SUBSEQUENCE RATIONAL ERGODICITY OF RANK-ONE TRANSFORMATIONS

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ABSTRACT. We show that all rank-one transformations are subsequence boundedly rationally ergodic and that there exist rank-one transformations that are not weakly rationally ergodic.

1. PRELIMINARIES

We consider standard Borel measure spaces, denoted $(X, \mathcal{B}, \mu)$, where μ is a nonatomic σ-finite measure and are interested in the case when μ is infinite. We study invertible measure-preserving transformations $T : X \to X$. A transformation T is **ergodic** if for every invariant set A, $\mu(A) = 0$ or $\mu(X \setminus A) = 0$, and **conservative** if for every measurable set of positive measure A, there exists $n \in \mathbb{N}$ such that $\mu(A \cap T^{-n}A) > 0$. Then T is conservative and ergodic if and only if every set A of positive measure sweeps out: $\bigcup_{n=0}^{\infty} T^{-n}A = X$.

When T is ergodic and finite measure-preserving, the ergodic theorem gives a quantitative estimate for the average number of visits to a measurable set for almost every point; for example, the law of large numbers can be obtained as a consequence of the ergodic theorem. In infinite measure, however, the averages given by the ergodic theorem (of visits to a finite measure set) converge to 0, and Aaronson has shown that there is no normalizing sequence of constants for which the ergodic averages (using this sequence) of visits to a set converge to the measure of the set, see [3, 2.4.1]. In [1], Aaronson defined the notions of weak rational ergodicity and rational ergodicity as giving quantitative estimates for ergodic averages for sets satisfying certain conditions, and extended them to bounded rational ergodicity in [2]. More recently, in [4], he defined a notion called rational weak mixing that is stronger than weak rational ergodicity. Not every conservative ergodic infinite measure-preserving transformation is rationally ergodic

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[1], [3], and thus there has been interest in transformations that satisfy this property. Examples of such transformations can be seen in the references in [3], [7].

In this paper, we first prove that all rank-one infinite measure-preserving transformations are subsequence boundedly rationally ergodic (Theorem 2.2), a notion where the bounded rational ergodicity condition is satisfied only for a subsequence. Our work builds on the paper by Dai, Garcia, Păduraru and Silva [7], where these properties are proved for a large class of rank-one transformations satisfying a condition called exponential growth. We extend the techniques of [7] and remove these assumptions. While in [7] the authors study first the weak rational ergodicity property, we work directly with the bounded rational ergodicity property. A consequence of Theorem 2.2 is that all rank-one transformations are not squashable, thus Maharam transformations are not rank-one. We also prove that there exist rank-one transformations that are not weakly rationally ergodic, so not rationally ergodic (the first examples of non-rationally ergodic transformations were Maharam transformations [1]).

Rank-one transformations have been studied extensively in ergodic theory and are a source of important examples and counterexamples (see, e.g. [10] and [9] for the finite measure-preserving and infinite measure-preserving cases, respectively). It is well known that in the finite measure-preserving case, rank-one transformations are generic under the weak (also called coarse) topology on the group of invertible transformations on a standard space. While it is expected that this result is also true in infinite measure, we have not found a reference with a proof so we have included a proof of this fact in Section 3. As Aaronson [4, §10] has shown that the class of weakly rationally ergodic transformations is meager in the group of measure-preserving transformations, a consequence of our results is that there exist rank-one transformations that are subsequence boundedly rationally ergodic but not weakly rationally ergodic. Rank-one examples were not known earlier and while we know existence we do not know an explicit construction.

Finally, in Section 4 we study the more recent property of rational weak mixing and give conditions for rank-one transformations not to be rational weak mixing.

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1.2. **Notions of Rational Ergodicity.** Given an invertible, measure-preserving transformation T and a set $F \subset X$ of positive finite measure, define the **intrinsic weight sequence** of F to be

$$u_k(F) = \frac{\mu(F \cap T^k F)}{\mu(F)^2} \quad \text{and write} \quad a_n(F) = \sum_{k=0}^{n-1} u_k(F).$$

A set F is said to **sweep out** if $\mu\left(X \backslash \bigcup_{i=0}^{\infty} T^i F\right) = 0$. By Maharam's theorem, if there is a sweep out set of finite measure the transformation is conservative. A transformation T is said to be **weakly rationally ergodic** [1] if there exists a set $F \subset X$ of positive finite measure that sweeps out such that for all measurable $A, B \subset F$,

$$\frac{1}{a_n(F)} \sum_{k=0}^{n-1} \mu(A \cap T^k B) \to \mu(A)\mu(B) \tag{1.1}$$

as $n \to \infty$. If T is weakly rationally ergodic it is conservative ergodic; furthermore any set $T^i(F)$ will work in place of F.

For $f : X \to X$ a measurable function, define

$$S_n(f) := \sum_{k=0}^{n-1} f \circ T^k.$$

A transformation T is said to be **rationally ergodic** if there exists an $M < \infty$ and $F \subset X$ that sweeps out of positive finite measure such that **Renyi's Inequality** is satisfied:

$$\int_F (S_n(1_F))^2 d\mu \leq M \left(\int_F S_n(1_F) d\mu \right)^2 \tag{1.2}$$

for all $n \in \mathbb{N}$. Furthermore, T is **boundedly rationally ergodic** if there exists an $F \subset X$ of positive finite measure that sweeps out such that

$$\sup_{n \geq 1} \left\| \frac{1}{a_n(F)} S_n(1_F) \right\|_\infty < \infty. \tag{1.3}$$

If (1.2) holds only for a subset $\{n_i\} \subset \mathbb{N}$, then we say that T is **subsequence rationally ergodic** and if (1.3) holds only for a subset $\{n_i\} \subset \mathbb{N}$, then we say that T is **subsequence boundedly rationally ergodic**. It was shown in [1] that rational ergodicity implies weak rational ergodicity and in [2] that bounded rational ergodicity implies rational ergodicity.

Additionally, T is **rationally weakly mixing** [4] if there exists a sweep out set $F \subset X$ of finite measure such that for all measurable $A, B \subset F$, we have that

$$\frac{1}{a_n(F)} \sum_{k=0}^{n-1} |\mu(A \cap T^k B) - \mu(A)\mu(B)u_k(F)| \to 0. \tag{1.4}$$

If the limit (1.1) only holds along a subsequence $\{n_i\} \subset \mathbb{N}$, then we say that the transformation T is **subsequence weakly rationally ergodic** along $\{n_i\}$ and if (1.4) holds along a subsequence, then we say that T is **subsequence rationally weak mixing** along $\{n_i\}$. Rational weak mixing along a sequence implies weak rational ergodicity along the same sequence [4].

1.3. **Rank-One Transformations.** Let T be an invertible measure-preserving transformation on X. A **Rokhlin column**, or **column**, is a collection of pairwise disjoint measurable sets $B, T(B), \ldots, T^{h-1}(B)$. We call any single such set in the column a **level**, and h the height of the column. In a column we imagine the levels being stacked on top of each other, with B at the bottom and $T^{h-1}(B)$ at the top.

An invertible measure-preserving transformation T is said to be **rank-one** if there exits a sequence of Rokhlin columns $C_n = \{B_n, \ldots, T^{h_n-1}(B_n)\}$ such that for any measurable set $A \subset X$ of finite measure and $\varepsilon > 0$, there exists an N such that for all $n \geq N$, we have that $\mu(A \triangle B'_n) < \varepsilon$ for some B'_n a union of levels of C_n. It follows that such a T is conservative ergodic.

It can be shown that every rank-one transformation can be constructed so that the sequence of columns $\{C_n\}$ is refining in the sense that every level of C_n is a union of levels in the previous column C_{n-1}, and $B_{n+1} \subset B_n$. This was shown in the finite measure-preserving case in [6, Lemma 9] and the same ideas work in the infinite case–full details are included in a version of this paper on the archives. Thus, we may assume that $B_{n-1} = \bigcup_{i \in E} T^i(B_n)$ for some $E \subset \{0, 1, \ldots, h_n - 1\}$.

1.4. **Rank-Ones as Cutting and Stacking.** When constructing rank-one examples it is useful to think of a process resembling cutting and stacking. Our first column C_0 consists of a single measurable set of positive finite measure.

In each step, given C_n, we cut the column into r_n subcolumns, where $r_n \geq 2$. That is, we divide B_n, the base of column C_n, into r_n sets of equal measure. If we label these sets as $B_{n,0}, B_{n,1}, \ldots, B_{n,r_n-1}$, then our first subcolumn would be $B_{n,0}, T(B_{n,0}), \ldots, T^{h_n-1}(B_{n,0})$ and our other $r_n - 1$ subcolumns would be defined similarly. Above any subcolumn, we may add any number of new levels, called **spacers**, under the condition that these new levels are also pairwise disjoint. Then, C_{n+1} is constructed by stacking each subcolumn with its associated spacers under the next subcolumn. If A is a level below a level B in C_{n+1}, then we must have that $T(A) = B$, $\mu(A) = \mu(B)$, and T is invertible on A. Thus, C_{n+1} will consist of r_n copies of C_n possibly separated by spacers.

Suppose that T is a rank-one transformation. Given a column C_n of T, we let h_n be the height of the column, w_n be the width (the measure of each level), and r_n be the number of subcolumns that C_n is cut into. Given a level J in C_n, we denote the height of J in C_n by $h(J)$. If we fix J a level in C_n, and an $m \geq n$, we define the **descendants** of J in C_m to be the set of levels in C_m whose disjoint union is C_n. We then define $D(J, m)$ as the set of the heights of these levels.

To form C_{n+1} from C_n, we first cut C_n into r_n subcolumns, which we denote by $C_n[0], C_n[1], \ldots, C_n[r_n - 1]$. Then, before stacking, we add $s_{n,k}$ spacers above each $C_n[k]$, $0 \leq k \leq r_n - 1$, where each $s_{n,k}$ is in $\mathbb{Z}_{\geq 0}$.

For full generality here, we should allow for the possibility of spacers beneath the first subcolumn $C_n[0]$. However, one can see that this is not necessary as any spacers placed under the first subcolumn $C_n[0]$ can be added as spacers above other subcolumns in later columns C_m, $m \geq n$.

Then define $h_{n,k} = h_n + s_{n,k}$ for each k. If we let

$$H_n = \{0\} \cup \left\{ \sum_{k=0}^{i} h_{n,k} : 0 \leq i < r_n - 1 \right\},$$

we have that, for J in C_j and $N \geq j$,

$$D(J, N) = h(J) + H_j \oplus H_{j+1} \oplus \cdots \oplus H_{N-1}.$$

With this notation we can easily find the number of elements in $D(J, N)$ to be $|D(J, N)| = |H_j| \cdot |H_{j+1}| \cdots |H_{N-1}| = r_j \cdot r_{j+1} \cdots r_{N-1}$. For a level J in C_n and $m \geq n$, we define the maximum height of its descendants in C_m to be $M_m = \max\{D(J, m)\}$.

2. Bounded Rational Ergodicity

We begin with the following lemma that will allow us to bound how many times the intervals $\{T^k I\}$ cover I for $|k| \leq M_m$.

Lemma 2.1. *Let T be a rank-one transformation and I the level in C_0. Then, the sets $\{T^k I\}$ for $|k| \leq M_m$ cover almost every point of I between $|D(I,m)|$ and $2 \cdot |D(I,m)|$ times.*

Proof. We first show that for almost every $x \in I$,

$$|D(I,m)| \leq |\{k \in [-M_m, M_m] : \exists y \in I, T^k y = x\}| \leq 2|D(I,m)|. \tag{2.1}$$

We fix an $x \in I$ and let J be the level in C_m that contains x and J' be any descendant of I in C_m. Set $d = h(J) - h(J')$. Clearly, $-M_m \leq d \leq M_m$ and so $T^d J' = J$. This holds for all $J' \in D(I,m)$. By construction, T^d is a bijection between J and J', so for all $x \in I$ we have $|D(I,m)| \leq |\{k \in [-M_m, M_m] : \exists y \in I, T^k y = x\}|$.

To show the upper bound, consider $J, J' \in D(I,m)$. We pick x to be a non-endpoint of J. We may assume that J is above J'. Let C_N be the first column such that the copy of C_m in C_N containing x is not the bottom or top copy of C_m in C_N. That is, the copy of C_m containing x is $C_{m,N}[n]$ and there are copies $C_{m,N}[n-1]$ below and $C_{m,N}[n+1]$ above it.

However, we see that any descendent of J' in $C_{m,N}[\ell]$ for $\ell \geq n+2$ is at least $2h_m - (h(J) - h(J')) > h_m$ levels from x in C_N and any descendent of J' in $C_{m,N}[\ell]$ for $\ell \leq n-1$ is at least $h_m + (h(J) - h(J')) > h_m$ levels from x in C_N. Hence, as $M_m \leq h_m$, only the descendants of J' in $C_{m,N}[n]$ and $C_{m,N}[n+1]$ can cover x for $|k| \leq M_m$. Thus, as each non-endpoint x in J can be covered at most 2 times by any descendant of J' and there are $|D(I,m)|$, for a.e $x \in I$ we have that (2.1) as desired.

Hence, the sets $\{T^k I\}$ for $|k| \leq M_m$ cover almost every point of I at least $|D(I,m)|$ times and at most $2 \cdot |D(I,m)|$ times. $\square$

The following proposition will also be useful in determining when rank-one transformations are (subsequence) boundedly rationally ergodic.

Proposition 2.1. *Let T be a rank-one transformation and I be the level in C_0. If for all measurable $B \subset I$ and for a fixed $n \in \mathbb{N}$, we have that*

$$\frac{1}{a_n(I)} \sum_{k=0}^{n-1} \mu(I \cap T^k B) \leq c\mu(B), \tag{2.2}$$

then for that n,

$$\left\| \frac{1}{a_n(I)} S_n(1_I) \right\|_\infty \leq c \tag{2.3}$$

Proof. We fix an x in our space and show that

$$\frac{1}{a_n(I)}\sum_{k=0}^{n-1} 1_I \circ T^k(x) \leq c. \tag{2.4}$$

We notice that it suffices to show (2.4) holds for $x \in I$. For $x \notin I$, we either have that $T^j(x) = y$ for some $y \in I$, $0 \leq j \leq n-1$ or there are no such y and j. In the former case, we take the least such j and have that $\frac{1}{a_n(I)}\sum_{k=0}^{n-1} 1_I \circ T^k(x) \leq \frac{1}{a_n(I)}\sum_{k=0}^{n-1} 1_I \circ T^k(y) \leq c$, and in the latter we have that $\frac{1}{a_n(I)}\sum_{k=0}^{n-1} 1_I \circ T^k(x) = 0$.

Now, for every $x \in I$, and a fixed $n \in \mathbb{N}$, we have that there exists a column C_m for which if $J \ni x$ is a level in C_m, there are at least n levels above J. Then, we see that

$$\frac{1}{a_n(I)}\sum_{k=0}^{n-1} 1_I \circ T^k(x) = \frac{1}{a_n(I)}\sum_{k=0}^{n-1} \frac{\mu(I \cap T^k J)}{\mu(J)} \leq c.$$

Now, since (2.4) holds for all $x \in I$ and therefore all $x \in X$, we have that (2.3) holds. □

Now, we move on to our main theorem, which establishes the subsequence bounded rational ergodicity of all rank-one transformations. This was shown in [7, Theorem 2.4] under the assumption of exponential growth, and the details there are only given for subsequence rational ergodicity.

Theorem 2.2. *All rank-one transformations are subsequence boundedly rationally ergodic.*

Proof. Given a rank-one transformation, we claim that it is subsequence boundedly rationally ergodic along the subsequence $\{M_m + 1\}$. By Proposition 2.1, it suffices to show that for all $B \subset I$

$$\frac{1}{a_{n_m}(I)}\sum_{k=0}^{n_m-1} \mu(I \cap T^k B) \leq c\mu(B) \tag{2.5}$$

holds for $n_m = M_m + 1$. As

$$\sum_{k=0}^{n_m-1} \mu(I \cap T^k I) = a_{n_m}(I) \cdot \mu(I)^2,$$

to show (2.5) it suffices to show that there is a c for which

$$\sum_{k=0}^{M_m} \mu(I \cap T^k B) \leq c\frac{\mu(B)}{\mu(I)^2}\sum_{k=0}^{M_m} \mu(I \cap T^k I). \tag{2.6}$$

Since by Lemma 2.1, the sets $\{T^k I\}$ for $-M_m \le k \le M_m$ cover almost every point of I at most $2\cdot|D(I,m)|$ times and at least $|D(I,m)|$ times, we get that

$$\sum_{k=-M_m}^{M_m} \mu(I\cap T^k B) = \sum_{k=-M_m}^{M_m} \mu(T^k I\cap B) \le 2\mu(B)\cdot|D(I,m)| \tag{2.7}$$

and

$$\sum_{k=-M_m}^{M_m} \mu(I\cap T^k I) \ge |D(I,m)|\mu(I). \tag{2.8}$$

As a result, we have that

$$\sum_{k=-M_m}^{M_m} \mu(I\cap T^k B) \le 2\left(\frac{|D(I,m)|}{|D(I,m)|}\right)\left(\frac{\mu(B)}{\mu(I)}\right)\left(\sum_{k=-M_m}^{M_m} \mu(I\cap T^k I)\right).$$

So

$$\begin{aligned}\sum_{k=0}^{M_m} \mu(I\cap T^k B) &\le 2\left(\frac{|D(I,m)|}{|D(I,m)|}\right)\left(\frac{\mu(B)}{\mu(I)}\right)\left(2\left(\sum_{k=0}^{M_m} \mu(I\cap T^k I)\right)-1\right)\\ &\le 4\frac{\mu(B)}{\mu(I)}\sum_{k=0}^{M_m} \mu(I\cap T^k I).\end{aligned}$$

Setting $c = 4\mu(I)$, we satisfy (2.6) and therefore (2.5) for all $B\subset I$. $\square$

Through a similar sequence of steps, we can also show that all rank-one transformations with bounded cuts are boundedly rationally ergodic, Theorem 2.3. This was shown in [7, 2.3] under the assumption of exponential growth. In [5], it was shown that all rank-one cutting and stacking transformations T with a bounded number of cuts satisfy the property that implies bounded rational ergodicity (see Remark 2.4).

Theorem 2.3. *Let T be a rank-one transformation with a bounded number of cuts. Then T is boundedly rationally ergodic.*

Remark 2.4. *(1). It was recently proven in* [5] *that all rank-one cutting and stacking transformations T with a bounded number of cuts satisfy the property:*

$$\exists A, 0<\mu(A)<\infty,\ \textit{for which}\ \sum_{|k|\le n} 1_A(T^k x) \asymp a_n(A),\ \ a.e.,$$

where for positive sequences a_n and b_n, $a_n \asymp b_n$ means the existence of $M>0$ such that $\frac{1}{M} < \frac{a_n}{b_n} < M$, $\forall n$ large. This property implies bounded rational ergodicity [5].

(2). The property in (1) but formulated for subsequences can be proved for all rank-one transformations using the ideas of Proposition 2.1. Namely, if T is a rank-one transformation, then

$$\exists A, 0 < \mu(A) < \infty, \text{ for which } \sum_{|k|\leq n} 1_A(T^k x) \asymp a_n(A), \text{ a.e., for } n \in \kappa,$$

which implies subsequence bounded rationally ergodic along the subsequence $\kappa = \{M_m + 1\}$.

(3). It was shown in [7, Theorem 2.2] *that for all normal rank-one transformations, the weak ergodicity condition holds when A and B are finite unions of levels. One can lift the normality assumption and prove this for all rank-one transformations (a detailed argument is given in an earlier version of this paper posted in the archives). Thus one could ask if all rank-one transformations are weakly rationally ergodic. In Section 3, we will see that this is not true as we will use genericity results to show that there exist rank-one transformations that are not weakly rationally ergodic. Thus, for a rank-one transformation that is not weakly rationally ergodic, the sets where the property fails cannot be levels in a cutting and stacking construction of the transformation.*

Recall that $S : X \to X$ is an **invertible nonsingular** transformation if it is invertible, measurable and $\mu(A) = 0$ if and only if $\mu(S(A)) = 0$. The (nonsingular) **centralizer** $C(T)$ of a transformation T consists of all invertible nonsingular S such that $ST = TS$. It was shown by Aaronson [1] that if an invertible S is in the nonsingular centralizer of a weakly rationally ergodic T, then S is measure-preserving. In fact, this result holds even if T is only subsequence weakly rationally ergodic, with essentially the same argument. As a consequence, we have that any invertible, nonsingular S in the centralizer of a rank-one transformation is measure-preserving. Recall that T is called **squashable** if it commutes with a non-measure-preserving S. Regarding the centralizer we note that Ryzhikov [14] has shown that the centralizer of a zero type conservative ergodic infinite measure-preserving transformation is trivial, i.e., consists just of powers of the transformation. In the finite measure-preserving case, it is well known that for rank-one transformations an element S of the centralizer is a limit in the weak topology of powers of T [11].

Thus using Theorem 2.2, we have the following corollary.

Corollary 2.5. *It T is a rank-one transformation and S is in the centralizer of T, then S is measure-preserving. Thus T is not squashable.*

Our result has an interesting consequence for Maharam transformations, which are a special class of infinite measure-preserving transformations arising from nonsingular transformations. If T is an invertible nonsingular transformation on X, we can define the Radon-Nikodym derivative $\omega(x) = \frac{d\mu\circ T}{d\mu}(x)$. Then define a space $X^* = X \times \mathbb{R}^+$ with measure $\mu * = \mu \times \lambda$. The **Maharam transformation** T^* is defined on X^* by $T^*(x, y) = (Tx, y/\omega(x))$. It can be shown that T^* is measure-preserving with respect to the infinite measure $\mu*$ and is conservative if and only if T is [13]. There are cases when T^* is also ergodic, namely when the nonsingular transformation T is type III (see e.g. [9]). There are also conservative ergodic Maharam $\mathbb{Z}$-extensions (see e.g., [3]), and our result would also apply to them.

Corollary 2.6. *Let T be a conservative nonsingular transformation. The Maharam transformation T^* is not rank-one.*

Proof. It is clear when T^* is not ergodic. Then assume T^* is ergodic. It is well known that T^* commutes with the non-measure-preserving transformations $Q(x, y) = (x, ay)$ for any positive constant a. Then use Corollary 2.5. ☐

3. Genericity of Rank-One in Infinite Measure

3.1. Some Definitions and Preliminaries.

We first discuss the **weak topology** on the space of all invertible, measure-preserving transformations on $X = \mathbb{R}_{\geq 0}$, with Lebesgue measure, which we call $\mathcal{M}$. As mentioned in the introduction, in the finite measure case this result is well known (see e.g. [12] and the references therein). The first discussion of genericity properties in infinite measure is probably in [15]. We say that a sequence of transformations $\{T_n\}$ converges to T if and only if $\{T_n A\}$ converges to TA for all $A \subset X$ such that $\mu(A) < \infty$. That is, $\mu(T_n A \triangle TA) + \mu(T_n^{-1} A \triangle T^{-1} A) \to 0$.

If $\{A_i\}$ is a dense collection of sets for the finite measure sets in the Borel sigma algebra $\mathcal{B}$ of X, then define a metric on $\mathcal{M}$ as follows:

$$d(T, S) = \sum_{i=1}^{\infty} \frac{1}{2^i} \left(\mu(TA_i \triangle SA_i) + \mu(T^{-1}A_i \triangle S^{-1}A_i) \right).$$

This metric generates the weak topology on the space. For completeness, we explicitly define $\{A_i\}$ here. We define the sets in stages. In the first stage, we let A_1 be $[0, 1)$. In the second stage, we let the next sixteen sets, $A_2, \ldots, A_{17}$, be all combinations of unions of the dyadic intervals $[0, 1/2)$, $[1/2, 1)$, $[1, 3/2)$, $[3/2, 2)$. Inductively, in the nth stage, we suppose that all sets in the previous $n - 1$ stages have been defined and we define the next $2^{n\cdot 2^{n-1}}$ sets A_i to be all unions of combinations of dyadic intervals of length $\frac{1}{2^{n-1}}$ in $[0, n)$. Any finite union of dyadic intervals is in $\{A_i\}$, and the $\{A_i\}$ are dense in the collection of finite measurable sets in $\mathcal{B}$. We can verify that under this choice of $\{A_i\}$, the metric always returns a finite distance.

We say that a transformation T is a **cyclic permutation of rank** k if T is the identity on $[k, \infty)$ and there exists I_k a dyadic interval in $[0, k)$ of length $1/2^{k-1}$ such that $I_k, TI_k, T^2I_k, \ldots, T^{k\cdot 2^k-1}I_k$ is a partition of $[0, k)$ consisting of disjoint half-open dyadic intervals of length $1/2^{k-1}$. We call the set of cyclic permutations of rank k, O_k.

3.2. **Genericity of Rank-ones.** We prove the genericity of rank-one transformations in $\mathcal{M}$ with the weak topology. We say that a (Rohklin) column approximates a measurable set of finite measure A with accuracy $\varepsilon > 0$ if there exists B', a union of levels, such that $\mu(A \triangle B') < \varepsilon$. A sequence of columns $\{C_n\}$ approximates A if for any $\varepsilon > 0$ there exists some $N_\varepsilon \in \mathbb{N}$ such that for all $n \geq N_\varepsilon$, C_n approximates A with accuracy ε. So T is rank-one if and only If there exists a sequence of columns $\{C_n\}$ such that $\{C_n\}$ approximates every finite measurable set $A \subset X$.

Theorem 3.1. *The rank-one transformations are generic in $\mathcal{M}$ under the weak topology.*

Proof. We show that the rank-one transformations contain a dense G_δ set in $\mathcal{M}$. To find this set, define

$$\mathcal{R}_j := \{T \in \mathcal{M} : T \text{ has a Rokhlin column that approximates } A_1, \ldots, A_j \text{ with accuracy } 1/j\}.$$

We claim now that

$$\mathcal{R} = \bigcap_{j=1}^{\infty} \mathcal{R}_j$$

is our desired dense G_δ set. It suffices to show that each $T \in \mathcal{R}$ is rank-one and each $\mathcal{R}_j$ is open and dense. Then, since $\mathcal{M}$ is a Polish and therefore a Baire space (see e.g., [3]), $\mathcal{R}$ is also dense. (We note here that Danilenko has informed us that the proof in [8] that the finite measure-preserving rank-ones are G_δ can be adapted to the infinite measure-preserving case.)

Now let $T \in \mathcal{R}$; we show that it is rank-one. For each j, there exists a Rokhlin column C_j for T, $B_j, TB_j, T^2B_j, \ldots, T^{h_j-1}B_j$, that approximates $A_1, \ldots, A_j$ with accuracy $1/j$. But then, given any finite measurable set $A \subset X$ and any $\varepsilon > 0$, we can find an A_i such that

$\mu(A \triangle A_i) < \varepsilon/2$. Now, choosing an $N > \max\{i, 2/\varepsilon\}$, we have that for all $j \geq N$, there exists some B'_j a finite union of levels of C_j that approximates A_i within $1/j < \varepsilon/2$ and therefore A within ε. Hence, T is rank-one.

We next fix an $\mathcal{R}_j$ and first show that it is dense. For O_k the set of cyclic permutations of rank k, the cyclic permutations $\bigcup_{k=1}^{\infty} O_k$ are dense ([15]). By construction, there exists a K such that for all $k \geq K$, each $A_i, 1 \leq i \leq j$ is a finite union of the dyadic intervals

$$[0, 1/2^k), [1/2^k, 2/2^k), \ldots, [k - 1/2^k, k).$$

But for any $T \in O_k$, there exists a Rokhlin column $I_k, TI_k, T^2I_k, \ldots, T^{k \cdot 2^k - 1}I_k$ of exactly these dyadic intervals. Hence, each A_i may be written as a union of the levels in this column, so $O_k \subset \mathcal{R}_j$ for all $k \geq K$. Since all but finitely many of the cyclic permutations are in any $\mathcal{R}_j$, $\mathcal{R}_j$ contains a dense set, so is dense.

Now we want to show that each $\mathcal{R}_j$ is open in $\mathcal{M}$. Let $T \in \mathcal{R}_j$. Then, there exists a Rokhlin column $C_j = \{B, TB, T^2B, \ldots, T^{h-1}B\}$ for T that approximates each $A_1, \ldots, A_j$ within some $a < 1/j$.

Fix an $\epsilon > 0$. Then, for any level $L = T^\ell B$, the ℓth level of C_i, we can approximate L within $\epsilon/4h$ by some A_k. Then, setting $\delta_\ell = \frac{\epsilon}{2^{k+1} \cdot h}$, we have that if $d(T, S) < \delta_\ell$, then

$$\mu(T(L) \triangle S(L)) < \mu(T(A_k) \triangle S(A_k)) + 2\mu(L \triangle A_k) < \frac{\epsilon}{h}.$$

Setting $\delta = \min_{0 \leq \ell \leq h-1}\{\delta_\ell\}$, we have that $\mu(T(L) \triangle S(L)) < \epsilon/h$ for all levels L of C_i.

Then, letting $C_0 = B, C_1 = SC_0 \cap TB, C_2 = SC_1 \cap T^2B, \ldots$, all the way up to $C_{h-1} = SC_{h-2} \cap T^{h-1}B$, we then define $S^{h-1}C = C_{h-1}$. As S is invertible, we can then define $C = S^{1-h}C_{h-1}$ and notice that $S^iC \subset T^iB$ for all such i. Furthermore, these levels would be disjoint and of the same measure.

Then, as $\mu(T(L) \triangle S(L)) < \epsilon/h$ for each $L = T^iB$, $0 \leq i \leq h-1$, we have that the measure of $S^iC \subset T^iB$ is at least

$$\mu(B) - (i+1) \cdot \frac{\epsilon}{h} \leq \mu(B) - \epsilon,$$

as $\mu(SC_i \cap T^{i+1}B) = \mu(SC_i \cap T(T^iB)) \geq \mu(C_i) - \epsilon/h$ for all i. Then, as $S^iC \subset T^iB$ for all i and S is measure-preserving, we have that $\mu(S^iC \triangle T^iB) < \epsilon$ for $0 \leq i \leq h-1$.

We let $\epsilon = \alpha/h$ for $\alpha = 1/j - a$. Then, we have that as the sets $\{B, TB, \ldots, T^{h-1}B\}$ approximate each $A_1, \ldots, A_j$ within $a < 1/j$, and we can find a δ such that for all transformations S, $d(T, S) < \delta$, S has a Rokhlin column $\{C, SC, \ldots, S^{h-1}C\}$ with $\mu(S^iC \triangle T^iB) <$

ϵ, we have that the column $\{C, SC, \ldots, S^{h-1}C\}$ approximates each $A_1, \ldots, A_j$ within $a + h\epsilon = 1/j$. Hence, each S in the ball of radius δ around T is in $\mathcal{R}_j$, showing that $\mathcal{R}_j$ is open. □

3.3. **Existence of a Rank-one Transformation that is not Weakly Rationally Ergodic.** Weak rational ergodicity is meager in the space of measure-preserving, invertible transformations under the weak topology, Aaronson [4, proof of Theorem F]. Thus we have the following corollary:

Corollary 3.2. *There exist rank-one transformations that are not weakly rationally ergodic.*

4. Rational Weak Mixing

In this section, we will give various conditions that imply that a transformation is not rationally weakly mixing. We first introduce some notation. We fix a non-negative sequence $u = \{u_k\}$ such that $\sum_{k=0}^{\infty} u_k = \infty$ and let $a_u(n)$ denote $\sum_{k=0}^{n-1} u_k$. Then, for a subset K of $\mathbb{N}$, we define $a_u(K, n) = \sum_{k \in K \cap [0,n-1]} u_k$.

Now, we define some notions of smallness and density. If $\kappa \subset \mathbb{N}$, we say that a set K is a (u, κ)-**small set** if

$$\frac{a_u(K, n)}{a_u(n)} \underset{n \in \kappa}{\longrightarrow} 0$$

as $n \to \infty$. Furthermore, we say a sequence $x_k \to L$ in (u, κ)-**density** to $L \in \mathbb{R}$ if there exists a (u, κ)-small set K such that $x_k \to L$ as $k \to \infty$ for $k \in \kappa \backslash K$. We denote this type of convergence as $x_k \underset{(u,\kappa)-d}{\longrightarrow} L$. We say that x_k converges (u, κ)-**strongly Cesaro** to $L \in \mathbb{R}$ if

$$\frac{1}{a_u(n)} \sum_{k=0}^{n-1} u_k |x_k - L| \underset{k \to \infty, k \in \kappa}{\longrightarrow} 0.$$

We denote this type of convergence as $x_k \underset{(u,\kappa)-s.C.}{\longrightarrow} L$.

We also say that two sequences $u = \{u_k\}$ and $v = \{v_k\}$ are κ-**asymptotic** (denoted by $u \overset{\kappa}{\approx} v$) if

$$\frac{1}{a_u(n)} \sum_{k=0}^{n-1} |u_k - v_k| \underset{k \to \infty, k \in \kappa}{\longrightarrow} 0.$$

Similarly, a sequence $u = \{u_k\}$ is called κ-**smooth** if

$$\frac{1}{a_u(n)} \sum_{k=0}^{n-1} |u_{k+1} - u_k| \underset{k \to \infty, k \in \kappa}{\longrightarrow} 0.$$

We start with the following theorem that gives conditions which lead to a transformation being not rationally weakly mixing.

Theorem 4.1. *If T is a normal, rank-one transformation such that for all $N \in \mathbb{N}$,*

$$\sum_{i=0}^{N-1}(d_i - d_i') = \sum_{i=0}^{N-1}(e_i - e_i')$$

for $d_i, d_i', e_i, e_i' \in H_i$ implies that $d_i - d_i' = e_i - e_i'$ for all $0 \leq i \leq N-1$, then T is not rationally weakly mixing.

Proof. We suppose that we have such a transformation T, but that it is rationally weakly mixing. Then, it is subsequence rationally weakly mixing along the sequence $\kappa = \{M_m + 1\}$.

It can be shown that the sets $F \subset X$ that we have rational weak mixing along κ with respect to is exactly $R_\kappa(T)$, the sets that we have weak rational ergodicity along κ with respect to ([4]). We proved earlier that T is subsequence weakly ergodic along κ with respect to I the level in C_0 and therefore any $J \subseteq I$ of positive measure. Define $u = \{u_k = \mu(J \cap T^k J)\}$.

We know by Aaronson [4] that if

$$\frac{\sum_{k=0}^{n-1} |\mu(J \cap T^k J) - \mu(J)^2 u_k(I)|}{a_n(I)} \to 0$$

then

$$\frac{\mu(J \cap T^k J)}{\mu(I \cap T^k I)} \underset{(u,\kappa)-d}{\longrightarrow} \frac{\mu(J)^2}{\mu(I)^2}.$$

Now we pick J a descendant in column C_j for some $j > 0$. Then, since T is normal, by an argument similar to that in Lemma **??** (see [7], Lemma 2.1), we have that for some N large,

$$\frac{\mu(J \cap T^k J)}{\mu(I \cap T^k I)} = \frac{w_N |D(J,N) \cap (k + D(J,N))|}{w_N |D(I,N) \cap (k + D(I,N)|}.$$

But then if $\mu(J \cap T^k J) > 0$, we can write $k = \sum_{i=0}^{N-1} k_i$, where $k_i \in D(J,i) - D(J,i)$. Moreover, $k = \sum_{i=j}^{N-1}(d_i - d_i') = \sum_{i=0}^{N-1}(e_i - e_i')$. Since we have that $d_i - d'_i = e_i - e'_i$ for all $j \le i \le N-1$, we have that

$$\begin{aligned}\frac{\mu(J \cap T^k J)}{\mu(I \cap T^k I)} &= \frac{\prod_{i=j}^{N} |\{(d_i, d'_i) : d_i - d'_i = k_i\}|}{\prod_{i=0}^{N} |\{(e_i, e'_i) : e_i - e'_i = k_i\}|} \\ &= \frac{1}{\prod_{i=0}^{j-1} \{(e_i, e'_i) : e_i - e'_i = 0\}|} \\ &= \frac{1}{\prod_{i=0}^{j-1} |H_i|}.\end{aligned}$$

Also, for k such that $\mu(J \cap T^k J) = 0$, we have that $\frac{\mu(J \cap T^k J)}{\mu(I \cap T^k I)} = 0$. Hence, since $\frac{\mu(J)^2}{\mu(I)^2} = \frac{1}{\prod_{i=0}^{j-1} |H_i|^2}$, we have that

$$\frac{\mu(J \cap T^k J)}{\mu(I \cap T^k I)} \underset{(u,\kappa)-d}{\not\to} \frac{\mu(J)^2}{\mu(I)^2}.$$

Hence, T cannot be rationally weakly mixing. □

Now, we use this theorem to give a few examples and classes of transformations that cannot be rationally weakly mixing. The following was shown in [7] under the normality assumption.

Corollary 4.2. *If the transformation T is steep, then T is not rationally weakly mixing.*

Proof. If T is steep with steepness rate equal to 5, then $\sum_{i=0}^{N}(d_i - d'_i) = \sum_{i=0}^{N}(e_i - e'_i)$ with $d_i, d'_i, e_i, e'_i \in H_i$ implies that $d_i - d'_i = e_i - e'_i$. It follows from Theorem 4.1 that T is not rationally weakly mixing. □

Theorem 4.3. *Let T be a rank-one transformation. If for all columns C_j and J a level of C_j with $J \subset I$, we have that*

$$\frac{\mu(J \cap T^{k+1} J)}{\mu(J \cap T^k J)} \underset{(u,\kappa)-d}{\not\to} 1,$$

where $\kappa = \{M_m + 1\}_{m \ge 1}$, then T is not rationally weakly mixing.

Furthermore, if there exists a level of a column C_j, $J \subset I$, such that $1 \notin (D(J,N) - D(J,N)) - (D(J,N) - D(J,N))$, for all sufficiently large N and T is a rank-one transformation, then we have that T is not rationally weakly mixing.

Proof. Suppose, by contradiction, that T is rationally weak mixing. By the same argument as in 4.1, T is subsequence rationally weakly mixing along κ and $I = C_0 \in R_\kappa(T)$. If $J \subseteq I$, then $J \in R_\kappa(T)$ and $T(J) \in R_\kappa(T)$. This proves that $\mu(J \cap T^k J) \overset{\kappa}{\approx} \mu(J \cap T^{k+1} J)$, meaning that $\mu(J \cap T^k J)$ is a κ-smooth sequence for levels $J \subseteq I$ of the columns C_j. Denote $v_k = \mu(J \cap T^k J)$ and let $v = (v_0, v_1, \ldots)$. As T is subsequence rationally weakly mixing along κ, then $u \overset{\kappa}{\approx} v$.

By Proposition 3.2 in [4], this means that the sequence

$$\frac{\mu(J \cap T^{k+1}J)}{\mu(J \cap T^k J)} \underset{(v,\kappa)-s.C.}{\longrightarrow} 1,$$

which implies by Remark 3.3 iii) of the same paper that

$$\frac{\mu(J \cap T^{k+1}J)}{\mu(J \cap T^k J)} \underset{(v,\kappa)-d}{\longrightarrow} 1.$$

By Remark 3.3 ii) in [4], it follows that

$$\frac{\mu(J \cap T^{k+1}J)}{\mu(J \cap T^k J)} \underset{(u,\kappa)-d}{\longrightarrow} 1,$$

as $u \overset{\kappa}{\approx} v$, and we have a contradiction.

Now we prove the second half of the theorem. By the first part, it suffices to show for J that

$$\frac{\mu(J \cap T^{k+1}J)}{\mu(J \cap T^k J)} \underset{(u,\kappa)-d}{\not\to} 1.$$

But as $1 \notin (D(J,N) - D(J,N)) - (D(J,N) - D(J,N))$ we get that for any fixed k, $\mu(J \cap T^{k+1}J)$ and $\mu(J \cap T^k J)$ cannot be both be greater than 0 at the same time. Hence, the limit cannot go to 1 and we are done. $\square$

Remark 4.4. *We note that the above proof shows more: it shows that T is not subsequence rationally weakly mixing along the sequence $\kappa = \{M_m + 1\}$. The importance of this sequence comes from the fact that all rank-one transformations are subsequence weakly rationally ergodic along κ.*

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