank-one transformations that implies power conservativity, and use it show show that some infinite measure-preserving staircases are power product conservative.

Acknowledgments. This paper is based on research in the Ergodic Theory group of the 2004 SMALL Undergraduate Summer Research Project at Williams College, with Silva as faculty advisor. Support for the project was provided by a National Science Foundation REU Grant and the Bronfman Science Center of Williams College. We would like to thank the referee for several remarks that improved our manuscript.

2. Construction of the Transformations

Fix a countable discrete Abelian group G. We will construct transformations that are G extensions of rank-one transformations produced by a standard cutting and stacking procedure. Let Γ be the set of all elements that are of the form

$$(\gamma_e, s_{e,0}, ..., s_{e,\gamma_e-1}, g_{e,0}, ..., g_{e,\gamma_e-1})$$

where $\gamma_e > 1$ is natural number and the remaining entries are an element of $\mathbb{N}_0^{\gamma_e} \times G^{\gamma_e}$. For clarity, we sometimes write the subscript $s_{e,0}$ as $s(e,0)$, etc. We think of Γ as the set of possible operations to go from one generation to the next. γ is the number of pieces that we cut each level into, $s_{e,i}$ describes the numbers of spacers added (i.e,, new levels), and g_{e_i} describe how the G-component of the column changes. Let

$$F : \mathbb{N}_0 \to \Gamma$$

be a function. We think of F as the map from generation numbers to what operation is performed in that generation. We require that F have the property that for any natural numbers n and d, there are infinitely many natural numbers m so that $F(n+i) = F(m+i)$ for all $0 \le i < d$. In other words any sequence that appears in F does so infinitely often. Let

$$F(n) = (\gamma_n, s(n,0), ..., s(n,\gamma_n - 1), g(n,0), ..., g(n,\gamma_n - 1)).$$

Given F, we define a (at most rank-$|G|$) transformation T as follows:

A **column** consists of a finite (ordered) sequence of intervals of the same length, called the **levels** of the column; the number of levels is the **height** of the column. We begin with generation-0 columns $C_{0,g}$ for $g \in G$, each consisting of an interval of mass 1. To obtain the generation-$(N+1)$ columns from the generation-N columns, first write each generation-N column $C_{N,g}$ $(g \in G)$ as

$$C_{N,g} = (I^{(0)}_{N,g}, I^{(1)}_{N,g}, ..., I^{(h_N-1)}_{N,g}),$$

where we think of h_N as the height of the column. Next, cut each level or interval $I^{(i)}_{N,g}$ into γ_N equal mass subintervals

$$I^{(i)}_{N,g,0}, \ldots, I^{(i)}_{N,g,\gamma_N-1},$$

and set

$$\begin{aligned} C_{N+1,g} =(&I^{(0)}_{N,g+g(N,0),0}, \ldots, I^{(h_N-1)}_{N,g+g(N,0),0}, S^{(0)}_{N,g,0}, \ldots, S^{(s(N,0)-1)}_{N,g,0}, \\ &I^{(0)}_{N,g+g(N,1),1}, \ldots, I^{(h_N-1)}_{N,g+g(N,1),1}, S^{(0)}_{N,g,1}, \ldots, S^{(s(N,1)-1)}_{N,g,1}, \ldots, \\ &I^{(0)}_{N,g+g(N,\gamma_N-1),\gamma_N-1}, \ldots, I^{(h_N-1)}_{N,g+g(N,\gamma_N-1),\gamma_N-1}, \\ &\qquad\qquad S^{(0)}_{N,g,\gamma_N-1}, \ldots, S^{(s(N,\gamma_N-1)-1)}_{N,g,\gamma_N-1}) \end{aligned}$$

where each $S^{(j)}_{N,g,i}$ is a spacer level, i.e., a new subinterval of the same length as any of the subintervals in its column. The resulting transformation is defined on the intervals of each column by sending that interval by translation to the interval above it if there is one. In the limit, the lengths of the intervals in each column converge to zero, so the transformation is defined in the union of all the levels. We thus obtain a transformation T that is measure-preserving. Furthermore, one can arrange the subintervals in each column so that T is defined on a finite or infinite subinterval of $\mathbb{R}$.

We will assume without loss of generality that $g(N,0) = 0$ for all $N \in \mathbb{N}_0$. We may do this as addition of a constant to all of $g_{N,i}$ for fixed N is equivalent to relabeling the generation-$(N+1)$ columns, and does not change the transformation T.

We prove the following theorems:

Theorem 2.1. *For all such F, T is multiply recurrent.*

Theorem 2.2. *T is power weakly mixing if and only if the following conditions are both satisfied,*

(1) $\{g(N,i) - g(N,0) : N \in \mathbb{N}_0, 0 \leq i \leq \gamma_N - 1\}$ *generate* G
(2) *For all N, $(1,0)$ is in the integer span of*

$$\{(s(N,i) + h_N, g(N,i+1) - g(N,i)) : 0 \leq i \leq \gamma_N - 2\} \cup$$
$$\{((s(M+1,i) + s(M,\gamma_M - 1) - s(M,0)),$$
$$(g(M+1,i+1) - g(M+1,i) + 2g(M,0) - g(M,\gamma_M - 1) - g(M,1)))$$
$$: M \in \mathbb{N}_0, 0 \leq i \leq \gamma_{M+1} - 2\}$$

in $\mathbb{Z} \times G$.

The first condition essentially states that it is possible to get from any column to any other column. The $\mathbb{Z} \times G$ that appears in the second condition should be thought of as a group acting on our space with G acting by changing column index, and $1 \in \mathbb{Z}$ acting as T. Let us call the terms in the second condition

$$t_{N,i} = (s(N,i) + h_N, g(N,i+1) - g(N,i))$$

and

$$c_{M,i} = ((s(M+1,i) + s(M,\gamma_M - 1) - s(M,0)),$$
$$(g(M+1,i+1) - g(M+1,i) + 2g(M,0) - g(M,\gamma_M - 1) - g(M,1))).$$

They each represent distances in this action between copies of columns as will be discussed later. The condition that $(1,0)$ be in their span essentially says that we have the control to shift things by T.

Note that if we assume that $G = \{1\}$, we get the following result:

Corollary 2.3. *If $G = \{1\}$, T is power weakly mixing if and only if*

$$\gcd\left(\{s(N,i) + h_N\} \cup \{(s(M+1,i) + s(M,\gamma_M - 1) - s(M,0))\}\right) = 1.$$

3. Some Machinery Involving copies of Columns

If I is a level of a generation-n column, $n > 1$, we say that a level K in a generation-$(n+m)$ column is a **copy** of I if K corresponds to a subset of level I. We define a copy of a column C, in some column of later generation, to be a union of consecutive levels that are, in order, copies of the levels of C. We would like to be able to index the copies of generation-N columns in a particular generation-$(N+M)$ column. If C is a copy of $C_{N+1,g}$, then we let $P_i(C)$ be the copy of $C_{N,g+g(N,i)}$ contained in C produced by the $(i+1)^{st}$ part of C from the cutting and stacking procedure. In particular, for $0 \leq i \leq \gamma_N - 1$ $P_i(C)$ is the $(i+1)^{st}$ copy of a generation-N column contained in $C_{N,g}$. Let

$$P_{N,g}[a_0, a_1, ..., a_n] = P_{a_0}(P_{a_1}(\dots P_{a_n}(C_{N,g})\dots)),$$

where $C_{N,g}$ is thought of as a copy of itself.

Notice that the $P_{N,g}[a_0, \dots, a_{n-1}]$ index all of the copies of generation-$(N+n)$ columns in $C_{N,g}$. Their relative positions are given by the radix ordering on the a_i with a_0 being the most significant.

Lemma 3.1. *$P_{N+n,g}[a_0, a_1, ..., a_{n-1}]$ is a copy of $C_{N,g+\sum_{i=0}^{n-1} g(N+i,a_i)}$.*

Proof. We proceed by induction on n. The $n = 0$ case is trivial. Assuming that our statement holds for $n-1$, we have, letting $C'_{M,h}$ denote a copy of $C_{M,h}$ for any $M \in \mathbb{N}_0$ and $h \in G$, that

$$\begin{aligned} P_{N+n,g}[a_0, a_1, ..., a_{n-1}] =& P_{a_0}\left(C'_{N+1,g+\sum_{i=1}^{n-1} g(N+i,a_i)}\right) \\ =& C'_{N,g+\sum_{i=0}^{n-1} g(N+i,a_i)}. \end{aligned}$$

This completes our inductive step and proves our lemma. □

Lemma 3.2. *$T^k(P_{N+n,g}[a_0, \ldots, a_{n-1}]) = P_{N+n,g}[b_0, \ldots, b_{n-1}]$ where*

$$k = \sum_{i=0}^{n-1}\left(h_{N+i}(b_i - a_i) + \sum_{j=0}^{b_i-1} s(N+i,j) - \sum_{j=0}^{a_i-1} s(N+i,j)\right).$$

Proof. We proceed by induction on $\sum_{i=0}^{n-1} |a_i - b_i|$. The statement is clearly true when this is 0. Otherwise, assume our hypothesis for smaller values of $\sum_{i=0}^{n-1} |a_i - b_i|$. Without loss of generality we may assume that $b_i > a_i$. Then we have that

$$\begin{aligned} P_{N+n,g}[b_0, \ldots, b_{n-1}] &= T^{h_{N+i}+s_{b_i-1}}(P_{N+n,g}[b_0, \ldots b_{i-1}, b_i - 1, b_{i+1}, \ldots, b_{n-1}]) \\ &= T^k(P_{N+n,g}[a_0, \ldots, a_{n-1}]). \end{aligned}$$

This completes our inductive step and proves the lemma. □

Lemma 3.3. *$T^k(P_{N+n,g}[\gamma_N - 1, \gamma_{N+1} - 1, \ldots, \gamma_{N+m} - 1, a_{m+1}, \ldots, a_{n-1}]) = P_{N+n,g}[0, \ldots, 0, a_{m+1} + 1, a_{m+2}, \ldots, a_{n-1}]$, where $a_{m+1} \leq \gamma_{N+m+1} - 2$ and*

$$k = h_N + \sum_{i=0}^{m} s(N+i, \gamma_{N+i} - 1) + s(N+m+1, a_{m+1}).$$

Proof. By Lemma 3.2 we have that

$$k = \sum_{i=0}^{m}\left(-(\gamma_{N+1} - 1)h_{N+i} - \sum_{j=0}^{\gamma_{N+i}-2} s(N+i,j)\right) + h_{N+m+1} + s(N+m+1, a_{m+1}).$$

Using the fact that $h_{n+1} = \gamma_n h_n + \sum_{j=0}^{\gamma_n-1} s(n,j)$, we have that

$$\begin{aligned} k &= h_{N+m+1} + s(N+m+1, a_{m+1} + \sum_{i=0}^{m} h_{N+1} + s(N+i, \gamma_{N+i} - 1) - h_{N+i+1} \\ &= h_N + \sum_{i=0}^{m} s(N+i, \gamma_{N+i} - 1) + s(N+m+1, a_{m+1}). \end{aligned}$$

Thus proving our lemma. □

Here $t_{N,i}$ represents the change in location of the copy when the index corresponding to generation-N is changed from i to $i+1$. $c_{M,i}$ represents the change in location of the copy corresponding to the pair of indices corresponding to generations M and $M+1$ changing from $(\gamma_M - 1, i)$ to $(0, i+1)$.

4. Necessity of the Conditions

Lemma 4.1. *T is ergodic only if condition 1 is satisfied.*

Proof. Suppose that $\{g(N,i) : N \in \mathbb{N}_0, 0 \leq i \leq \gamma_N - 1\}$ generate $H \subsetneq G$. Let $g_1, g_2 \in G$ be in different cosets of H. Consider $A = C_{0,g_1}, B = C_{0,g_2}$. Assume for sake of contradiction, that for some n, $\mu(T^n(A) \cap B) > 0$. This would imply that there is some column that contains both a copy of C_{0,g_1} and a copy of C_{0,g_2}. Suppose these copies are $P_{N,g_3}[a_0, \ldots, a_{n-1}]$ and $P_{N,g_3}[b_0, \ldots, b_{n-1}]$. Then by Lemma 3.1 we have that $g_1 = g_3 + \sum_{i=0}^{n-1} g(i, a_i)$ and $g_2 = g_3 + \sum_{i=0}^{n-1} g(i, b_i)$. Hence we have that $g_2 - g_1 = \sum_{i=0}^{n-1} g(i, b_i) - g(i, a_i) \in H$, but this is not the case. Hence T is not ergodic. □

Remark 4.2. *This condition is actually sufficient for T being ergodic. This fact will follow from Lemma 5.2.*

Lemma 4.3. *T is totally ergodic only if condition 2 is satisfied.*

Proof. Pick an N for which the condition does not hold. Let $t_{N,i} = t_i$. Let

$$H = \operatorname{span}(\{t_i : 0 \leq i \leq \gamma_N - 2\} \cup \{c_{M,i} : M \in \mathbb{N}_0, 0 \leq i \leq \gamma_{M+1} - 2\})$$

Let $H \cap (\mathbb{Z} \times \{0\}) \subset \mathbb{Z}(D, 0)$ for $D > 1$. We will prove that there is no integer n, so that $\mu(T^{nD}(I_{N,0}^{(0)}) \cap I_{N,0}^{(1)}) > 0$. Suppose for sake of contradiction that this is not the case. Then there must exist copies C_1, C_2 of $C_{N,0}$ that are in the same column, and with $T^{nD-1}(C_1) = C_2$. Suppose that these copies are $P_{N+l,g}[a_0, \ldots, a_{l-1}]$ and $P_{N+l,g}[b_0, \ldots, b_{l-1}]$. For two copies of generation-N columns in $C_{N+l,g}$, α, β define $\Delta(\beta, \alpha) = (k, h)$ where $T^k(\alpha) = \beta$, and α and β are copies of $C_{N,g'}$ and $C_{N,g'+h}$ respectively. Notice that $\Delta(\gamma, \beta) + \Delta(\beta, \alpha) = \Delta(\gamma, \alpha)$. Notice also that $\Delta(C_2, C_1) = (nD-1, 0)$. Lastly, notice that C_1 and C_2 are connected by some chain of copies where each pair of consecutive copies are of the form given in Lemma 3.3. We have by Lemmas 3.1 and 3.3 that

$$\begin{aligned}
&\Delta(P_{N+l,g}[0, \ldots, 0, a_{m+1}+1, a_{m+2}, \ldots, a_{l-1}],\\
&\quad P_{N+l,g}[\gamma_N - 1, \gamma_{N+1} - 1, \ldots, \gamma_{N+m} - 1, a_{m+1}, \ldots, a_{l-1}]) =\\
&(h_N + s(N+m+1, a_{m+1}) + \sum_{i=0}^{m} s(N+i, \gamma_{N+i} - 1),\\
&\quad g(N+m+1, a_{m+1}+1) - g(N+m+1, a_{m+1}) - \sum_{i=0}^{m} g(N+i, \gamma_{N+i} - 1)) =\\
&\quad t_0 + \sum_{i=0}^{m-1} c_{N+i,0} + c_{N+m,a_{m+1}} \in H.
\end{aligned}$$

Combining these facts with the fact that H is additively closed, we have that $\Delta(C_2, C_1) \in H$. But by assumption, $(nD-1, 0) \notin H$. This is a contradiction. Hence T^D is not ergodic, proving our lemma.

□

5. Sufficiency of Conditions

Lemma 5.1. *If condition 2 is satisfied, then for any $N \in \mathbb{N}_0$ there exists some $D \neq 0$ so that $(D, 0, 0)$ is in the integer span of*

$$\{(s(N,i), g(N,i+1) - g(N,i), 1) : 0 \leq i \leq \gamma_N - 2\} \cup$$
$$\{(c_{M,i}, 0) : M \in \mathbb{N}_0, 0 \leq i \leq \gamma_{M+1} - 2\}$$

in $\mathbb{Z} \times G \times \mathbb{Z}$.

Proof. Let the intersection of the integer span of this with $\mathbb{Z} \times \{0\} \times \mathbb{Z}$ be H. Consider the homomorphism, $\phi_M : \mathbb{Z} \times G \times \mathbb{Z} \to \mathbb{Z} \times G$ defined by $\phi_M(a, b, c) = (a + h_M c, b)$. Notice that ϕ sends $\mathbb{Z} \times \{0\} \times \mathbb{Z}$ to $\mathbb{Z} \times \{0\}$. Hence for all M with $F(M) = F(N)$, we have that $(1, 0) \in \phi_M(H)$. Suppose for sake of contradiction that $H \cap (\mathbb{Z} \times \{0\} \times \{0\}) = \{(0, 0, 0)\}$. Then H must have infinite index in $\mathbb{Z} \times \{0\} \times \mathbb{Z}$. So $H = \mathbb{Z}(a, 0, b)$ for some a and b. But then, $\phi_M(H) = \mathbb{Z}(a + bh_M, 0)$. But this implies that for infinitely many values of h_M that $a + bh_M = \pm 1$. This implies that $b = 0$ and $a = \pm 1$. But then $(1, 0, 0) \in H \cap (\mathbb{Z} \times \{0\} \times \{0\})$. This is a contradiction. □

The remainder of this section is devoted to proving that $T^{k_1} \times \cdots \times T^{k_m}$ is ergodic. We will let A and B be arbitrary sets of positive measure in the m-fold product of the space on which T is defined.

Given a measurable set of positive measure A and $\epsilon > 0$, we say that an interval I, is more than $(1-\epsilon)$-**full** of A if $\mu(A \cap I) > (1-\epsilon)\mu(I)$; a similar notion is defined for product sets.

We use a standard technique from measure theory, sometimes called double approximation. We describe it first in the case of subsets of $\mathbb{R}$. Suppose we are given a refining family of measurable sets that approximate the measurable sets; for example the family of dyadic intervals $[k, 2^n, k + 1/2^n), (k \in \mathbb{Z}, n \in \mathbb{N})$, or the intervals obtained in a rank-one cutting and stacking construction (or more generally a sufficient semi-ring, see e.g. [13]). It is useful to specify an *order* or *stage* for sets in the family; for example, the intervals $[k, 2^n, k + 1/2^n)$, or all levels in a column C_n, are considered of order n. Then given any set of positive measure A and any interval I in the family that is more than half-full of A, for any $\epsilon > 0$, there exists an integer N so that for each $n \geq N$, more than half of the subintervals of I of order n are more than $(1-\epsilon)$-full of A. We will use this in the case when I and J are products of levels, more than half-full of measurable sets A and B, respectively. Then if we have generations of partitions of I and J into equal numbers of subsets of equal measure, each generation a refinement of the previous one, with bijections between the N^{th} generation subsets of I and the N^{th} generation subsets of J, then for any $\epsilon > 0$, there exist corresponding subsets of I and J of some generation that are $(1-\epsilon)$-full of A and B, respectively (see e.g. [13, 6.5.4]).

Lemma 5.2. *If condition 1 holds, then there exist an integer N and levels $I_1, \ldots, I_m$, $J_1, \ldots, J_m$ from generation-N columns so that I_i and J_i are in the same column and so that $I_1 \times \cdots \times I_m$ and $J_1 \times \cdots \times J_m$ are more than $\left(\frac{1}{2}\right)$-full of A and B respectively.*

Proof. We may find an integer N_1 and generation-N_1 columns

$$I'_1, \ldots, I'_m, \; J'_1, \ldots, J'_m$$

so that $I'_1 \times \cdots \times I'_m$ and $J'_1 \times \cdots \times J'_m$ are more than $\left(\frac{1}{2}\right)$-full of A and B respectively. Let I'_i be in $C_{N_1,h_{1,i}}$ and let J'_i be in $C_{N_1,h_{2,i}}$. By condition 1, we may write

$$h_{1,i} + \sum_{j=1}^{r_{1,i}} g(e_{1,i,j}, l_{1,i,j}) = h_{2,i} + \sum_{j=1}^{r_{2,i}} g(e_{2,i,j}, l_{2,i,j})$$

for some values of $r \in \mathbb{N}_0$, $e \in S$ and $l \in \mathbb{N}_0$ (since along with the assumption that $g(N,0) = 0$, condition 1 states that the $g(N,i)$ generate G). Since F attains all values in Γ infinitely often, we may find some sequence of consecutive integers, $a, a+1, \ldots, a+b$, so that for each $t \in \{1,2\}$, $1 \le i \le m$, and $1 \le j \le r_{t,i}$ there is a distinct $0 \le \alpha_{t,i,j} \le b$ so that $F(a + \alpha_{t,i,j}) = e_{t,i,j}$. Using double approximation, we may find an integer N_2 so that $F(N_2 + i) = F(a + i)$ for all $0 \le i \le b$ and generation N_2 levels $I''_1, \ldots, I''_m, J''_1, \ldots, J''_m$ so that $I''_1 \times \cdots \times I''_m$ and $J''_1 \times \cdots \times J''_m$ are more than $\left(\frac{1}{2\prod_{j=a}^{a+b} \gamma_j^m}\right)$-full of A and B respectively. Furthermore, we can ensure that if I''_i and J''_i are in columns $C_{N_2,h'_{1,i}}$ and $C_{N_2,h'_{2,i}}$ respectively, that $h'_{1,i} - h'_{2,i} = h_{1,i} - h_{2,i}$. Then we let $N = N_2 + b + 1$ and let I_i be the copy of I''_i in

$$P_{N,h'_{1,i} - \sum_{j=1}^{r_{2,i}} g(e_{2,i,j}, l_{2,i,j})}[d_0 \ldots d_b]$$

where

$$d_p = \begin{cases} l_{2,i,j} & \text{if } p = \alpha_{2,i,j}; \\ 0 & \text{if } p \neq \alpha_{2,i,j}\ \forall j; \end{cases}$$

and J_i the copy of J''_i in

$$P_{N,h'_{2,i} - \sum_{j=1}^{r_{1,i}} g(e_{1,i,j}, l_{1,i,j})}[d'_0 \ldots d'_b]$$

where

$$d'_p = \begin{cases} l_{1,i,j} & \text{if } p = \alpha_{1,i,j}; \\ 0 & \text{if } p \neq \alpha_{1,i,j}\ \forall j. \end{cases}$$

These are copies of the correct columns by Lemma 3.1. They are clearly in the same column. Furthermore we have that $I_1 \times \cdots \times I_m$ and $J_1 \times \cdots \times J_m$ are more than $\left(\frac{1}{2}\right)$-full of A and B respectively, proving our lemma. □

Lemma 5.3. *If conditions 1 and 2 hold, there exist levels $I_1, \ldots, I_m, J_1, \ldots, J_m$ that satisfy the conditions from Lemma 5.2 with the additional property that some power of T^D (dependent on i) sends I_i to J_i, where D satisfies the statement of Lemma 5.1 for some N_0*

Proof. The proof follows the same lines as that of Lemma 5.2. We start with the levels given to us by Lemma 5.2 and then use double approximation to get the levels that we need. Suppose that I_i and J_i are separated by T^{r_i}. Using double approximation we know that for any $\epsilon > 0$ and all sufficiently large generation numbers N, we can find generation-N copies I'_i and J'_i of I_i and J_i respectively, so that $T^{r_i}(I'_i) = J'_i$ and so that $I'_1 \times \ldots \times I'_m$ and $J'_1 \times \ldots \times J'_m$ are at least $(1 - \epsilon^m/2)$-full of A and B respectively. What we wish to show is that for some $\epsilon > 0$, and for arbitrarily large generation numbers N, given any such intervals I'_i and J'_i, we can find copies I''_i and J''_i of these in generation-$(N+n)$, that are of size at least ϵ that of the original, and so that I''_i and J''_i are in the same column, separated by a power of T^D. The result would then follow since $I''_1 \times \ldots \times I''_m$ and $J''_1 \times \ldots \times J''_m$ would be at least $\left(\frac{1}{2}\right)$-full of A and B respectively.

For the above to work, we need only show that for any separation $r = r_i$, that for some sufficiently small $\epsilon > 0$ and sufficiently large generation M, that we can find two copies of $C_{M,g}$ that are of size at least ϵ that of the original, are in the same column, and are separated by a power of T congruent to r modulo D. This allows us to produce the necessary copies of I'_i and J'_i for each i.

For each congruence class, c modulo D such that for infinitely many N, h_N is in c and $F(N) = F(N_0)$, we can, by condition 2, find some integer combination of $t_{N,i}$ and $c_{M,i}$ that add up to $(r, 0)$ in $(\mathbb{Z}/D\mathbb{Z}) \times G$. Suppose that the sum of the absolute values of the multiples of terms of the form $t_{N,i}$ needed is at most X. For $a, b \in \Gamma$ suppose that the sum of the absolute values of multiplies terms of the form $c_{M,i}$ where $F(M) = a$ and $F(M + 1) = b$ needed is at most $Y_{a,b}$. Find a string of consecutive integers, I, so that on this string the following hold:

There are at least XD values $n \in I$ so that $F(n) = e$.

There are a number of non-overlapping pairs of consecutive integers in I which do not intersect any of the n used in the previous condition, so that for at least $2DY_{a,b}$ of these pairs, F evaluated at these values yields a and b in that order.

Then for any interval of sufficiently large numbers, I' on which F agrees with the values it takes on I, we can find one of these congruence classes, c for which there are at least X values $n \in I'$ for which $F(n) = e$ and $h_n \equiv c \pmod{D}$.

For each a and b we can find at least $2Y_{a,b}$ pairs of consecutive integers $n, n+1 \in I'$ so that $F(n) = a$, $F(n + 1) = b$ and h_n has the same value modulo D for all of these pairs.

Now if M' is the smallest value in I', we can construct two copies of $C_{M',g}$ whose size is at least $\prod_{j \in I} \frac{1}{\gamma_j}$ of the original. Suppose that

$$\sum_{i=0}^{n-2} \alpha_i(s(N, i) + c, g(N, i + 1) - g(N, i)) + \sum_{M,i} \beta_{M,i} c_{M,i} = (r, 0)$$

in $(\mathbb{Z}/D\mathbb{Z}) \times G$. Then we consider copies of the form

$$P_{M'+k+1,g}[d_0 \dots d_k], P_{M'+k+1,g}[d'_0 \dots d'_k]$$

where $M' + k$ is the largest value in I'. We define the d_i and d'_i as follows:

There are α_i values $n \in I'$ for which $F(n) = e$ and $h_n \equiv c \pmod{D}$ where $d'_{n-M'} = i + 1$ and $d_{n-M'} = i$ (if α_i is negative, we reverse the values and do it $|\alpha_i|$ times).

There are $\beta_{M,i}$ values $n \in I'$ where $F(n) = F(M)$, $F(n + 1) = F(M + 1)$, $d'_{n-M'} = 0$, $d_{n-M'} = \gamma_M - 1$, $d_{n-M'+1} = i$, $d'_{n-M'+1} = i + 1$, and the same number of such values of n so that h_n has the same congruence class modulo D where $d_{n-M'} = 1$ and $d'_{n-M'} = 0$. (again, if $\beta_{M,i}$ is negative, we reverse the values of d and d' and use the absolute value).

By Lemmas 3.1 and 3.2 these copies have the properties that we want. □

Lemma 5.4. *Given $a, b \in \Gamma$, with some n where $F(n) = a, F(n+1) = b$, and given $k \in \mathbb{N}$, there exists an interval I of natural numbers, and functions $f_0, f_1, \dots, f_k : I \to \mathbb{N}_0$ so that:*

(1) $0 \le f_i(l) < \gamma_{F(l)}$. *This allows us to think of the f_i as indexing copies of a generation* $\min(I)$ *column.*
(2) *When such copies are considered, they are in the same column with consecutive copies separated by the same power of T.*

(3) *For every* $1 \leq i \leq k$, $0 \leq x \leq \gamma_a - 1$ *and* $0 \leq y \leq \gamma_b - 2$, *there exists* $n' \in I$ *so that* $F(n') = a, F(n'+1) = b$ *and the values on* $(n', n'+1)$ *of* f_0 *and* f_i *are* (x, y) *and* $(x+1, y)$ *(or* $(0, y+1)$ *if* $x = \gamma_a - 1$*) respectively.*

The significance of Lemma 5.4 is that it allows us to produce several equally spaced copies of a given interval. In particular the ability to do this is absolutely necessary to prove Theorem 2.1. Additionally, the last of the desired properties will be necessary to make slight modifications to these copies in order to prove Theorem 2.2.

Proof. Our basic construction will involve a several of counters that are incremented for each f_i. If all we want is the spacing between the corresponding columns to form an arithmetic progression (ignoring the group action for the moment) we can do so as follows. Suppose for concreteness that we have 5 n_i so that the $s(n_i, j)$ are the same, and so that $\gamma_{n_i} = 5$. We construct our sequence of columns as copies corresponding the following sequences (which are in turn represented as $f_i : I \to \mathbb{N}_0$).

$$\begin{gathered}
00 \ldots 01 \ldots 02 \ldots 03 \ldots 04 \\
01 \ldots 02 \ldots 03 \ldots 04 \ldots 10 \\
02 \ldots 03 \ldots 04 \ldots 10 \ldots 11 \\
03 \ldots 04 \ldots 10 \ldots 11 \ldots 12 \\
04 \ldots 10 \ldots 11 \ldots 12 \ldots 13 \\
10 \ldots 11 \ldots 12 \ldots 13 \ldots 14
\end{gathered}$$

where the pairs of listed entries correspond to $f(n_i)$ and $f(n_i + 1)$, and where the implied entries are all 0 (or at least the same for each sequence).

Note that incrementing $f(n_i)$ increases the spacing by $h_{n_i} + s(n_i, f(N + n_i))$. Also since $h_{n_i+1} = 5h_{n_i} + s(n_i, 0) + s(n_i, 1) + s(n_i, 2) + s(n_i, 3) + s(n_i, 4)$, changing $f(n_i + 1), f(n_i)$ from 04 to 10 increases the spacing by $h_{n_i} + s(n_i, 4)$. Since the $s(n_i, j)$ are the same, call them $s(j)$. Then the distance between adjacent pairs of the copies specified above is $h_{n_1} + h_{n_2} + h_{n_3} + h_{n_4} + h_{n_5} + s(0) + s(1) + s(2) + s(3) + s(4)$.

Unfortunately, this does not account for the group action. If on the other hand, we were lucky enough to have pairs of $F(n_i) = F(N_{i+1}) = e$ with $\gamma_e = 5$, we could then let our f_i be the functions

$$\begin{gathered}
13 \ldots 22 \ldots 31 \ldots 40 \\
23 \ldots 32 \ldots 41 \ldots 01 \\
33 \ldots 42 \ldots 02 \ldots 11 \\
43 \ldots 03 \ldots 12 \ldots 21 \\
04 \ldots 13 \ldots 22 \ldots 31
\end{gathered}$$

where again the pairs shown correspond to $f(n_i), f(n_{i+1})$, and f is 0 elsewhere. Notice that in each transition exactly one 0 is changed to a 1 one 1 to a 2,..., and one 4 changed to a 0. This implies that all of these copies are in the same column. Furthermore the spacing between adjacent copies is easily seen to be $h_{n_1} + h_{n_2} + h_{n_3} + h_{n_4} + s(0) + s(1) + s(2) + s(3) + s(4)$.

Our final construction will be a generalization of the one above.

Find a sequence of consecutive values of F of the form $e_1, e_2, \ldots, e_w, a, b$ with $\prod_{i=1}^{w} \gamma_{e_i} > k$. Extend this to a sequence of the form

$$e_1, e_2, \ldots, e_w, a, b, d_1, \ldots, d_z, e_1, e_2, \ldots, e_w.$$

Find an interval I so that F applied to I yields $\left(\prod_{i=1}^{n} \gamma_{e_i}\right) \gamma_a \gamma_b \left(\prod_{i=1}^{l} \gamma_{d_i}\right) - 1$ non-intersecting copies of the above sequence. We will make our f_i all be 0 off of these subsequences.

We define f_0 on these subintervals so that it takes every possible set of values on the first $w + z + 2$ entries (as limited by property 1) except for all 0's. On each such block we let f_0 take values on the two instances of e_i that add up to $\gamma_{e_i} - 1$.

We think of the values of an f_i on such a block as an appropriate radix representation (leftmost digit least significant) of a natural number. We inductively define f_{i+1} to represent the number one larger. In particular if on some such block f_i takes the values $\gamma_1 - 1, \ldots, \gamma_s - 1, v_{s+1}, \ldots, v_{2w+z+2}$ where γ_j is the appropriate γ for the j^{th} term and $v_{s+1} < \gamma_{s+1} - 1$, then on this block f_{i+1} takes values $0, \ldots, 0, v_{s+1} + 1, v_{s+2}, \ldots, v_{2w+z+2}$.

We note that Property 1 is clearly satisfied. Property 3 is satisfied because if we consider the blocks on which f_0 has values $\gamma_{e_1} - 1, \ldots, \gamma_{e_w} - 1, x, y$, then f_0 and f_i have the appropriate values on the a, b terms. (Using the fact that $\prod_{i=1}^{n} \gamma_{e_i} > i$.)

We note that by Lemma 3.3 that the difference in heights of the consecutive copies indexed by the f_i is a fixed sum of h_N corresponding to the beginnings of blocks, plus a correction term based on changes in the number of N for which $f_i(N)$ and $F(N)$ have particular given values. Combining this with Lemma 3.1 we need only show that the number of such N remains constant.

For each f_i and each block we associate the three numbers corresponding to the natural numbers given by the appropriate radix representations $f_i(n_0+1), \ldots, f_i(n_0+w)$, and $f_i(n_0+w+1), \ldots, f_0(w+z+2)$ and $f_i(n_0+w+z+3), \ldots, f_0(2w+z+3)$, where n_0+1 is the beginning of the subinterval. It suffices to show that the multiplicities with which numbers show up in either of the first and third places remain constant, and that the multiplicities with which numbers show up in the second place remain constant.

In f_i the first and second places take all possible values except for $i, 0$. Now if M_1, M_2 are one more than the maximum possible values in the first and second places, then f_i and f_0 agree in the third place except when the value of f_0 in the first two are $M_1 - 1 - l, M_2 - 1$ with $0 \leq l < i$. In that case we have carry over to the third place and there f_i has the value of $l + 1$ instead of l. So in the third place, f_i and an extra i and one fewer 0. This completes our proof.

□

We can now prove Theorem 2.1.

Proof. Let A be a set of positive measure. Let k be an integer. From Lemma 5.4 we can see that there is an $\epsilon > 0$ and columns of arbitrarily high generation so that these columns have $k + 1$ copies whose size is more than ϵ of that of the original, so that these copies are in the same column and consecutive copies are separated by the same amount. Take a level of such a generation that is more than $\left(1 - \frac{\epsilon}{k+1}\right)$-full of A. Then the copies of this level in those copies of its column are each more than $\left(\frac{k}{k+1}\right)$-full of A and have the property that for some n, T^{in}

of the bottom level is another one of the levels for $1 \leq i \leq k$. Hence for this n, $\mu(A \cap T^n(A) \cap \cdots \cap T^{kn}(A)) > 0$. □

We now prove Theorem 2.2.

Proof. We begin with the levels given to us by Lemma 5.3. We wish to show that there is an $\epsilon > 0$ so that for m pairs of levels of arbitrarily high generation with the same separation between corresponding levels as we have between I_i and J_i, we can find copies of these levels of size more than ϵ times that of the original so that corresponding copies are in the same column, and so that the difference in heights between the i^{th} pair of copies is proportional to k_i. This would prove our theorem with a simple application of double approximation.

Notice that the intersection of the span of the set in Lemma 5.1 with $\mathbb{Z} \times G \times \{0\}$ is the span of $(c_{M,i}, 0)$, since $t_{N,i} - t_{N,j} = c_{N-1,i} - c_{N-1,j}$. Therefore, $(D, 0)$ is in the span of the $c_{M,i}$. Hence if $T^{Dd_i}(I_i) = J_i$ then we can write

$$(Dd_i, 0) = \sum_{M,j} \alpha_{M,j,i} c_{M,j}$$

Using the f_0, f_{k_i} from Lemma 5.4 to index copies of columns, we can find an $\epsilon > 0$ so that for arbitrarily large generations of columns, we can find m pairs of copies satisfying:

- Each pair of copies is in the same column.
- Each pair of copies are separated by a power of T proportional to k_i.
- Each copy is at least ϵ the size of the original.
- For any M, j with there are at least $|\alpha_{M,j,i}|$ integers n so that $F(n) = F(M)$, $F(n+1) = F(M+1)$ and so that in the indexing of the first and second copy in the i^{th} pair, the index of the copy at the digits corresponding to n and $n+1$ are either 0,j and 1,j or $\gamma_n - 1$,j and 0,$j+1$ respectively.

If we change $|\alpha_{M,j,i}|$ of one of these types described in the last point to the other, then we keep these pairs of copies in the same column, but alter their relative height difference by Dd_i. This provides what we need for the double approximation. □

Remark 5.5. *Note that the proof of Theorem 2.2 implies that T is power weakly mixing if and only if it is totally ergodic. Note also that to check condition 2, it is sufficient to first check to see if a D from Lemma 5.1 exists, and if one does, to check condition 2 modulo D for a particular N so that infinitely often $F(M) = F(N)$ and $h_N \equiv h_M \pmod{D}$. This reduces checking condition 2 to a finite computation.*

Notice also that if $\mathrm{Im}\ (\mathrm{F}) = \{(n, s_0, \ldots, s_{n-1}, 0, g_1, \ldots, g_{n-1})\}$, *then condition 2 can be written in the simple form that* $(1, 0)$ *is in the span of* $\{(s_i + h_N, g_{i+1} - g_i), (s_{n-1}, -g_{n-1})\}$ *for all* N.

6. Examples

These first few will be where Γ has a single element as in the last remark.

Consider for example the Chacón-m transformation for $m \geq 2$. Following N. Friedman, a Chacón-m transformation is a rank-one transformation where column C_{n+1} is obtained from column C_n but cutting each level of C_n into m sub-levels, stacking from left to right and placing a spacer on top of the last level; see Section 9 for more details on rank-one constructions. We can define a Chacón-m transformation by letting $G = \{0\}$, $n = m$ and $s_i = 0$ for $0 \leq i \leq n-2$, $s_{n-1} = 1$, and $g_i = 0$ for $0 \leq i \leq n-1$. Clearly, $\{g_i\}$ generates G, so condition 1 is satisfied. Since

$(1,0) = (s_{n-1}, -g_{n-1})$ it is in the span of $\{(s_i + h_N, g_{i+1} - g_i), (s_{n-1}, -g_{n-1})\}$. Therefore, it is power weakly mixing.

Consider the transformation defined by $G = \{0\}$, $n = 3$ and the sequence $(1, 1, 0, 0, 0, 0)$. Condition 2 states that $(1, 0)$ is in the span of $\{(1 + h_N, 0), (1 + h_N, 0), (0, 0)\} = (1 + h_N)\mathbb{Z} \times \{0\}$. Which does not hold for any, h_N. Therefore this transformation is not power weakly mixing (in fact it is not T^2 ergodic).

Consider the transformation, T defined by the group $G = \mathbb{Z}$, $n = 5$ and the set $(0, 0, 0, 1, 0, 0, 1, 0, 0, 0)$. It satisfies condition 1 since $g_1 = 1$ generates G. Condition 2 states that $(1, 0) \in \operatorname{span}\{(h_N, 1), (h_N, -1), (h_N, 0), (1+h_N, 0), (0, 0)\}$. This clearly holds since $(1, 0) = (1 + h_N, 0) - (h_N, 0)$. Therefore, T is a power weakly mixing, infinite measure-preserving transformation.

Lastly consider the group G to be any countably generated Abelian group with generators e_i for $i \in \mathbb{N}_0$. If $n \in \mathbb{N}$, let $e(n) = k$ where k is the largest power of 2 such that 2^k divides n. Let $F(n) = (4, 0, 0, 0, 1, 0, 0, e_{e(n+1)}, 0)$. F clearly satisfies the necessary condition. Notice that T is a G-extension of Chacón-4. Condition 1 is clearly satisfied. Condition 2 is satisfied since

$$(1, 0) = ((s(1, 0) + s(0, 3) - s(0, 0)), (g(1, 1) - g(1, 0) - g(0, 3) - g(0, 1))).$$

Therefore, T is power weakly mixing.

7. Non-totally ergodic 2-point extension

As an example of the above we analyze what happens in the particular case where $G = \mathbb{Z}/2\mathbb{Z}$ and $F(n) = (2, 0, 1, 0, 1)$.

Consider the two-point extension T of the Chacon-2 transformation formed as follows. Begin with two intervals of equal size – call them columns $C_{0,0}$ and $C_{0,1}$. These will be known as the generation zero columns. To define the generation $n+1$ columns, cut each of the generation n columns in half, stacking the right half of $C_{n,1}$ over the left half of $C_{n,0}$ and vice versa. Then add a spacer to the top of the two columns thus formed to yield the generation $n + 1$ columns. The transformation T is defined to map each point to the point directly above it. To see that T is indeed a two-point extension of Chacon-2, associate each point in $C_{n,0}$ with the corresponding point in $C_{n,1}$ for all n. The resulting space and transformation are exactly Chacon-2. It is well-known that Chacon-2 is a weakly mixing transformation, therefore totally ergodic, but we will show that the two-point extension T is not even T^2 ergodic.

Let I and J be the top and middle levels, respectively, in $C_{1,0}$. Suppose for some m we have $\mu(T^m(I) \cap J) > 0$. Then there must be some generation in which there is a copy J' of J above a copy I' of I by a distance m levels. That is, $T^m(I') = J'$, and we will write $d(I', J') = m$. We will show that this cannot be the case if m is even.

We prove by induction on n that for all n, any two copies of I in one of the generation n columns must be an even distance apart. This is vacuously true for $n = 1, 2$. Assume it is true for generation n. Label the left halves of the top copies of I in $C_{n,0}$ and $C_{n,1}$ as I_0 and I_1, respectively. Label the right halves of the bottom copies of I as I_2 and I_3, respectively. To prove the claim we must show that the distances from I_0 to I_3 and from I_1 to I_2 are both even. For convenience label the right halves of the bottom levels of $C_{n,0}$ and $C_{n,1}$ by K_0 and K_1 respectively. Then $d(K_0, I_2) = 2$ and $d(K_1, I_3) = 5$ since this is true for $n = 2$ and since the bottoms of columns are preserved through later generations. The distance from I_0 to K_1 is given by:

$$d(I_0, K_1) = \begin{cases} n+3 & \text{if } n \text{ is even;} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

Similarly, we have

$$d(I_1, K_0) = \begin{cases} n & \text{if } n \text{ is even;} \\ n+3 & \text{if } n \text{ is odd.} \end{cases}$$

These are true because they hold for $n = 2$ and by induction on n. The fundamental idea is that the top of $C_{n,1}$ looks like the top of $C_{n-1,0}$ with an extra spacer added on top. From this the above statements are easily shown by induction.

Both $d(I_0, I_3) = d(I_0, K_1) + d(K_1, I_3)$ and $d(I_1, I_2) = d(I_1, K_0) + d(K_0, I_2)$ must then be even, independent of n. By induction, $\mu(T^m(I) \cap I) > 0$ implies m is even. Since each copy of J lies directly below a copy of I, this means that $\mu(T^m(I) \cap J) > 0$ implies m is odd. Therefore $\mu(T^{2m}(I) \cap J) = 0$ for all m, so T^2 is not ergodic.

8. Conservativity and Recurrence on a Sufficient Class

In this section all transformations are assumed to be infinite measure-preserving, and not necessarily invertible. If for any measurable set A we have $\mu(A \setminus \bigcup_{i=1}^{\infty} T^{-i}A) = 0$ then T is said to be **recurrent**. For sets of finite measure this condition is equivalent to $\mu(A \cap \bigcup_{i=1}^{\infty} T^{-i}A) = \mu(A)$. A class $\mathcal{C}$ of subsets of X is called a **sufficient class** if it satisfies the following approximation property for all measurable $A \subset X$:

$$\mu(A) = \inf \left\{ \sum_{j=1}^{\infty} \mu(I_j) : \{I_j\} \text{ cover A and } I_j \in \mathcal{C} \right\}$$

A transformation is said to be **conservative on** $\mathcal{C}$ or **recurrent on** $\mathcal{C}$ if the condition for conservativity or recurrence holds for all $I \in \mathcal{C}$ of positive measure, but not necessarily for all measurable sets. While conservativity and recurrence are known to be equivalent, we show in this section that conservativity and recurrence on a sufficient class $\mathcal{C}$ are not equivalent. In particular recurrence on a sufficient class implies recurrence, but the same is not true for conservativity.

Consider the following infinite measure-preserving transformation $T : \mathbb{R} \to \mathbb{R}$ which is conservative on the sufficient class $\mathcal{C} = \{I : I \text{ is a finite open interval}\}$. It is well known that there exist sets $K \subset [0,1)$ and $K^c = [0,1) \setminus K$ of positive measure such that $\mu(I \cap K) > 0$ if and only if $\mu(I \cap K^c) > 0$ for all $I \in \mathcal{C}$. Define T by:

$$T(x) = \begin{cases} x & \text{if } x \bmod 1 \in K; \\ x+1 & \text{if } x \bmod 1 \in K^c. \end{cases}$$

Then T is conservative on $\mathcal{C}$, but $\mu(T^{-n}(K^c) \cap K^c) = 0$ for all nonzero n so T is not conservative. Note however that T is not recurrent on $\mathcal{C}$.

Proposition 8.1. *Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space, and let T be an infinite measure-preserving transformation. If T is recurrent on a sufficient class, then T is recurrent.*

Proof. Let $\mathcal{C}$ be a sufficient class, and suppose that T is not recurrent. Then T is not conservative, so there exists a set A of positive measure such that $\mu(A \cap T^{-n}(A)) = 0$ for all $n > 0$. Perhaps taking a subset, we may assume that A has finite measure.

We can then find a set $I \in \mathcal{C}$ such that $\mu(A \cap I) > \frac{1}{2}\mu(I)$. Note that if a subset of $I \setminus A$ with positive measure is mapped into A by T^{-i} for some i it will never be mapped into A for any $j > i$ (by hypothesis, any subset of A of positive measure is never mapped into A under iteration by T^{-1}), so we have

$$\begin{aligned}\mu\left(I \cap \bigcup_{i=1}^{\infty} T^{-i}(I)\right) &= \mu\left((I \setminus A) \cap \bigcup_{i=1}^{\infty} T^{-i}(I)\right) + \mu\left((I \cap A) \cap \bigcup_{i=1}^{\infty} T^{-i}(I \setminus A)\right)\\ &\leq \mu\left(I \setminus A\right) + \mu\left(I \setminus A\right)\\ &< \frac{1}{2}\mu(I) + \frac{1}{2}\mu(I) = \mu(I)\end{aligned}$$

Thus T is not recurrent on $\mathcal{C}$. □

To see that a regularity condition on the space $(X, \mathcal{B}, \mu)$ is necessary, let $X = \mathbb{R}$ and $\mathcal{B} = 2^{\mathbb{R}}$. Define $\mu(A)$ to be zero if A is finite or countable and infinity otherwise. The collection $\mathcal{C}$ of all singletons in $\mathbb{R}$ along with the set $\mathbb{R}$ itself is a sufficient class for this space. Any bijective map on X is measure-preserving and recurrent on $\mathcal{C}$, but in general it need not be recurrent.

9. Power Product Conservative

In this section we obtain a condition for rank-one transformations that implies power product conservativity. A notion that has been used to study conservativity of products is that of positive type. A transformation T is of **positive type** if $\limsup_{n\to\infty} \mu(T^n(A) \cap A) > 0$ for all sets A of positive measure. Clearly, if T is of positive type, then it is conservative. It was shown in [2] that if T is of positive type, then for each positive integer d, the Cartesian product of d copies of T is of positive type, so positive type implies infinite conservative index. But it is easily verified that the transformation T_1 of [3] is of positive type but as already mentioned $T_1 \times T_1^2$ is not conservative, so T_1 is not power product conservative, showing that positive type does not imply power product conservative. Our condition can be used to show show that some infinite measure-preserving staircases are power product conservative.

First, we introduce some notation for constructing measure-preserving rank-one transformations. We start with a column C_0, which is a unit interval. Let r_n be a sequence of integers with $r_n \geq 2$. At each stage n, we have a column C_n that consists of h_n intervals (h_n denotes the height of column C_n, which is defined to be the number of intervals in the column). We denote the intervals in C_n by $I_{n,0}, I_{n,1}, ..., I_{n,h_n-1}$, where $I_{n,0}$ is the interval at the lowest level. A column determines a map on all of its levels but the top, where each interval is mapped to the interval above it by the canonical translation. Column C_{n+1} is obtained from column C_n by cutting and stacking according to the following procedure. Cut all intervals of column C_n into r_n subintervals of the same measure to form subcolumns, $C_n^{[0]}, C_n^{[1]}, ..., C_n^{[r_n-1]}$. Then we may put spacers on top of the subcolumns. Let $\{s_{n,i}\}_{i=0}^{r_n-1}$ be a doubly indexed sequence of non-negative integers. The sequence $s_{n,0}, s_{n,1}, ..., s_{n,r_n-1}$ specifies the number of spacers on each respective subcolumn. Then stack subcolumns on top of one another, with each subcolumn going underneath its adjacent subcolumn to the right, so that the rightmost subcolumn goes on the very top. This cutting and stacking procedure obtains column C_{n+1}. This defines a sequence of columns C_n and as the width of the column approaches 0, it defines a measure-preserving transformation on a finite or an infinite interval.

We prove the following Theorem:

Theorem 9.1. *Let T be a rank-one transformation with sequence of cuts $\{r_n\}$. If for all $d > 0$,*

$$\liminf_{n\to\infty} \frac{h_n^{d-1}}{\prod_{i=0}^{n-1} r_i^d} = 0$$

then T is power product conservative.

We do this by proving the following more precise theorem.

Theorem 9.2. *Let T be a rank-one transformation with sequence of cuts $\{r_n\}$. Let $\{k_i\}_{i=1}^d$ be integers. If*

$$\liminf_{n\to\infty} \frac{h_n^{d-1}}{\prod_{i=0}^{n-1} r_i^d} = 0,$$

then $T^{k_1} \times T^{k_2} \times \ldots \times T^{k_d}$ is conservative.

Fix the k_i. Let $S = T^{k_1} \times T^{k_2} \times \ldots \times T^{k_d}$. For a column C_n define an equivalence relation $\sim_n$ on d-fold products of levels so that $I \sim_n J$ if and only if for some integer N, $S^N(I) = J$.

Lemma 9.3. *Suppose that $A \subset X^d$ is a set where $\mu(S^n(A) \cap A) = 0$ for all $n \neq 0$. Let L be the $\sim_n$ equivalence class of products of levels equivalent to I. Then*

$$\mu\left(A \cap \left(\bigcup_{J\in L} J\right)\right) \leq \mu(I).$$

Proof. First modify A by a set of measure 0 so that $S^n(A) \cap A = \emptyset$ for all $n \neq 0$. Note that $\bigcup_{J\in L} J$ is a subset of $\bigcup_{N\in\mathbb{Z}} S^N(I)$. In fact for some subset $P \subset \mathbb{Z}$, we can write

$$\bigcup_{J\in L} J = \coprod_{N\in P} S^N(I).$$

We therefore, think of this set as $P \times I$. Let χ_A be the characteristic function of A on $P \times I$. We note that since $S^n(A) \cap A = \emptyset$ for $n \neq 0$, that $\chi_A(a, x)$ and $\chi_A(b, x)$ cannot both be 1 for $a \neq b$. Hence

$$\int_P \chi_A(n, x)dn = \sum_{n\in P} \chi_A(n, x) \leq 1.$$

So by changing the order of integration, we get that

$$\begin{aligned}
\mu\left(A \cap \left(\bigcup_{J\in L} J\right)\right) &= \int_{P\times I} \chi_A(n, x)d\mu \\
&= \int_I \int_P \chi_A(n, x)dndx \\
&\leq \int_I dx \\
&= \mu(I).
\end{aligned}$$

$\square$

Lemma 9.4. *The number of equivalence classes of $\sim_n$ is at most*

$$\left(\sum_{i=1}^{d} k_i\right) h_n^{d-1}.$$

Proof. To each product of levels, we can associate a d-tuple of integers in the range $[1, h_n]$ representing the heights of the levels. It is clear that the product associated with $\{a_i\}$ and the product associated with $\{a_i+k_i\}$ are equivalent. Each equivalence class has at least one element with $\sum_{i=1}^{d} a_i$ minimal. We will bound the number of such sequences.

Clearly, for $\sum_{i=1}^{d} a_i$ to be a minimal representative of an equivalence class, $\{a_i - k_i\}$ cannot be a valid sequence. Therefore $a_i \leq k_i$ for some i. The number of such sequences with $a_i \leq k_i$ for a given i is at most $k_i h_n^{d-1}$. Summing over i gives our result. □

We are now prepared to prove Theorem 9.2.

Proof. Suppose that

$$\liminf_{n\to\infty} \frac{h_n^{d-1}}{\prod_{i=0}^{n-1} r_i^d} = 0,$$

and that $A \subset X^d$ satisfies $\mu\left(S^n(A) \cap A\right) = 0$ for all $n \neq 0$. We wish to bound $\mu\left(A \cap C_n^d\right)$. Lemma 9.3 says that the intersection of A with any equivalence class of levels is at most the size of a level, or

$$\left(\prod_{i=0}^{n-1} r_i^d\right)^{-1}$$

. Lemma 9.4 says that there are at most $K h_n^{d-1}$ equivalence classes where $K = \sum_{i=1}^{d} k_i$. Therefore,

$$\mu\left(A \cap C_n^d\right) \leq \frac{K h_n^{d-1}}{\prod_{i=0}^{n-1} r_i^d}.$$

Hence

$$\liminf_{n\to\infty} \mu\left(A \cap C_n^d\right) = 0.$$

Since C_n^d exhausts X^d, this implies that $\mu(A) = 0$. □

A rank-one transformation such that $s_{n,i} = i$ for $0 \leq i \leq r_n - 1$, is called a staircase transformation (i.e., the spacers are added in a staircase fashion). It may be finite or infinite measure-preserving depending on the growth of r_n. In the case that it is finite measure-preserving, Adams showed in [1] that T is mixing provided that $r_n^2/h_n \to 0$ as $n \to \infty$.

Corollary 9.5. *There exists an infinite measure-preserving staircase transformation T such that T is power product conservative.*

Proof. Let T be the classical staircase transformation with $r_m = 2^{2^m}$. Direct computation shows that $h_m \leq \frac{m+1}{2} r_m$ holds for $m = 0$, and we will prove that it holds for all m by induction. If this holds for m we can apply the definition of h_{m+1} for a staircase transformation to yield

$$h_{m+1} = h_m r_m + \frac{1}{2} r_m(r_m - 1) \leq \frac{m+1}{2} r_m^2 + \frac{1}{2} r_m^2 = \frac{m+2}{2} r_m^2 = \frac{m+2}{2} r_{m+1}.$$

Then we can bound

$$\frac{h_m^{d-1}}{(\prod_{i=0}^{m-1} r_i)^d} = \frac{h_m^{d-1}}{\left(\frac{r_m}{2}\right)^d} = \frac{1}{h_m}\left(\frac{2h_m}{r_m}\right)^d \leq \frac{(m+1)^d}{h_m} \leq \frac{(m+1)^d}{r_m}$$

which approaches 0 as $m \to \infty$ for all $d \geq 1$. Hence, this transformation T satisfies the condition of Theorem 9.1. ☐

Corollary 9.6. *Let T be a rank-one transformation with $r_n = 2^{2^n}$ and such that*

$$s_{n,i} = \begin{cases} 2i & \textit{if } \ 2|i; \\ 0 & \textit{otherwise.} \end{cases}$$

Then T is an infinite measure-preserving transformation that is power product conservative but such that T^2 is not ergodic.

Proof. An argument similar to that in Corollary 9.5 shows that T is power product conservative. Let I_1 and I_2 be two levels in some column C_m such that $\alpha(I_1)-\alpha(I_2)$ is a positive odd number. Now consider sublevels of I_1 and I_2, denoted by I_1' and I_2' respectively, in some column C_n for $n > m$. From the construction, $\alpha(I_1') - \alpha(I_2')$ will always be an odd number. It follows that there does not exist an integer t such that $T^{2t}(I_1') = I_2'$. Hence, T^2 is not ergodic. ☐

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