## A Mathematical Seduction

First let me say that I was a normal math student—certainly no genius. I got all the way through high school pre-calculus and did okay, I guess. Of course just like many healthy teenagers, I used to sneak off into the garage where I had found a hidden stack of old *Math Horizons* from the mid 1970's. While it was always exciting to look at the articles and illustrations, the best parts were the letters to the "Math Forum." Those letters got me and my other adolescent friends through some tense trigonometric times. But never in my wildest math dreams did I think I would ever write a letter to "Math Forum." That all changed in 1992 when I left home to attend a small, liberal arts Connecticut college (I don't want to reveal its name since some of the math profs might remember me).

I took a year of calculus in college. Even though I had great profs, I hated infinite series with a passion. How could anyone like all those stupid tests: The Comparison Test; the Limit Comparison Test; the Ratio Test; the Alternating Series Test; the *n*th Root Test; the Integral Test; the Blah Blah Blah Test... the list went on and on. Being a healthy college calculus student, I had every calculus student's fantasy:

$$\textit{If } \lim_{n\to\infty}|a_n|=0,$$

$$\textit{then } \sum_{n=1}^{\infty} a_n \textit{ must converge.}$$

It was incredibly frustrating for me to face the fact that my fantasy was just that—a fantasy. The harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ actually diverges to infinity even though the individual terms head to zero. The whole thing was a huge turn off for me and I decided to abstain from mathematics forever. But everything changed in the fall of 1993. That's when I met this incredible math major (I'll call her Emmy).

It was a Friday night in late October. Everyone I knew had taken off to Hartford for the Vanilla Ice concert. I had an econ midterm coming up on Monday so I was cramming away in the deserted library. I always studied on the top floor of the library which housed all the language books. I worked at a large oak table in a secluded corner behind the Esperanto section.

Marginal demand drifted into marginal propensity to consume and I drifted into a deep sleep. I must have been asleep for over an hour when I was startled by a loud thud. As my head lifted off the oak table, I saw Emmy. "Oh, I'm sorry to have disturbed your studying," she whispered as she reached down to pick her Hewlett-Packard calculator off the floor. I looked at her fallen calculator and replied, "No problem. You know, I never understood how to use those. You have to punch the keys in the wrong order and hit 'Enter' instead of '='." Emmy giggled and said, "I guess you're not a math major…" But Emmy was—she was an exchange student from Germany and at that moment I would have never guessed that this exotic stranger would soon, right there on that oak table, make all my fantasies come true.

## Getting Close Together

I told her about my calculus classes and how I hated infinite series and all those dumb tests. As I summed up, Emmy smiled, looked into my eyes, and in a soft voice asked, "Have you ever engaged in *real* analysis with someone?'' Of course, as I had stopped (perhaps prematurely) at calculus, I just blushed and answered, "No…" With a quick glance, she asked if I was up for trying it right there and then for the first time. I was so nervous that my palms were sweating. At that moment I couldn't think straight. I just stammered, "Yeah… but be gentle." With that Emmy stood up and slowly slid her chair to my side of the table. She opened up her notebook to expose a ream of pure white (unlined) paper. With one stroke of a pencil she began.

It was incredible. It was as though there were only two mathematical objects in the entire world: the rational numbers $\mathbb{Q}$, and the absolute value on $\mathbb{Q}$, $|\ |$. It felt so natural—everyone loves the fractions and the absolute value allows us to measure how close things are to each other. Little did I know that things were about to get *really* close.

## My First Real Experience

Emmy showed me an infinite list of rational numbers $a_1, a_2, a_3, \ldots$ . I immediately had the urge to sum them up as an infinite series, but Emmy wouldn't let me. She teased me—forcing me to slow down and build up the sequence of partial sums, $S_1 = a_1$, $S_2 = a_1 + a_2, \ldots,$ $S_N = a_1 + a_2 + \cdots + a_N$ and so on. Then Emmy moved toward me and whispered in my ear, "If those rational numbers $S_1$, $S_2$, $S_3$, … eventually all get really really close together, then let's say that the infinite series $\sum_{n=1}^{\infty} a_n$ *converges*." She then translated that intimate declaration onto the page using provocative math symbols: *Suppose that given any tiny number* $\varepsilon > 0$, *there exists an integer N so that for all indices r, t* $\geq N$, *we have* $|S_r - S_t| < \varepsilon$. *Then we say that the associated infinite series* $\sum_{n=1}^{\infty} a_n$ *converges (to some sum).* Emmy told me that if we wanted to be within $\varepsilon$, then $N$ is how long we would have to hold off until we were destined to be that close and stay that close forever.

It all made complete sense and there was a rush to my head as I experienced an awakening. At that moment, Emmy revealed that the collection of all possible convergent sums of rational numbers is nothing more nor less than the set of real numbers $\mathbb{R}$. I couldn't believe what I

**"Math Forum" has been a nonexistent, regular feature of *Math Horizons* since the late 1960's containing letters from readers depicting intimate mathematical experiences. This story may not be appropriate for all readers—especially those who are offended by explicit math acts.**

was hearing. My mind was reeling. Wow! We just conceived the real numbers with just $\mathbb{Q}$ and | |! It was my first real experience with analysis and I was never going to forget it. I thought that would be it. But Emmy had other ideas. As I wiped a bead of sweat from my forehead, Emmy leaned over and devilishly grinned, "What if we replace | | by another absolute value on $\mathbb{Q}$? Are you up for it?'' Another absolute value on $\mathbb{Q}$? There is no other, that's why we call it *the* absolute value! I knew that I was out of my league and I should turn away, but I couldn't resist so I breathlessly replied, "I don't know…"

## Absolute Pleasure

After Emmy composed herself, she revealed that a function $\|\ \|: \mathbb{Q} \to [0, +\infty)$ is an absolute value if

1) $\|x\| \geq 0$ for all $x \in \mathbb{Q}$ and $\|x\| = 0$ if and only if $x = 0$.
2) $\|xy\| = \|x\|\ \|y\|$ for all $x, y \in \mathbb{Q}$.
3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathbb{Q}$. (Affectionately known as the *triangle inequality*.)

Of course none of this shocked me since I'd had a lot of experience with the absolute value | |. But then Emmy laid a number on me: she let $p$ be a fixed prime number and I felt a strange sensation. She introduced me to a new move—a map $|\ |_p \to [0,+\infty)$ and she declared that $|0|_p = 0$. She took a nonzero $\frac{r}{s} \in \mathbb{Q}$, and, before anyone could have said "The Fundamental Theorem of Arithmetic" she had ripped off the prime $p$ to reveal

$$\frac{r}{s} = p^n \frac{r'}{s'},$$

where $n \in \mathbb{Z}$ and both $r'$ and $s'$ were stripped of any factors of $p$. She then tenderly announced that

$$\left|\frac{r}{s}\right|_p = p^{-n}.$$

I was confused but Emmy quickly turned my puzzlement into pleasure by going all the way to illustrate her technique with $^{140}/_{297} = 2^2 \cdot 3^{-3} \cdot 5 \cdot 7 \cdot 11^{-1}$:

$$\left|\frac{140}{297}\right|_2 = 2^{-2},\quad \left|\frac{140}{297}\right|_3 = 3^3,\quad \left|\frac{140}{297}\right|_7 = 7^{-1},$$

and even

$$\left|\frac{140}{297}\right|_{19} = 19^{-0} = 1.$$

I nearly flipped over when I got it. "So $|^{140}/_{297}|_5 = 5^{-1}$!'' I shouted. Emmy smiled knowingly and nodded. It was the best five-play I ever had. I realized I was experiencing something incredible. "In fact, if you plug in *any integer* the value cannot exceed 1!'' I excitedly exclaimed. "You're absolutely right,'' she said as she wrote: $|\text{any integer}|_p \leq 1$. Emmy looked around to make sure we were still alone and asked in a low voice, "Now, are you ready to go wild?" I didn't know what to say but she sure did.

"*Let p be a prime number. Then* $|\ |_p$ *is an absolute value on* $\mathbb{Q}$."

She then said that they call $|\ |_p$ the *p-adic absolute value*. Well it was easy for me to see that $|x|_p \geq 0$ for all $x \in \mathbb{Q}$, $|x|_p = 0$ if and only if $x = 0$, and even that $|xy|_p = |x|_p\ |y|_p$ for all $x, y \in \mathbb{Q}$. But the triangle inequality, $|x + y|_p \leq |x|_p + |y|_p$ for all $x, y \in \mathbb{Q}$, was, for me, taboo. In the end, its allure was too much and I pleaded with Emmy, "Give me the reason… Why does the triangle inequality hold?'' "Okay, I'll give it to you… but only for integers $x$ and $y$.'' I figured that if I experienced Emmy's rationale for the integers, I could do it with the rationals by myself.

She wanted to role play. She said we should pretend that $x = p^n x'$ and $y = p^m y'$, where the integers $x'$ and $y'$ had no factors of $p$ and that $0 \leq n \leq m$. I readily agreed to play along and added that in this scenario, $|y|_p = p^{-m} \leq p^{-n} = |x|_p$ and so $|x|_p = \max\{|x|_p, |y|_p\}$. I could see Emmy was excited that I was into it. She then showed me the light:

$$\begin{aligned}
|x+y|_p &= |p^n x' + p^m y'|_p \\
&= |p^n (x' + p^{m-n} y')|_p \\
&= |p^n|_p\, |\text{some integer}|_p \\
&\leq p^{-n} \cdot 1 \\
&= |x|_p = \max\{|x|_p, |y|_p\} \\
&\leq |x|_p + |y|_p\,!
\end{aligned}$$

I was in absolute $p$-adic ecstasy. But she pushed me over the edge when she showed me that we went even further than I had thought. Together we actually reached a heightened awareness:

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}.$$

While I was completely surprised, it was clear that this was not the first time for Emmy. In a wispy voice she calmed me down, "It's okay. Just relax now, it's always intense when you first encounter the ***strong*** *triangle inequality*.''

It was true—Emmy offered me a new and wonderfully strange measure of closeness in the world of rational numbers. In fact, Emmy and I had gone the distance: the only nontrivial absolute values on the rational numbers are the $p$-adics, $|\ |_p$ for each prime $p$, and the usual one $|\ |$. My heart was racing as I tried to catch my breath. The oak table in front of us was now completely covered with tousled sheets of paper. I closed my eyes to take it all in. I wanted to remember that moment forever.

## Living Out My Fantasy

It was a while before either of us spoke. I felt exhausted and my clothes were still moist with perspiration. I rolled my head to the side to look at Emmy—she was glowing and bright. "Hey there,'' I said with some awkwardness; I didn't know if I should stay there with her or leave so that she could do her homework. But Emmy knew what she wanted. She smiled at me and asked, "Are you ready to make your fantasy a reality?'' I felt a tingle and I jumped. What was she going to do next? I was no longer nervous. I realized that Emmy knew what she was doing and whatever it was she wanted to do, it was going to rock my world.

Emmy wanted me to do some of the work and I was ready. "So what would it mean for an infinite series of rational numbers to converge $p$-adically?" she posed. I said exactly what she wanted to hear. "Suppose that $a_1, a_2, \ldots \in \mathbb{Q}$ and $S_N = \sum_{n=1}^{N} a_n$. If given any tiny number $\varepsilon > 0$, there exists an integer $N$ so that for all indices $r, t \geq N$, we have $|S_r - S_t|_p < \varepsilon$, then we say that the associated infinite series $\sum_{n=1}^{\infty} a_n$ *converges p-adically* (to some sum)." Emmy was tickled and laughed with delight. She then added, "We call the collection of all possible convergent sums the set of *p-adic numbers* $\mathbb{Q}_p$." It was, at the same time, comfortably familiar and excitingly new. But suddenly I felt an urge within me. It was so natural I could not hold back, "How do we determine if an infinite series converges $p$-adically?" Emmy just stared at me and beamed. I yearned for her mathematical touch but she just kept looking into my eyes and grinning. It was as though she could read every thought inside my head. Her silence made me feel uneasy. Slowly she opened her lips and softly whispered words that continue to sing in my ears.

"*If* $\lim_{n\to\infty} |a_n|_p = 0$, *then the infinite series* $\sum_{n=1}^{\infty} a_n$ *converges p-adically.*"

I was overcome. Emmy made every calculus student's fantasy a reality for me! Here we need only *one* test to determine convergence…

If the terms $p$-adically go to 0,
the series converges!

With tears of pleasure rolling down my cheeks, I looked into her eyes and declared my love for all that is $p$-adic. I then offered her an untouched, pure white piece of computer paper. She took it in her hand and there was a surge of excitement as we both held that same piece of paper. She looked at me longingly and said, "Let's do it together." I just wanted to satisfy her as best as I could. So we got down to business.

We let $\varepsilon > 0$. Emmy then said that since $\lim_{n\to\infty}|a_n|_p = 0$, there must exist an integer $N$ so that for all $n \geq N$, we have $|a_n|_p < \varepsilon$. I was completely with her; for that instant, we were one. I jumped on the fact that for any indices $r \geq t \geq N$, the strong triangle inequality brings us to

$$|S_r - S_t|_p = \left|\sum_{n=1}^{r} a_n - \sum_{n=1}^{t} a_n\right|_p = \left|\sum_{n=t+1}^{r} a_n\right|_p = |a_{t+1} + a_{t+2} + \cdots + a_r|_p \leq \max\{|a_{t+1}|_p, |a_{t+2}|_p, \ldots, |a_r|_p\}.$$

Emmy was going absolutely crazy. She finished the deed by shouting out that each of those indices is at least as large as $N$, and so the maximum $p$-adic value of all those terms is less than $\varepsilon$. So for all $r, t \geq N$, $|S_r - S_t|_p < \varepsilon$, and we completed the act of making my fantasy come true.

## Sum Enchanted Evening…

The mood quickly changed as we began to frolic in Emmy's amazing garden of convergent delights. Emmy wrote $\sum_{n=1}^{\infty} 3^n = 3 + 3^2 + 3^3 + \cdots$. She asked if the series converged. I laughed; before I met Emmy I would have thought, 'Of course it didn't—those terms are drifting off to infinity!' But Emmy had turned this calculus kid into a mathematical man. I could tell she wanted an answer. In a deep voice I replied, "It depends: With respect to what absolute value?" At that point, we both lost control of our senses. For any prime $p \neq 3$, $\lim_{n\to\infty} |3^n|_p = \lim_{n\to\infty} 1 = 1$. Since the terms are not even approaching zero, there is no way for the series to converge $p$-adically for $p \neq 3$. But, $\lim_{n\to\infty}|3^n|_3 = \lim_{n\to\infty} 3^{-n} = 0$. So 3-adically, the series converges! That series converges with respect to $|\ |_3$ but diverges with respect to all other absolute values on $\mathbb{Q}$!

"Sum it! Sum it! Sum it!" Emmy cried out. I couldn't help myself. I was swept up in the moment. She wanted me to sum it 3-adically and I wasn't going to let her down. I took my arm and forcefully thrusted the papers off the oak table onto the floor and began to sum. It was just like *Good Will Hunting* but a million times better—I was doing it with Emmy rather than Matt Damon. I used a trick I remembered from calc class: I wrote $S = 3 + 3^2 + 3^3 + 3^4 + \cdots$, which, after multiplying by 3, gave rise to $3S = 3^2 + 3^3 + 3^4 + \cdots$. I subtracted the two identities to get $2S = -3$, and I summed it: $S = -3/2$! Pretty wild (and way wrong) if you're just living life on the real line $\mathbb{R}$. But 3-adically I had all the right moves! Emmy loved it.

We didn't stop there. We threw convergence to the wind —if it felt right, we did it. Together we soon found that $\sum_{n=1}^{\infty}\frac{1}{n}$ diverged with respect to every absolute value on $\mathbb{Q}$, and that $\sum_{n=0}^{\infty}\frac{1}{n!}$ converged to $e$ in the eyes of $|\ |$ but diverged like exploding fireworks in the midnight sky for every $p$-adic absolute value. We then switched who's on top and saw that $\sum_{n=0}^{\infty} n!$ converged with respect to $|\ |_p$ for every prime $p$, but obviously raced off like mad to infinity when viewed in $\mathbb{R}$.

Seemed as if we did it all: we had series that converged with respect to some absolute values but not others, and series that converged nowhere. But something was missing and we both knew it. We looked at each other and wondered. Could we create an infinite series of positive rational numbers that converged with respect to every single absolute value on $\mathbb{Q}$?

We had to try. Emmy wrote down $\sum_{n=0}^{\infty} n!$ since that series converged everywhere except with respect to the usual absolute value. Under her breath I heard her moan, "Make it converge in $\mathbb{R}$." I didn't know what to do, so I modified her sum to:

$$\sum_{n=0}^{\infty}\frac{n!}{n!^2}$$

since I knew that converged with respect to $|\ |$. Unfortunately I also knew I had let Emmy down since that series diverged everywhere else. Much to my amazement, Emmy yelled, "Yes! Yes! Yes!'' and with one final stroke, she added 1:

$$\sum_{n=0}^{\infty}\frac{n!}{n!^2+1}.$$

It was unbelievable. We had done it: that sum converges with respect to each and every absolute value on $\mathbb{Q}$!

My body was tingling from head to toe. I had never in my life experienced pleasure like that before. I just fell back in my chair and collapsed into a deep deep sleep. I have no idea how long I slept. I was awakened by a janitor who yelled at me for making a huge mess and also for being in the library after it had closed. I looked around but Emmy was nowhere to be found.

## The Point of No Return

That encounter changed my life. The following semester I started taking more mathematics courses and ended up becoming a math major. I went on to a earn a PhD in $p$-adic analysis. I later learned that the world of $p$-adic numbers holds more surprises than just my old series-test fantasy. For instance, all $p$-adic triangles are isosceles and every point inside a $p$-adic circle is its center. Even closer to my night with Emmy, I now realize that

$$\sum_{n=0}^{\infty}\left(\frac{n!}{n!^2+1}\right)^{n!^3}$$

converges to a *transcendental* number in $\mathbb{R}$ and also in $\mathbb{Q}_p$ for each and every prime $p$. I wish I could have shared all these other discoveries with Emmy. But perhaps she already knows them.

I've had many mathematical experiences since that night, some have even involved convex bodies. But none of them have compared to that first night of steamy mathematics. I'll always remember Emmy—that exotic stranger who first exposed me to a world of $p$-adic pleasures.

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**Not really at Williams College**

[Editor's Note: For detailed proofs of the $p$-adic convergence of Drew and Emmy's series see Does $\sum_{n=0}^{\infty}\frac{1}{n!}$ Really Converge? Infinite Series and $p$-adic Analysis by Edward Burger and Tom Struppeck in the August 1996 issue of *The American Mathematical Monthly*. To learn more about $p$-adic analysis in general consult *Exploring the Number Jungle: A Journey into Diophantine Analysis* by Edward Burger (whose name, curiously enough, is a permutation of Drew Aderburg.)]

The proof consists of two copies of the triangle, with their legs placed on a common straight line, as shown in the figure. The trapezoid is then completed by joining the remaining vertices of the triangles, computing the area of the trapezoid, and comparing it to the sum of the areas of the three triangles into which it is decomposed.

Adjoining the mirror reflection of this trapezoid, one obtains the square found in the ancient Chinese text, the *Arithmetic Classic of the Gnomon and the Circular Paths of Heaven*; however, there is no indication that Garfield or any of his colleagues in Congress had any knowledge of this or any similar historical proofs; indeed, what we have seen of Garfield's biography shows that his mathematical training, especially in geometry, was limited and largely self-taught. Thus he can be credited with some innate instinct and a worthy degree of geometrical curiosity, perhaps not often found in the halls of legislatures. ■

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## For Further Reading

Read Allen Peskin's *Garfield: a Biography*, Kent State University Press, for more details of Garfield's life. *The Pythagorean Proposition* by Elisha S. Loomis, published by the National Council of Teachers of Mathematics, contains hundreds of proofs of the theorem: Garfield's is number 231. You can read about the *Arithmetic Classic of the Gnomon and the Circular Paths of Heaven* in David Burton's *The History of Mathematics: An Introduction*.