Compact plane waves with parallel Weyl curvature

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Abstract. This is an exposition of recent results — obtained in joint work with Andrzej Derdzinski — on *essentially conformally symmetric* (ECS) manifolds, that is, those pseudo-Riemannian manifolds with parallel Weyl curvature which are not locally symmetric or conformally flat. In the 1970s, Roter proved that while Riemannian ECS manifolds do not exist, pseudo-Riemannian ones do exist in all dimensions $n \geq 4$, and realize all indefinite metric signatures. The local structure of ECS manifolds is known, and every ECS manifold carries a distinguished null parallel distribution $\mathcal{D}$, whose rank is always equal to 1 or 2. We review basic facts about ECS manifolds, briefly discuss the construction of compact examples, and outline the proof of a topological structure result: outside of the locally homogeneous case and up to a double covering, every compact rank-one ECS manifold is a bundle over $\mathbb{S}^1$ whose fibers are the leaves of $\mathcal{D}^\perp$. Finally, we mention some classification results for compact rank-one ECS manifolds.

Keywords: Parallel Weyl tensor, conformally symmetric manifolds, compact pseudo-Riemannian manifolds

1 Introduction and History

By an *essentially conformally symmetric* (ECS) manifold [11], we mean a pseudo-Riemannian manifold $(M, \mathtt{g})$ of dimension $n \geq 4$ with parallel Weyl curvature ($\nabla W = 0$), which is not locally symmetric ($\nabla R \neq 0$) or conformally flat ($W \neq 0$). The reason why the latter conditions are excluded from the definition of an ECS manifold is because both $\nabla R = 0$ and $W = 0$ trivially imply that $\nabla W = 0$. As Roter showed in 1977, this is not really something artificial to do: a *Riemannian* manifold with parallel Weyl curvature must necessarily be locally symmetric or conformally flat [10, Theorem 2]. Roter also showed that ECS manifolds do exist in all dimensions $n \geq 4$, and realizing all possible indefinite metric signatures [37, Corollary 3].

The condition $\nabla W = 0$, being one of the natural differential conditions to be imposed in the irreducible components of the curvature tensor (in the sense of Besse, cf. [4, Chapter 16]), has been investigated and used by a number of authors — these include Cahen and Kerbrat [5], Mantica and Suh [33], Suh et al. [39], Hotloś [31], Deszcz et al. [24,25], among others. Some techniques used in the study of ECS manifolds also have been applied to obtain results for more general classes of manifolds, as shown, for instance, in [2], [6], [7], [23], and [40].

ECS manifolds are naturally sorted into two types. Namely, we define the *rank* of $(M, \mathtt{g})$ [17] to be the rank of its *Olszak distribution* $\mathcal{D}$, which is given by

$$\mathcal{D}_x = \{v \in T_xM \mid \mathtt{g}_x(v, \cdot) \wedge W_x(v', v'', \cdot, \cdot) = 0, \text{ for all } v', v'' \in T_xM\} \tag{1}$$

for all $x \in M$. The distribution $\mathcal{D}$ was originally introduced by Olszak [35] for the study of conformally recurrent manifolds but, in the ECS case, it turns out that $\mathcal{D}$ is a null parallel distribution whose rank is always equal to 1 or 2. For this reason, one speaks of *rank-one/rank-two ECS manifolds.*

Along with $\mathcal{D}$, the quotient vector bundle $\mathcal{D}^\perp/\mathcal{D}$ over M also plays a central role in the study of ECS manifolds. Namely,

> the Levi-Civita connection of $(M, \mathtt{g})$ induces connections in both $\mathcal{D}$ and $\mathcal{D}^\perp/\mathcal{D}$, with the latter connection being always flat, and the former one being flat when $(M, \mathtt{g})$ is of rank one, (2)

cf. [14, Lemma 2.2]. There, it is also shown that the Ricci endomorphism of $(M, \mathtt{g})$ is $\mathcal{D}$-valued, regardless of the rank of $(M, \mathtt{g})$.

Every Lorentzian ECS manifold $(M, \mathtt{g})$, unable to carry a null distribution of rank two, must be of rank one. Flatness of the connections induced in $\mathcal{D}$ and $\mathcal{D}^\perp/\mathcal{D}$ then means that, up to a double isometric covering (taken to globally trivialize $\mathcal{D}$ if needed), $(M, \mathtt{g})$ must be a *pp-wave spacetime*. Such spacetimes, introduced by Ehlers and Kundt in the 1960s [26], remain an active topic of research: see [8,27], [1,30], [3,36], and also the recent preprints [28,29]. While solving the partial differential equation $\nabla W = 0$ in Brinkmann coordinates does lead to a local classification result for Lorentzian ECS manifolds, the local structure of ECS manifolds of either rank has been determined in full generality by Derdzinski and Roter [14].

In view of the above, our attention will be mostly focused on global features of rank-one ECS manifolds; little is known about the rank-two case beyond their local structure and Theorem 4 in Section 4. One very natural question is whether compact manifolds can admit ECS metrics or not. The first examples of compact ECS manifolds were provided by Derdzinski and Roter in 2010 [15, Theorem 1.1], in all dimensions of the form $n = 3k+2$ with $k \geq 1$, and they realize all indefinite metric signatures. Such examples, arising as isometric quotients of what we call here *standard ECS plane waves* (see Section 3), also present distinct geometric and topological features: they are all of rank one, geodesically complete, not locally homogeneous, and diffeomorphic to total spaces of torus bundles over $\mathbb{S}^1$. The completeness conclusion in Lorentzian signature is not surprising: as shown by Leistner and Schliebner, compact pp-wave spacetimes are always complete [32]; more generally, Mehidi and Zeghib have shown that a compact Lorentzian manifold carrying a null parallel vector field must be complete [34].

Some further questions arise:

(i) Are there compact ECS manifolds of dimensions $n \geq 5$ other than those of the form $n = 3k + 2$ with $k \geq 1$?
(ii) Are there compact four-dimensional ECS manifolds?
(iii) Must compact ECS manifolds be geodesically complete?
(iv) Can a compact ECS manifold be locally homogeneous?
(v) Must compact rank-one ECS manifolds be bundles over $\mathbb{S}^1$?

Our recent progress in the topic consists in providing answers and partial answers to the above questions. In more detail:

(i) Yes. There are compact rank-one ECS manifolds of all dimensions $n \geq 5$, realizing all indefinite metric signatures [16, Theorem A].
(ii) There are no four-dimensional compact rank-one ECS manifolds [19, Corollary F]. However, it is still an open question whether there are four-dimensional compact rank-two ECS manifolds.
(iii) No. There are geodesically incomplete compact rank-two ECS manifolds of all *odd* dimensions $n \geq 5$, with semi-neutral metric signature [20, Theorem 6.1]. However, in a *generic* sense, compact rank-one ECS manifolds are complete [19, Theorem E]. (We elaborate on the meaning of 'genericity' in Section 6.)
(iv) Yes. The examples from [20, Theorem 6.1] mentioned in (iii) happen to be all locally homogeneous, although we also present incomplete non locally-homogeneous ones in [20, Theorem B.1].
(v) Outside of the locally homogeneous case and up to a double isometric covering, yes [17, Theorem A]. In addition, the fibers of the resulting bundle projection $M \to \mathbb{S}^1$ are the leaves of $\mathcal{D}^\perp$. It is an open question whether these conclusions can be extended to the locally homogeneous case as well without the additional assumption that $\mathcal{D}^\perp$ has a compact leaf.

Outline: In Section 2, we review the *translational-dilational* terminology [18], used when referring to the two distinct classes of compact ECS manifolds mentioned in the answers to (i) and (iii)-(iv) above.

In both [16] and [20], the resulting compact ECS manifolds again arise as isometric quotients of standard ECS plane waves, which we review in Section 3. In particular, we also describe the isometry group of such manifolds.

Next, in Section 4, we introduce a family $\{\mathrm{G}(\sigma)\}_{\sigma\in\mathrm{S}}$ of subgroups of the isometry group of a standard ECS plane wave $(\widehat{M}, \widehat{\mathfrak{g}})$, and present in Theorem 2 a criterion for the existence of subgroups Γ of a given $\mathrm{G}(\sigma)$ for which the quotient $\widehat{M}/\Gamma$ is smooth and compact. Using it, we then outline the constructions given in [16] and [20].

Section 5 is devoted to discussing the proof of the topological structure result mentioned in (v) above [17], and the concepts used in it; special emphasis is given to the dichotomy property [21] for a codimension-one foliation $\mathcal{V}$ in a smooth manifold M, which is established and applied for $\mathcal{V} = \mathcal{D}^\perp$.

Finally, we conclude the presentation with Section 6, elaborating on the notion of *genericity* mentioned in the answer to (iii), and stating some classification results for generic compact rank-one ECS manifolds [18,19]. Such results culminate in the answer to (ii).

Acknowledgements: It is a pleasure to thank Andrzej Derdzinski and Paolo Piccione for their very helpful comments on earlier drafts of this text.

2 The translational-dilational holonomy dichotomy

The new compact ECS examples presented in [16, Theorem A] and [20, Theorems 6.1 and B.1], discussed in Section 4, are sorted out into two classes. Here, we discuss them in more generality: a *weakly pp-wave manifold* $(M, \mathtt{g})$, that is, a pseudo-Riemannian manifold carrying a null parallel distribution $\mathcal{P}$ of rank one satisfying (2) (for $\mathcal{D}$ replaced with $\mathcal{P}$), will be called *translational* or *dilational*, according to whether the holonomy group of the flat connection induced in $\mathcal{P}$ is finite or infinite. Every *Ricci-recurrent* ECS manifold — that is, those for with $\nabla \mathrm{Ric} = \theta \otimes \mathrm{Ric}$ for some 1-form θ, outside of the zero-set of Ric — is weakly pp-wave, with $\mathcal{P}$ being the Olszak distribution $\mathcal{D}$ or the image of the Ricci endomorphism, according to whether its rank is one or two; this is a direct consequence of Roter's local structure theorem [37], and we describe the local model in Section 3.

To understand the reason for the names 'translational' and 'dilational' more concretely, let $\pi\colon \widetilde{M} \to M$ be the universal covering of M and set $\widetilde{\mathtt{g}} = \pi^*\mathtt{g}$. Then $(\widetilde{M}, \widetilde{\mathtt{g}})$ is also weakly pp-wave, with a distribution $\widetilde{\mathcal{P}}$ projecting onto $\mathcal{P}$ and also satisfying (2). As $\widetilde{M}$ is simply connected, we may fix

$$\text{a smooth function } t\colon \widetilde{M} \to I, \text{ surjective onto some open interval } I \subseteq \mathbb{R}, \text{ whose gradient is null, parallel, and spans } \widetilde{\mathcal{P}}. \tag{3}$$

Writing $M = \widetilde{M}/\Gamma$ for some group $\Gamma \cong \pi_1(M)$ acting on $(\widetilde{M}, \widetilde{\mathtt{g}})$ freely and properly discontinuously by deck isometries, we see that for every $\gamma \in \Gamma$ there is $(q, p) \in \mathrm{Aff}(\mathbb{R})$ such that $t \circ \gamma = qt + p$. This gives rise to two group homomorphisms

$$\text{i) } \Gamma \ni \gamma \mapsto (q, p) \in \mathrm{Aff}(\mathbb{R}) \quad \text{and} \quad \text{ii) } \Gamma \ni \gamma \mapsto q \in \mathbb{R} \smallsetminus \{0\}. \tag{4}$$

As the image of (4-ii) equals the holonomy group of the flat connection induced in $\mathcal{P}$, cf. [18, Lemma 3.1] (whose proof does not use compactness of M or the parallel Weyl curvature condition), the two possibilities for $(M, \mathtt{g})$ read:

$$\begin{aligned} &\text{(i) translational: } |q| = 1 \text{ for every } \gamma \in \Gamma, \text{ and} \\ &\text{(ii) dilational: } |q| \neq 1 \text{ for some } \gamma \in \Gamma. \end{aligned} \tag{5}$$

On the other hand, the image of (4-i) consists (up to an index-two subgroup) only of translations, or of dilations from a same fixed point, according to whether $(M, \mathtt{g})$ is translational or dilational.

There are two last relevant consequences of (5) for ECS manifolds.

Proposition 1. *Let (M,g) be a Ricci-recurrent ECS manifold. Then:*

(i) *if (M,g) is Lorentzian, it is translational.*
(ii) *if (M,g) is translational and compact, it cannot be locally homogeneous.*

Proof. Consider first (i), with $\mathcal{P}=\mathcal{D}$, as well as the vector space V of parallel sections of $\widetilde{\mathcal{D}}^{\perp}/\widetilde{\mathcal{D}}$. Note that $\dim V = n-2$, and that $\widetilde{\mathrm{g}}$ induces a pseudo-Euclidean inner product $\langle\cdot,\cdot\rangle$ on V. If Z is any vector field on $\widetilde{M}$ such that $\mathrm{d}t(Z)=1$, and W is the Weyl tensor of $(\widetilde{M},\widetilde{\mathrm{g}})$, the operator

$$A\colon V\to V,\ \text{given by}\ A(X+\widetilde{\mathcal{D}})=W(Z,X)Z+\widetilde{\mathcal{D}}, \tag{6}$$

is well-defined, traceless, self-adjoint, and independent of the choice of Z; see [13, Section 4] and [19, Section 5]. The derivative of any element $\gamma\in\mathit{\Gamma}$, regarded as a deck isometry of $(\widetilde{M},\widetilde{\mathrm{g}})$, induces an isometry C of $(V,\langle\cdot,\cdot\rangle)$ with $CAC^{-1}=q^2A$, where q is the (4-ii)-image of γ. When (M,g) is dilational, A must necessarily be nilpotent, being conjugate to a nontrivial multiple of itself, cf. (5-ii). On the other hand, $(V,\langle\cdot,\cdot\rangle)$ is Euclidean when (M,g) is Lorentzian, and in this case the only self-adjoint nilpotent operator is $A=0$. As $W=0$ whenever $A=0$, (i) must hold.

To address (ii), we first observe that

$$\begin{aligned}&\mathrm{Ric}=(2-n)f(t)\,\mathrm{d}t\otimes\mathrm{d}t\ \text{in}\ \widetilde{M},\ \text{for a suitable smooth}\\ &\text{function}\ f\colon\widetilde{M}\to\mathbb{R},\ \text{which is locally a function of}\ t.\end{aligned} \tag{7}$$

Here, $f(t)$ stands for the composition $f\circ t$. Indeed, the Ricci endomorphism of $(\widetilde{M},\widetilde{\mathrm{g}})$ is $\widetilde{\mathcal{D}}$-valued and self-adjoint, while the relation $\mathrm{d}^{\nabla}\mathrm{Ric}=0$ (implied by $\nabla W=0$) trivially leads to $\mathrm{d}f\wedge\mathrm{d}t=0$. In addition, we have

$$q^2f(qt+p)=f(t)\ \text{for all}\ \gamma\in\mathit{\Gamma},\ \text{with}\ (q,p)\ \text{being the (4-ii)-image of}\ \gamma, \tag{8}$$

due to $\mathit{\Gamma}$-invariance of Ric. Thus, when (M,g) is locally homogeneous, $|f|^{1/2}\mathrm{d}t$ is a closed and $\mathit{\Gamma}$-invariant 1-form without zeros. The existence of such a 1-form, in turn, implies that the level sets of t are connected and coincide with the leaves of $\widetilde{\mathcal{D}}^{\perp}$, making f a global function of t [17, Lemma 7.2]. As f is nowhere-vanishing and $(|f|^{-1/2})^{\cdot\cdot}=0$ by [17, Theorem 7.3], we may assume that $f(t)=f(1)/t^2$ after replacing t with an affine function of t if necessary. On the other hand, if at the same time (M,g) is translational, f is $\mathit{\Gamma}$-invariant by (8) and (5-i), and thus survives on the compact quotient $M=\widetilde{M}/\mathit{\Gamma}$ as an unbounded and continuous function. This contradiction proves (ii).

3 Standard ECS plane waves

3.1 Setup and terminology

In this section, following [37], we fix the following data: an integer $n\geq 4$, a pseudo-Euclidean vector space $(V,\langle\cdot,\cdot\rangle)$ of dimension $n-2$, a nonzero traceless and self-adjoint operator $A\colon V \to V$, an open interval $I \subseteq \mathbb{R}$, and a nonconstant smooth function $f\colon I \to \mathbb{R}$. With this in place, we consider the simply connected n-dimensional pseudo-Riemannian manifold

$$(\widehat{M}, \widehat{\mathtt{g}}) = (I \times \mathbb{R} \times V,\ \kappa\,\mathrm{d}t^2 + \mathrm{d}t\,\mathrm{d}s + \langle\cdot,\cdot\rangle), \tag{9}$$

where $\kappa\colon I \times \mathbb{R} \times V \to \mathbb{R}$ is given by $\kappa(t,s,v) = f(t)\langle v,v\rangle + \langle Av, v\rangle$. Here, $\mathrm{d}t$, $\mathrm{d}s$, and $\langle\cdot,\cdot\rangle$ are identified with their pullbacks to $\widehat{M}$. Note that the null coordinate vector field ∂_s is parallel, due to its being obviously a Killing vector field and corresponding under $\widehat{\mathtt{g}}$ to the closed 1-form $\mathrm{d}t/2$. We will repeatedly refer to

$$\text{the null parallel rank-one distribution } \mathcal{P} \text{ on } \widehat{M} \text{ spanned by } \partial_s. \tag{10}$$

A routine computation also shows that

$$(\widehat{M}, \widehat{\mathtt{g}}) \text{ is geodesically complete if and only if } I = \mathbb{R}. \tag{11}$$

In addition, it is well-known that $(\widehat{M}, \widehat{\mathtt{g}})$ is an ECS manifold, and that every point in a rank-one ECS manifold has a neighborhood isometric to an open subset of a suitable $(\widehat{M}, \widehat{\mathtt{g}})$, cf. [14, Theorem 4.1]. As (t,s,v) are global Brinkmann coordinates for $(\widehat{M}, \widehat{\mathtt{g}})$ and the function κ is quadratic in the variable v, we will call $(\widehat{M}, \widehat{\mathtt{g}})$ a *standard ECS plane wave*. We emphasize that our standard ECS plane waves, which obviously include the Lorentzian ones, may have arbitrary indefinite metric signature.

In [18] and [19], such $(\widehat{M}, \widehat{\mathtt{g}})$ were called 'rank-one ECS models.' To justify the change in terminology here, we observe that $(\widehat{M}, \widehat{\mathtt{g}})$ does not necessarily have rank one as claimed in [14], but instead

$$\begin{array}{l}\text{the rank of } (\widehat{M}, \widehat{\mathtt{g}}) \text{ equals 1 or 2 according to} \\ \text{whether } \operatorname{rank}(A) > 1 \text{ or } \operatorname{rank}(A) = 1, \text{ respectively,}\end{array} \tag{12}$$

cf. [22]. Indeed, the only possibly-nonzero components of the Weyl operator of $(\widehat{M}, \widehat{\mathtt{g}})$ acting on bivectors are given by $W(\partial_t \wedge \partial_j) = 2\partial_s \wedge A\partial_j$, for all indices $j = 1, \ldots, n-2$, where $(x^1, \ldots, x^{n-2})$ are arbitrary linear coordinates in V [37, p. 93]. We may now compute the Olszak distribution $\mathcal{D}$ of $(\widehat{M}, \widehat{\mathtt{g}})$, using (1). If $u = a\partial_t + b\partial_s + u_0$ lies in $\mathcal{D}$, with $a, b \in \mathbb{R}$ and $u_0 \in V$, the conditions $u \wedge \partial_s \wedge A\partial_j = 0$ directly imply that $a = 0$ and $u_0 \wedge \partial_s \wedge w = 0$ for all $w \in \operatorname{Im}(A)$, so that $u_0 = 0$ whenever $\operatorname{rank}(A) > 1$, while b remains free; when $\operatorname{rank}(A) = 1$, we instead obtain that $u_0 \in \operatorname{Im}(A)$. Thus, for $\mathcal{P}$ given in (10), we have $\mathcal{D} = \mathcal{P}$ when $\operatorname{rank}(A) > 1$, and $\mathcal{D} = \mathcal{P} \oplus \operatorname{Im}(A)$ when $\operatorname{rank}(A) = 1$, finally proving (12).

The standard ECS plane waves of the form (9) with $\operatorname{rank}(A) = 1$ form a narrow — but relevant — class of rank-two ECS manifolds; see [12] for more details on the local structure of rank-two ECS manifolds.

3.2 The full isometry group

In order to describe the isometry group of an n-dimensional standard ECS plane wave $(\widehat{M}, \widehat{\mathtt{g}})$ defined as in (9), two ingredients are needed.

The first one is the subgroup S of the direct product $\mathrm{Aff}(\mathbb{R}) \times \mathrm{O}(V, \langle\cdot,\cdot\rangle)$, consisting of all triples $\sigma = (q, p, C)$ having $CAC^{-1} = q^2 A$, $qt + p \in I$ and $f(t) = q^2 f(qt + p)$ for all $t \in I$. The second one is

$$\text{the } 2(n-2)\text{-dimensional symplectic vector space } (\mathcal{E}, \Omega) \text{ of solutions } u\colon I \to V \text{ of the ordinary differential equation } \ddot{u}(t) = f(t)u(t) + Au(t). \tag{13}$$

Here, Ω is defined by $\Omega(u, w) = \langle \dot{u}, w\rangle - \langle u, \dot{w}\rangle$. Associated with $(\mathcal{E}, \Omega)$ is its Heisenberg group: the Cartesian product $\mathrm{H} = \mathbb{R} \times \mathcal{E}$ equipped with the group operation $(r, u)(\hat{r}, \hat{u}) = (r + \hat{r} - \Omega(u, \hat{u}), u + \hat{u})$.

All three quantities q, (q, p), and C depend homomorphically on σ. Therefore, we have left actions of S on I, $\mathbb{R}$, and $\mathrm{C}^\infty(I, V)$, given by

$$\text{i) } \sigma t = qt + p, \quad \text{ii) } \sigma s = q^{-1}s, \quad \text{and} \quad \text{iii) } (\sigma u)(t) = Cu(q^{-1}(t - p)), \tag{14}$$

respectively. As (14-i) and (14-ii) are at odds when $I = \mathbb{R}$, we adopt only (14-i) and explicitly write $q^{-1}s$ for (14-ii), understanding that q is the first component of σ. Note that as a particular case of (14-iii), S acts on V via $\sigma v = Cv$, and these actions are compatible in the sense that $(\sigma u)(t) = \sigma(u(\sigma^{-1}t))$. Finally, as a routine computation shows, (14-iii) also restricts to an action of S on $\mathcal{E}$ for which $\sigma^*\Omega = q^{-1}\Omega$ for every $\sigma \in \mathrm{S}$, whenever σ is regarded as a linear operator $\sigma\colon \mathcal{E} \to \mathcal{E}$.

The description of $\mathrm{Iso}(\widehat{M}, \widehat{\mathrm{g}})$ given below as a semidirect product was already known to Schliebner, at least in Lorentzian signature, cf. the preprint [38]. In general indefinite metric signature, it can be found in [19, Theorem 6.1]:

Theorem 1. *The isometry group of a standard ECS plane wave $(\widehat{M}, \widehat{\mathrm{g}})$ is isomorphic to the semidirect product $\mathrm{S} \ltimes_\rho \mathrm{H}$, where the structure morphism $\rho\colon \mathrm{S} \to \mathrm{Aut}(\mathrm{H})$ is given by $\rho(\sigma)(r, u) = (q^{-1}r, \sigma u)$. More precisely, $\Phi \in \mathrm{Iso}(\widehat{M}, \widehat{\mathrm{g}})$ corresponds to the triple (σ, r, u) characterized by*

$$\Phi(t, s, v) = \big(\sigma t,\ -\langle \dot{u}(\sigma t), 2\sigma v + u(\sigma t)\rangle + q^{-1}s + r,\ \sigma v + u(\sigma t)\big) \tag{15}$$

for all $(t, s, v) \in \widehat{M}$, while the group operation in $\mathrm{S} \ltimes_\rho \mathrm{H}$ reads

$$(\sigma, r, u)(\hat{\sigma}, \hat{r}, \hat{u}) = \big(\sigma\hat{\sigma}, r + q^{-1}\hat{r} - \Omega(u, \sigma\hat{u}), u + \sigma\hat{u}\big), \tag{16}$$

for all $(\sigma, r, u), (\hat{\sigma}, \hat{r}, \hat{u}) \in \mathrm{S} \ltimes_\rho \mathrm{H}$.

Remark 1. Whenever convenient, we identify the Heisenberg group H with the normal subgroup $\{(1, 0, \mathrm{Id}_V)\} \times \mathrm{H}$ of $\mathrm{Iso}(\widehat{M}, \widehat{\mathrm{g}})$, which also equals the kernel of the homomorphism $\mathrm{Iso}(\widehat{M}, \widehat{\mathrm{g}}) \ni (\sigma, r, u) \mapsto \sigma \in \mathrm{S}$. By normality, H is invariant under all conjugation mappings $\mathrm{C}_\Phi\colon \mathrm{Iso}(\widehat{M}, \widehat{\mathrm{g}}) \to \mathrm{Iso}(\widehat{M}, \widehat{\mathrm{g}})$. If $\Phi = (\sigma, b, w)$, the restriction $\mathrm{C}_\Phi\colon \mathrm{H} \to \mathrm{H}$ is explicitly given by $\mathrm{C}_\Phi(r, u) = \big(q^{-1}r - 2\Omega(w, \sigma u), \sigma u\big)$.

A proof of Theorem 1 is given in [18, Theorem 4.1], where we explain exactly how the proof of [9, Theorem 2] can be carried over to our current setting.

In order to explain geometric meaning of $(\mathcal{E}, \Omega)$ in an abstract rank-one ECS manifold, it will be convenient to use an explicit description of $\mathfrak{iso}(\widehat{M}, \widehat{\mathtt{g}})$. Note that the Lie algebra $\mathfrak{s}$ of S consists of all triples $(a, b, P) \in \mathfrak{aff}(\mathbb{R}) \times \mathfrak{so}(V, \langle\cdot,\cdot\rangle)$ having $[P, A] = 2aA$ and $2af(t) + (at + b)\dot{f}(t) = 0$ for all $t \in I$.

Corollary 1. *The Lie algebra $\mathfrak{iso}(\widehat{M}, \widehat{\mathtt{g}})$ of Killing vector fields on a standard ECS plane wave is isomorphic to the semidirect product $\mathfrak{s} \ltimes_{\rho_*} \mathfrak{h}$, with ρ as in Theorem 1. Namely, $X \in \mathfrak{iso}(\widehat{M}, \widehat{\mathtt{g}})$ corresponds to the triple $((a, b, P), \ell, w)$ characterized by*

$$X_{(t,s,v)} = (at + b)\partial_t|_{(t,s,v)} + (-\langle \dot{w}(t), 2v\rangle - as + \ell)\partial_s|_{(t,s,v)} + (Pv + w(t)), \quad (17)$$

for every $(t, s, v) \in \widehat{M}$. Furthermore, the Lie bracket of two Killing vector fields $X \sim ((a, b, P), \ell, w)$ and $\hat{X} \sim ((\hat{a}, \hat{b}, \hat{P}), \hat{\ell}, \hat{w})$ is given by

$$[X, \hat{X}] \sim \big((0, \hat{a}b - a\hat{b}, [\hat{P}, P]),\, 2\Omega(w, \hat{w}) - \hat{a}\ell + a\hat{\ell}\,,\, u\big), \quad (18)$$

where $u \in \mathcal{E}$ is defined by $u(t) = (at + b)\hat{w}^{\cdot}(t) - (\hat{a}t + \hat{b})\dot{w}(t) + \hat{P}w(t) - P\hat{w}(t)$.

Assume now that $(\widehat{M}, \widehat{\mathtt{g}})$ is of rank one. The Killing vector fields tangent to the leaves of $\mathcal{D}^\perp$ are obtained by setting $a = b = 0$ in (17), and P is the action of its local flow on the space of parallel sections of $\mathcal{D}^\perp/\mathcal{D}$ (so that such action is trivial precisely when $P = 0$). Taking equivalence classes of these Killing vector fields modulo $\mathcal{D}$, in turn, amounts to setting $\ell = 0$. The resulting quotient space is naturally identified with $\mathcal{E}$, and the Lie bracket between $X \sim ((0, 0, 0), 0, w)$ and $\hat{X} \sim ((0, 0, 0), 0, \hat{w})$ is readily seen to be given by $[X, \hat{X}] = 2\Omega(w, \hat{w})\partial_s$, cf. (18). The symplectic vector space $(\mathcal{E}, \Omega)$ makes sense for an abstract rank-one ECS manifold, once the objects A and f in (6) and (7) are in place. As a result, the symplectic form Ω is in fact a $\mathcal{D}$-valued Lie bracket, which becomes $\mathbb{R}$-valued once a null parallel vector field spanning $\mathcal{D}$ has been chosen.

4 Compact quotients of standard ECS plane waves

4.1 The groups $\mathrm{G}(\sigma)$

As mentioned in the Introduction, the compact rank-one ECS examples presented in [15] were built as isometric quotients of standard ECS plane waves, which all turned out to be of rank one. Finding subgroups Γ of $\mathrm{Iso}(\widehat{M}, \widehat{\mathtt{g}}) \cong \mathrm{S} \ltimes \mathrm{H}$ (see Theorem 1) for which the quotient $\widehat{M}/\Gamma$ is smooth and compact can be as complicated as the factor S is. Thus, we restrict our search for Γ to a certain family $\{\mathrm{G}(\sigma)\}_{\sigma \in \mathrm{S}}$ of subgroups of $\mathrm{Iso}(\widehat{M}, \widehat{\mathtt{g}})$. Namely, for each $\sigma \in \mathrm{S}$, we let

$$\mathrm{G}(\sigma) = \{(\sigma^k, r, u) \mid k \in \mathbb{Z} \text{ and } (r, u) \in \mathrm{H}\} \cong \mathbb{Z} \ltimes \mathrm{H}. \quad (19)$$

Abbreviating each (σ^k, r, u) simply to (k, r, u), the action of $\mathrm{G}(\sigma)$ on $\widehat{M}$ reads

$$(k, r, u)\cdot(t, s, v) = (\sigma^k t, -\langle \dot{u}(\sigma^k t), 2\sigma^k v + u(\sigma^k t)\rangle + q^{-k}s + r, \sigma^k v + u(\sigma^k t)), \quad (20)$$

while the operation in $\mathrm{G}(\sigma)$ can be written as

$$(k,r,u)(\hat{k},\hat{r},\hat{u}) = (k+\hat{k}, r+q^{-k}\hat{r} - \Omega(u,\sigma^k\hat{u}), u+\sigma^k\hat{u}), \tag{21}$$

for all $(k,r,u),(\hat{k},\hat{r},\hat{u}) \in \mathrm{G}(\sigma)$ and $(t,s,v) \in \widehat{M}$. Here, we use the notation introduced in (14) and the lines following it. Lastly, inspired by the discussion in Section 2, we will say that

$$\begin{array}{l}\sigma \text{ is } \textit{translational} \text{ if it has the form } \sigma = (1,p,C) \text{ and } I = \mathbb{R},\\ \text{and } \textit{dilational} \text{ if it has the form } \sigma = (q,0,C) \text{ and } I = (0,\infty),\end{array} \tag{22}$$

the interval I being the one used to define $(\widehat{M},\widehat{\mathrm{g}})$ in (9).

4.2 First-order subspaces, and the existence criterion

In both [16, Section 7] and [20, Section 5], for the sake of self-containedness, we have presented *ad hoc* proofs that once a suitable subgroup Γ of $\mathrm{Iso}(\widehat{M},\widehat{\mathrm{g}})$ (in fact, lying in some $\mathrm{G}(\sigma)$) has been found, the resulting quotient $\widehat{M}/\Gamma$ is indeed smooth and compact. Adapting the results from [15, Sections 4–7], we establish in Theorem 2 below a more general criterion for the existence of subgroups Γ of a given $\mathrm{G}(\sigma)$ producing smooth compact quotients $\widehat{M}/\Gamma$.

Consider again the symplectic vector space $(\mathcal{E},\Omega)$ introduced in (13), associated with a standard ECS plane wave $(\widehat{M},\widehat{\mathrm{g}})$. A vector subspace $\mathcal{L}$ of $\mathcal{E}$ is called a *first-order subspace* if, for every $t \in I$, the restriction $\delta_t|_{\mathcal{L}} \colon \mathcal{L} \to V$ is an isomorphism, where $\delta_t \colon \mathcal{E} \to V$ denotes the evaluation at t. The name 'first-order' is explained by the next result, which is a direct generalization of [15, Lemma 4.1]:

Lemma 1. *First-order subspaces $\mathcal{L}$ of $(\mathcal{E},\Omega)$ are in bijective correspondence with smooth curves $B \colon I \to \mathrm{End}(V)$ such that $\dot{B} + B^2 = f + A$, where f stands for the function $t \mapsto f(t)\,\mathrm{Id}_V$. The correspondence assigns to B the space $\mathcal{L}$ of all solutions $u \colon I \to V$ to the differential equation $\dot{u}(t) = B(t)u(t)$, and*

(i) *$\mathcal{L}$ is a Lagrangian subspace of $(\mathcal{E},\Omega)$ if and only if each $B(t)$ is self-adjoint.*

In addition, if $\sigma = (q,p,C)$ is chosen as in (22), then

(ii) *$\mathcal{L}$ is σ-invariant if and only if $B(\sigma t) = q^{-1}CB(t)C^{-1}$ for each $t \in I$;*
(iii) *whenever $\mathcal{L}$ is σ-invariant, the determinant of $\sigma|_{\mathcal{L}} \colon \mathcal{L} \to \mathcal{L}$ is given by*

$$\det\left(\sigma|_{\mathcal{L}}\right) = (\det C)\exp\left(-\int_{\varepsilon}^{\sigma\varepsilon} \mathrm{tr}(B(t))\,\mathrm{d}t\right), \tag{23}$$

where ε is 0 or 1 according to whether σ is translational or dilational.

Now, given $\sigma \in \mathrm{S}$, we let $\Pi \colon \mathrm{G}(\sigma) \to \mathbb{Z}$ and $\delta \colon \mathrm{Ker}\,\Pi \to \mathcal{E}$ be the natural projections, given by $\Pi(k,r,u) = k$ and $\delta(0,r,u) = u$. In addition, to each subgroup Γ of $\mathrm{G}(\sigma)$ we associate the following objects:

$$\begin{array}{rl}\text{(i)} & \text{the intersection } \Sigma = \Gamma \cap \operatorname{Ker} \Pi. \\ \text{(ii)} & \text{the image } \Lambda = \delta(\Sigma). \\ \text{(iii)} & \text{the subspace } \mathcal{L} \text{ of } \mathcal{E} \text{ spanned by } \Lambda.\end{array} \tag{24}$$

Note that Σ may be seen as a subset of $\mathrm{H} = \mathbb{R} \times \mathcal{E}$ in view of Remark 1, and we in fact have $\Sigma \subseteq \mathbb{R} \times \Lambda$.

The next result generalizes [15, Theorem 5.1], originally stated only for the case where $\sigma \in \mathrm{S}$ is translational and with $C = \mathrm{Id}_V$. Below, $\mathcal{P}$ is the distribution introduced in (10), and the conjugation mapping C_γ is as in Remark 1. Note that in the rank-two case, the distribution $\mathcal{P}$ also survives in the resulting compact quotients.

Lemma 2. *For a standard ECS plane wave $(\widehat{M}, \widehat{\mathrm{g}})$ and $\sigma \in \mathrm{S}$ chosen as in (22), let there be a subgroup Γ of $\mathrm{G}(\sigma)$ acting freely and properly discontinuously on $\widehat{M}$ with a compact quotient $M = \widehat{M}/\Gamma$. Then:*

(i) *There is $\theta \in [0, \infty)$ such that $\Sigma \cap (\mathbb{R} \times \{0\}) = \mathbb{Z}\theta \times \{0\}$ and $2\Omega(u, \hat{u}) \in \mathbb{Z}\theta$ for all $u, \hat{u} \in \Lambda$.*

(ii) *For every $k \in \Pi(\Gamma)$, we have $\sigma^k(\Lambda) = \Lambda$ and $\sigma^k(\mathcal{L}) = \mathcal{L}$. In addition, $\sigma^k|_{\mathcal{L}} = \mathrm{Id}_{\mathcal{L}}$ if k arises from a central element of Γ, and thus Γ is not virtually Abelian unless $\sigma^k|_{\mathcal{L}} = \mathrm{Id}_{\mathcal{L}}$ for some positive $k \in \Pi(\Gamma)$.*

(iii) *There is a locally trivial fibration $M \to \mathbb{S}^1$ whose fibers are the leaves of $\mathcal{P}^\perp$, all diffeomorphic to tori when $\theta = 0$ in* (i).

(iv) *$\mathcal{L}$ is a σ-invariant first-order subspace of $\mathcal{E}$ and, for θ in* (i), *one of the following situations must occur:*
   (a) *$\theta = 0$, $\mathcal{L}$ is Lagrangian, and Σ is a lattice in $\mathbb{R} \times \mathcal{L}$ projecting isomorphically onto Λ under the projection $\mathbb{R} \times \mathcal{L} \to \mathcal{L}$.*
   (b) *$\theta > 0$, while Λ and each $\Lambda_t = \delta_t(\Sigma)$ are lattices in $\mathcal{L}$ and V, respectively.*

(v) *If σ is dilational, then option* (a) *in* (iv) *is the one that holds and Γ is not virtually Abelian.*

Proof. The proof of [15, Theorem 5.1] can be applied here, *verbatim*, to obtain a major part of the claimed conclusions. Namely, (i), (ii) and (iv) here correspond to (c)-(d), (e)-(f), and (g)-(h) in [15], respectively.

Consider (iii). It is clear that

$$\begin{array}{l}\Sigma \text{ acts freely and properly discontinuously on} \\ \text{each hypersurface } \widehat{M}_t = \{t\} \times \mathbb{R} \times V, \text{ with } t \in I,\end{array} \tag{25}$$

and that the projection $\widehat{M} \ni (t, s, v) \mapsto t \in I$ becomes a Γ-equivariant map (the action of Γ on I is induced by the action of S, cf. (14-i)). Such projection then descends to a surjective submersion $M = \widehat{M}/\Gamma \to I/\Gamma \cong \mathbb{S}^1$, whose (automatically compact) fibers are precisely the closed images of the natural embeddings $\widehat{M}_t/\Sigma \hookrightarrow \widehat{M}/\Gamma$. Here,

$$\begin{array}{l}\text{the quotient } I/\Gamma \text{ equals either } \mathbb{R}/p\mathbb{Z} \text{ or } (0,\infty)/q^{\mathbb{Z}} \text{ according} \\ \text{to whether } \sigma \text{ in (22) is translational or dilational, respectively,} \\ \text{but both quotients are naturally isomorphic to the circle } \mathbb{S}^1.\end{array} \tag{26}$$

By Ehresmann's fiber theorem, $M \to \mathbb{S}^1$ is a fiber bundle projection. It remains to show that, when $\theta = 0$, each quotient $\widehat{M}_t/\Sigma$ is diffeomorphic to a torus. The surjective homomorphism $\delta|_\Sigma \colon \Sigma \to \Lambda$, having trivial kernel by (i), must be an isomorphism, allowing us to regard Σ (now Abelian) as a discrete subgroup of $\mathbb{R} \times \mathcal{L}$. Next, as $\mathcal{L}$ is Lagrangian by (i), the induced Heisenberg operation in $\mathbb{R}\times\mathcal{L}$ coincides with its obvious vector space addition. The mapping

$$\mathbb{R} \times \mathcal{L} \ni (r, u) \mapsto \big(t, r - \langle \dot{u}(t), u(t)\rangle, u(t)\big) \in \widehat{M}_t \tag{27}$$

is a Σ-equivariant diffeomorphism, making $(\mathbb{R} \times \mathcal{L})/\Sigma$ diffeomorphic to $\widehat{M}_t/\Sigma$, and hence compact. Thus Σ is a lattice in $\mathbb{R} \times \mathcal{L}$ and $(\mathbb{R} \times \mathcal{L})/\Sigma$ is a torus.

We now address (v). If σ is dilational, Γ must necessarily contain an element $\gamma = (k, r, u)$ with $k \neq 0$, so that $\gamma_*\partial_s = q^{-k}\partial_s$ with $q \neq 1$, while the complete flow ϕ of the Killing vector field ∂_s in $\widehat{M}$ is given in $\mathrm{G}(\sigma)$ by $\phi(\tau, \cdot) = (0, \tau, 0)$. If it were $\theta > 0$, we would have $\phi(\tau, \cdot) \in \Gamma$ for some $\tau \neq 0$ by (i), leading to a contradiction [18, Lemma 2.2]: for $a = q^{-k}$ we have $|a| \neq 1$, while $\gamma^m \circ \phi(\tau, \cdot) = \phi(a^m\tau, \cdot) \circ \gamma^m$ leads to $\phi(a^m\tau, \cdot) \in \Gamma$, for every $m \in \mathbb{Z}$. If $\eta = \mathrm{sgn}(1 - |a|)$ and $x \in \widehat{M}$ is arbitrary, $\{\phi(a^{m}\tau, x)\}_{m\geq 1}$ is a sequence in $\widehat{M}$ with mutually distinct terms for m large, converging to x, thus precluding proper discontinuity of the action of Γ on $\widehat{M}$. This proves that $\theta = 0$. Finally, if Γ had an Abelian subgroup Γ' with $[\Gamma : \Gamma'] < \infty$, then $\Gamma' \not\subseteq \Sigma$ as $[\Gamma \colon \Sigma] = \infty$, so that $\Pi(\Gamma')$ is not trivial, while $[\Gamma' \cap \Sigma : \Sigma] < \infty$ implies that $\Gamma' \cap \Sigma$ spans $\mathbb{R} \times \mathcal{L}$. A second contradiction follows: let $\gamma \in \Gamma' \smallsetminus \Sigma$ and note that while C_γ is trivial as Γ' is Abelian, its first component equals $q^{-1} \neq 1$, by (5) and the last line of Remark 1.

We are ready for the main result of this section.

Theorem 2. *For a standard ECS plane wave* $(\widehat{M}, \widehat{\mathrm{g}})$ *of either rank, and an isometry* $\gamma = (\sigma, b, w)$ *with* $\sigma \in \mathrm{S}$ *chosen as in* (22)*, the following conditions are equivalent:*

(a) *There is a discrete subgroup* $\Gamma \leq \mathrm{G}(\sigma)$ *acting freely and properly discontinuously on* $\widehat{M}$ *with a compact quotient* $M = \widehat{M}/\Gamma$*.*
(b) *There is a* σ*-invariant first-order subspace* $\mathcal{L}$ *of* $(\mathcal{E}, \Omega)$*, a lattice* $\Sigma \subseteq \mathbb{R} \times \mathcal{L}$ *with* $\mathrm{C}_\gamma(\Sigma) = \Sigma$*, and* $\theta \in [0, \infty)$ *such that* $\Sigma \cap (\mathbb{R} \times \{0\}) = \mathbb{Z}\theta \times \{0\}$ *and* $\Omega(u, \hat{u}) \in \mathbb{Z}\theta$ *for all* $u, \hat{u} \in \Lambda$*, where* Λ *is the image of* Σ *under the projection* $\mathbb{R} \times \mathcal{L} \to \mathcal{L}$*.*

If (b) *holds,* Γ *in* (a) *can be taken to be the group generated by* γ *and* Σ *and there is a locally trivial fibration* $M \to \mathbb{S}^1$ *whose fibers, all diffeomorphic to a torus or to a* 2*-step nilmanifold according to whether* $\mathcal{L}$ *is Lagrangian or not, are the leaves of* $\mathcal{P}^\perp$*. Finally,* M *equipped with its natural quotient metric is translational and complete, or dilational and incomplete, according to whether* $\sigma \in \mathrm{S}$ *itself is translational or dilational.*

Proof that (a) *implies* (b): Assume that there is a subgroup Γ of $\mathrm{G}(\sigma)$ as in (a). If σ is dilational, then items (i), (iii), (iv-a) and (v) in Lemma 2 amount to (b) with $\theta = 0$, and there is nothing else to prove. If σ is translational, we claim that replacing the compact quotient M with a suitable further quotient $M' \cong M/\mathbb{Z}_2$, we may assume that

$$\text{(i) } \Omega(u,\hat{u}) \in \mathbb{Z}\theta \text{ for all } u,\hat{u} \in \Lambda, \quad \text{and} \quad \text{(ii) } \Sigma \text{ is a lattice in } \mathbb{R}\times\mathcal{L}, \tag{28}$$

for θ and Σ as in Lemma 2(i), and (24-i), respectively, cf. [15, Section 6]. This completes the proof, but we justify (28) below for the reader's convenience.

First, assuming that $\theta > 0$, note that the element $\zeta = (0,\theta/2,0) \in \mathrm{G}(\sigma)$ is central, by (21) with $q = 1$, and hence induces an isometry ζ_0 of M (equipped with the quotient metric) such that $\zeta_0 \circ \pi = \pi \circ \zeta$, with $\pi\colon \widehat{M} \to M$ being the quotient projection. As $\zeta^2 \in \Gamma$, we have that $\zeta_0^2(\pi(x)) = \pi(\zeta^2(x)) = \pi(x)$ for every $x \in \widehat{M}$, allowing us to consider the (automatically proper) isometric action of $\mathbb{Z}_2 = \{\mathrm{Id}_M, \zeta_0\}$ on M, which we now claim is also free. Indeed, if it were to be $\zeta_0(\pi(x)) = \pi(x)$ for some $x = (t,s,v) \in \widehat{M}$, we would have that

$$(0,\theta/2,0)(t,s,v) = (k,r,u)(t,s,v), \quad \text{for some } (k,r,u) \in \Gamma. \tag{29}$$

By (20), comparing the I-components in (29) immediately yields $k = 0$, so that $u \in \Lambda \subseteq \mathcal{L}$, while comparing the V-components leads to $u(t) = 0$. Thus $u = 0$, as $\mathcal{L}$ is a first-order subspace by Lemma 2(iv). However, by Lemma 2(i), $(\theta/2, 0) = (r,0) \in \mathbb{Z}\theta \times \{0\}$ leads to $\theta = 0$, which is a contradiction. Hence the quotient $M' = M/\mathbb{Z}_2$ is indeed smooth and compact. Writing $M' = \widehat{M}/\Gamma'$, for the subgroup Γ' of $\mathrm{G}(\sigma)$ generated by Γ and ζ, we observe that $\theta' = \theta/2$ and $\Sigma' = \Sigma/2$ correspond to Γ' as θ and Σ corresponded to Γ, so that (28-i) follows from Lemma 2(i).

It remains to establish (28-ii), still under the assumptions that σ is translational and that Lemma 2(iv-b) holds. Fixing a basis $\{u_j\}_{j=3}^n$ of $\mathcal{L}$ which generates Λ, as well as $r_3,\ldots,r_n \in \mathbb{R}$ such that $(r_j,u_j) \in \Sigma$ for all $j = 3,\ldots,n$, we now claim that

$$\Sigma = \mathbb{Z}(\theta,0) \oplus \mathbb{Z}(r_3,u_3) \oplus \cdots \oplus \mathbb{Z}(r_n,u_n). \tag{30}$$

Denoting the right side of (30) by Σ'', we first note that $\delta|_\Sigma\colon \Sigma \to \Lambda$ and $\delta|_{\Sigma''}\colon \Sigma'' \to \Lambda$ are both surjective homomorphisms, and so

$$\begin{gathered}\text{every element of } \Sigma \text{ can be written in the specific form} \\ (\theta,0)^\ell (r_3,u_3)^{k_3}\cdots(r_n,u_n)^{k_n}, \text{ for some } \ell,k_3,\ldots,k_n \in \mathbb{Z}.\end{gathered} \tag{31}$$

Indeed, as $\delta((\theta,0)^\ell(r_3,u_3)^{k_3}\cdots(r_n,u_n)^{k_n}) = \sum_{j=3}^n k_j u_j$, one may reverse-engineer (31) by starting with $(r,u) \in \Sigma$, writing $u = \sum_{j=3}^n k_j u_j$ for suitable $k_3,\ldots,k_n \in \mathbb{Z}$, setting $(\widetilde{r},u) = (r_3,u_3)^{k_3}\cdots(r_n,u_n)^{k_n}$, and finally noting that $\delta(r,u) = \delta(\widetilde{r},u)$. (As $(\widetilde{r},u) \in \Sigma$ and $\mathrm{Ker}(\delta|_\Sigma) = \mathbb{Z}\theta\times\{0\}$ by Lemma 2(i), it follows that $(r,u) = (\theta,0)^\ell(\widetilde{r},u)$ for some $\ell \in \mathbb{Z}$.)

With (31) in place, observe that the mapping

$$\mathbb{R}\times\Lambda \ni (r,u) \mapsto (r+\mathbb{Z}\theta, u) \in \mathbb{R}/\mathbb{Z}\theta \times \Lambda \tag{32}$$

is a group homomorphism onto the direct product of $\mathbb{R}/\mathbb{Z}\theta$ and Λ, with kernel $\mathbb{Z}\theta \times \{0\}$, *regardless of the group structure considered in* $\mathbb{R} \times \Lambda$: the additive structure, or the Heisenberg one. Then, given $(r, u) \in \Sigma$ written as in (31), consider the linear combination $(r'', u) = \ell(\theta, 0) + \sum_{j=3}^{n} k_j(r_j, u_j) \in \Sigma''$, and observe that (32) maps both (r, u) — by (21) and (28-i) — and (r'', u) to the same image $(\sum_{j=3}^{n} k_j r_j + \mathbb{Z}\theta, u)$, allowing us to write $(r, u) = \ell'(\theta, 0) + (r'', u)$ for some $\ell' \in \mathbb{Z}$ (so that $\Sigma \subseteq \Sigma''$), or $(r'', u) = (\theta, 0)^{\ell''}(r, u)$ for some $\ell'' \in \mathbb{Z}$ (so that $\Sigma'' \subseteq \Sigma$). This establishes (30), and thus (28-ii), as required.

Proof that (b) *implies* (a): Assume that the objects $\mathcal{L}$, Σ, and θ are given as in (b). Our goal is to construct a smooth compact quotient of $\widehat{M}$. We start by showing that

the quotient space $N = (\mathbb{R} \times \mathcal{L})/\Sigma$, where the lattice Σ acts on the product $\mathbb{R} \times \mathcal{L}$ by *Heisenberg* left-translations, is a compact manifold. (33)

Indeed, if $\theta = 0$ (or, more generally, if $\mathcal{L}$ is Lagrangian), the induced Heisenberg operation in $\mathbb{R} \times \mathcal{L}$ agrees with its standard vector space addition, and so (33) follows from Σ being a lattice in $\mathbb{R} \times \mathcal{L}$ (in which case N is a torus). If $\theta > 0$ instead, the product $[0, \theta] \times Q$ is a compact fundamental domain for the *Heisenberg* action of Σ on $\mathbb{R} \times \mathcal{L}$ whenever $Q \subseteq \mathcal{L}$ is the image under the projection $\mathbb{R} \times \mathcal{L} \to \mathcal{L}$ of a compact fundamental domain for the *additive* action of Σ on $\mathbb{R} \times \mathcal{L}$: for any $(r, u) \in \mathbb{R} \times \mathcal{L}$ there is $(r_0, u_0) \in \Sigma$ such that $u + u_0 \in Q$, and choosing $k \in \mathbb{Z}$ such that $k\theta + r_0 + r - \Omega(u_0, u) \in [0, \theta]$, it follows from $(\theta, 0) \in \Sigma$ that $(\theta, 0)^k(r_0, u_0)(r, u) \in [0, \theta] \times Q$, as required.

Now, with ε being 0 or 1 according to whether σ in (22) is translational or dilational, we let $\widetilde{w} \in \mathcal{L}$ be the unique element with $\widetilde{w}(\sigma\varepsilon) = w(\sigma\varepsilon)$ and $\widetilde{b} \in \mathbb{R}$ be given by $\widetilde{b} = b - \langle \dot{w}(\sigma\varepsilon) - B(\sigma\varepsilon)w(\sigma\varepsilon), w(\sigma\varepsilon)\rangle$, where B corresponds to $\mathcal{L}$ as in Lemma 1. Note that $\widetilde{b} = b$ whenever $w = \widetilde{w} \in \mathcal{L}$. We claim that

$\phi\colon \mathbb{R} \times \mathcal{L} \to \mathbb{R} \times \mathcal{L}$ given by $\phi(r, u) = \big(q^{-1}r + \widetilde{b} - \Omega(\widetilde{w} - 2w, \sigma u), \sigma u + \widetilde{w}\big)$ is a Σ-equivariant mapping, and induces a diffeomorphism $\Phi\colon N \to N$. (34)

Indeed, σ-invariance of $\mathcal{L}$ says that ϕ is $(\mathbb{R} \times \mathcal{L})$-valued, while a routine computation shows that $\phi(r, u) = [\mathrm{C}_\gamma(r, u)](\widetilde{b}, \widetilde{w})$, so that C_γ-invariance of Σ together with the resulting relation $\phi((r_0, u_0)(r, u)) = [\mathrm{C}_\gamma(r_0, u_0)]\phi(r, u)$, valid for any $(r_0, u_0) \in \Sigma$, implies Σ-equivariance of ϕ.

We define an action of $\mathbb{Z}$ on $I \times N$ by setting $k \cdot (t, \Sigma(r, u)) = (\sigma^k t, \Phi^k \Sigma(r, u))$, so that the projection $I \times N \to I$ becomes $\mathbb{Z}$-equivariant and induces a locally trivial fibration $M \to \mathbb{S}^1$, where $M = (I \times N)/\mathbb{Z}$. It is clear that M so defined is compact. The projection $\pi\colon \widehat{M} \to M$, in turn, is then defined as the composition

$$\begin{array}{ccccccc} \widehat{M} & \longrightarrow & I \times \mathbb{R} \times \mathcal{L} & \longrightarrow & I \times N & \longrightarrow & M \\ (t, s, v) & \longmapsto & (t, r, u) & \longmapsto & (t, \Sigma(r, u)) & \longmapsto & \mathbb{Z} \cdot (t, \Sigma(r, u)), \end{array} \qquad (35)$$

where (t, r, u) and (t, s, v) are related via (27). Note that $\widehat{M}$ is the universal covering of M.

Letting Γ be generated by Σ and γ, it remains to be seen that the action of Γ on $\widehat{M}$ — given by (20) — is free and that the π-fibers coincide with the Γ-orbits. As $\gamma = (1, b, w)$ and Σ is C_γ-invariant, all elements of Γ are of the form $(1, b, w)^k(0, r, u)$ with $k \in \mathbb{Z}$ and $(r, u) \in \Sigma$. Freeness of the Γ-action on $\widehat{M}$ is now clear: if $(1, b, w)^k(0, r, u)(t, s, v) = (t, s, v)$ for some $(t, s, v) \in \widehat{M}$, projecting onto the I-component yields $k = 0$, so that $(0, r, u)(t, s, v) = (t, s, v)$ implies that $(r, u) = (0, 0)$ by freeness of the Σ-action on each $\widehat{M}_t$ (which corresponds under (27) to a manifestly-free Heisenberg action of Σ on $\mathbb{R} \times \mathcal{L}$). Finally, as Σ and γ act on $\widehat{M}$ by deck transformations, each Γ-orbit is contained in a π-fiber; conversely, if $\pi(t, s, v) = \pi(\widehat{t}, \widehat{s}, \widehat{v})$, we may write $\widehat{t} = \sigma^k t$ and replace $(\widehat{t}, \widehat{s}, \widehat{v})$ with $(1, b, w)^{-k}(\widehat{t}, \widehat{s}, \widehat{v})$ if necessary to assume that $k = 0$ and $t = \widehat{t}$, so that (t, s, v) and $(\widehat{t}, \widehat{s}, \widehat{v})$ are in the same Σ-orbit (and hence in the same Γ-orbit).

Clearly, the "type" of M (translational or dilational), equipped with its natural quotient metric, is the same as of σ in (22); its completeness (or, incompleteness) now follows directly from (11).

4.3 The translational construction

Here, we describe how Theorem 2 leads to compact rank-one translational ECS manifolds [16, Theorem A]:

Theorem 3. *There exist compact rank-one translational ECS manifolds of all dimensions $n \geq 5$ and all indefinite metric signatures, forming the total space of a nontrivial torus bundle over $\mathbb{S}^1$ with its fibers being the leaves of $\mathcal{D}^\perp$, all geodesically complete, and none locally homogeneous. In each fixed dimension and metric signature, there is an infinite-dimensional moduli space of local-isometry types.*

The proof strategy consists in *simultaneously* producing a standard ECS plane wave $(\widehat{M}, \widehat{\mathtt{g}})$ and a suitable subgroup Γ of $\mathrm{G}(\sigma)$, where the element $\sigma \in \mathrm{S}$ is translational and of the form $\sigma = (1, p, \mathrm{Id}_V)$ for some $p > 0$. Requiring the $\mathrm{O}(V, \langle\cdot,\cdot\rangle)$-component of σ to be trivial turns out to be a mild assumption which can be generically arranged for — see Section 6. As in (26), we set $\mathbb{S}^1 = \mathbb{R}/p\mathbb{Z}$, so that p-periodic functions defined on $\mathbb{R}$ (and valued anywhere) may be regarded as defined on $\mathbb{S}^1$, and integration from 0 to p becomes integration over $\mathbb{S}^1$.

Our goal is to find a σ-invariant first-order subspace $\mathcal{L}$ of $(\mathcal{E}, \Omega)$, a conjugation-invariant lattice $\Sigma \subseteq \mathbb{R} \times \mathcal{L}$, and some $\theta \geq 0$ satisfying certain additional conditions. By Lemma 1, such $\mathcal{L}$ corresponds to a smooth curve $B \colon \mathbb{S}^1 \to \mathrm{End}(V)$ solving the differential equation $\dot{B} + B^2 = f + A$. This is a crucial point: the objects f and A needed to define the metric $\widehat{\mathtt{g}}$ are in fact *determined* by B, as the scalar and traceless parts of $\dot{B} + B^2$. In addition, if each $B(t)$ is self-adjoint, $\mathcal{L}$ becomes Lagrangian and the only condition relating Σ and θ simply reads $\Sigma = \mathbb{Z}\theta \times \Lambda$. Lemma 2 suggests that Λ ought to arise as the lattice preserved by $\sigma|_{\mathcal{L}} = \exp\big(-\int_{\mathbb{S}^1} B\big)$ acting on $\mathcal{L}$.

In summary, it suffices to obtain a curve $B \colon \mathbb{S}^1 \to \mathrm{End}(V)$ of self-adjoint endomorphisms of $(V, \langle\cdot,\cdot\rangle)$ for which

$$\begin{array}{rl} \text{(i)} & \text{the trace of } \dot{B}+B^2 \text{ is nonconstant.}\\ \text{(ii)} & \text{the traceless part of } \dot{B}+B^2 \text{ is a nonzero constant.}\\ \text{(iii)} & \text{the operator } \exp\big(-\textstyle\int_{\mathbb{S}^1} B\big) \text{ maps a lattice } \Lambda\subseteq\mathcal{L} \text{ onto itself.}\end{array} \tag{36}$$

Conditions (36-i) and (36-ii) ensure, respectively, that the resulting standard ECS plane wave $(\widehat{M},\widehat{\mathrm{g}})$ is not locally symmetric or conformally flat. Then, we *define* f and A as the scalar and traceless parts of $\dot{B}+B^2$, the lattice Σ as the product $\mathbb{Z}\theta\times\Lambda$, and finally apply Theorem 2.

To narrow down the search for B we assume that $V=\mathbb{R}^{n-2}$, let $\Delta^{n-2}\cong\mathbb{R}^{n-2}$ denote the space of diagonal matrices of order $n-2$, and consider the arithmetic condition imposed on $(\lambda_1,\dots,\lambda_{n-2})\in\mathbb{R}^{n-2}$:

$$\begin{array}{l}\{\lambda_1,\dots,\lambda_{n-2}\}\subseteq(0,\infty)\smallsetminus\{1\} \text{ is not of}\\ \text{the form } \{\lambda\} \text{ or } \{\lambda,\lambda^{-1}\}, \text{ for any } \lambda>0.\end{array} \tag{37}$$

Whenever the entries of $\Theta\in\Delta^{n-2}$ satisfy (37), there is an infinite-dimensional submanifold of $\mathrm{C}^\infty(\mathbb{S}^1)$ whose elements can be realized as $(n-2)^{-1}\mathrm{tr}(\dot{B}+B^2)$ for some $B\in\mathrm{C}^\infty(\mathbb{S}^1,\Delta^{n-2})$ having $\exp(-\int_{\mathbb{S}^1}B)=\Theta$ and the traceless part of $\dot{B}+B^2$ be a nonzero constant [16, Theorem 6.2]. The proof of this fact combines a well-placed application of the Inverse Function Theorem in the lower-regularity Banach spaces $\mathrm{C}^k(\mathbb{S}^1,\Delta^{n-2})$, and the existence of local smooth-preserving retractions [16, Lemma 5.1], needed to deform the resulting C^k curves B into C^∞ ones. This takes care of conditions (36-i) and (36-ii). As for (36-iii), it suffices to note that for every $n\geq 5$ there is a matrix in $\mathrm{GL}_{n-2}(\mathbb{Z})$ with mutually distinct eigenvalues satisfying (37), cf. [16, Lemma 4.1], and hence conjugate to some $\Theta\in\Delta^{n-2}$ — which is then used as the starting point of the preceding argument.

4.4 The dilational construction

This time, we describe how Theorem 2 leads to odd-dimensional compact rank-two dilational ECS manifolds.

Theorem 4. *There exist compact rank-two dilational ECS manifolds of all odd dimensions $n\geq 5$ and with semi-neutral metric signature, including locally homogeneous ones, forming the total space of a nontrivial torus bundle over $\mathbb{S}^1$ with its fibers being the leaves of $\mathcal{P}^\perp$, all of them geodesically incomplete. In each fixed odd dimension, there is an infinite-dimensional moduli space of local-isometry types.*

Here, by a pseudo-Riemannian metric with *semi-neutral* metric signature, we mean one for which the difference between its positive and negative indices is at most one.

As before, the construction involves two different aspects: one analytical for f and A, and one combinatorial for Γ. The difference is that while the arithmetic condition (37) was simple, and finding f and A required deforming constant solutions of an ordinary differential equation, the roles are now reversed: f and A arise from the very explicit formulas (40) and (42), and the existence of Σ and $\mathcal{L}$ is established via an elaborate combinatorial structure, only available in odd dimensions [20, Section 2].

More precisely, a $\mathbb{Z}$-*spectral system* is a quadruple (m, k, E, J) where $m, k \geq 2$ are integers and, setting $\mathcal{V} = \{1, \ldots, 2m\}$, $E\colon \mathcal{V} \to \mathbb{Z} \smallsetminus \{-1\}$ and $J\colon \mathcal{V} \to \{0, 1\}$ are functions, with E injective, such that:

$$\begin{array}{rl} \text{(i)} & k+1 = 2E(1) \text{ (thus making } k \text{ odd).} \\ \text{(ii)} & E(i) + E(i') = -1 \text{ and } J(i) + J(i') = 1 \text{ whenever } i + i' = 2m+1. \\ \text{(iii)} & E(i) - E(i') = k \text{ and } J(i) + J(i') = 1 \text{ whenever } i' = i+1 \text{ is even.} \\ \text{(iv)} & \text{the set } Y = \{-1\} \cup \{E(i)\}_{i \in J^{-1}(1)} \text{ is symmetric about zero.} \end{array} \tag{38}$$

The *spectral selector* $S_1 = J^{-1}(1)$ is simultaneously a selector for both families

$$\begin{array}{rl} \text{(i)} & \{\{i, i'\} \in \wp_2(\mathcal{V}) \mid i + i' = 2m+1\} \text{ and} \\ \text{(ii)} & \{\{i, i'\} \in \wp_2(\mathcal{V}) \mid i' = i+1 \text{ is even}\}, \end{array} \tag{39}$$

where $\wp_2(\mathcal{V})$ stands for the collection of two-element subsets of $\mathcal{V}$. The function E, in turn, is to be thought of as an 'exponent function', which will give rise to the correct spectrum for the dilational element $\sigma \in \mathrm{S}$ to be described below.

Let $q \in (0, \infty) \smallsetminus \{1\}$ and $n \geq 5$ be odd, set $m = n-2$, and fix a $\mathbb{Z}$-spectral system (m, n, E, J) [20, Theorem 2.2]. In addition, let $(V, \langle\cdot,\cdot\rangle)$ be a m-dimensional pseudo-Euclidean vector space of semi-neutral metric signature and, for the scalars $a(1), \ldots, a(m) \in \mathbb{R}$ given by $a(i) = E(2i-1) + (1-n)/2$, for all i, we define A and C by

$$Ae_m = e_1 \text{ and } Ae_i = 0 \text{ for } i < m, \text{ and } Ce_i = q^{a(i)} e_i \text{ for all } i. \tag{40}$$

In (40), $(e_1, \ldots, e_m)$ is a basis for V with the property that, for some $\varepsilon = \pm 1$, $\langle e_i, e_k \rangle = \varepsilon \delta_{ij}$ for all $i, j \in \{1, \ldots, m\}$, where $k = m + 1 - j$ and we set $e_0 = 0$. With this, we have that A is self-adjoint, $C \in \mathrm{SO}(V, \langle\cdot,\cdot\rangle)$, and the relation $CAC^{-1} = q^2 A$ holds (cf. Section 3.2).

The element $\sigma = (q, 0, C)$, regarded as an endomorphism of $\mathcal{E}$ via (14-iii), has the spectrum $(\mu^+ q^{a(1)}, \mu^- q^{a(1)}, \ldots, \mu^+ q^{a(m)}, \mu^- q^{a(m)})$, where (μ^+, μ^-) is the spectrum of the endomorphism $y \mapsto (t \mapsto y(t/q))$ on the space of solutions $y\colon (0, \infty) \to \mathbb{R}$ of the ordinary differential equation $\ddot{y} = fy$. As a consequence,

$$\begin{array}{l} \text{when } \mu^{\pm} = q^{(-1 \pm n)/2}, \text{ the spectrum of} \\ \sigma \text{ becomes precisely } (q^{E(1)}, \ldots, q^{E(2m)}), \end{array} \tag{41}$$

cf. [20, formula (4.2)]. Condition (41) can be achieved, for instance, by choosing the function

$$f(t) = \frac{n^2 - 1}{4t^2}. \tag{42}$$

We may also require the eigenfunctions associated with μ^+ and μ^- to be positive. A straightforward computation — see Theorem 1 — shows that this choice of f makes the resulting standard ECS plane wave $(\widehat{M}, \widehat{\mathrm{g}})$ homogeneous.

Let $(u_1, \ldots, u_{2m}) = (u_1^+, u_1^-, \ldots, u_m^+, u_m^-)$ be a basis of $\mathcal{E}$ consisting of eigenvectors of σ, associated with the eigenvalues $q^{E(1)}, \ldots, q^{E(2m)}$, with the property that $\Omega(u_i, u_j) = 0$ whenever $i, j \in \{1, \ldots, 2m\}$ have $i + j \neq 2m + 1$, whose existence is ensured by [20, Lemma 4.1]. It now follows that the direct sum $\mathcal{L} = \bigoplus_{i \in S_1} \mathbb{R} u_i$ is a first-order σ-invariant Lagrangian subspace of $(\mathcal{E}, \Omega)$: σ-invariance is obvious, being Lagrangian is a consequence of S_1 being a selector for (39-i), and being first-order follows from the matrix representing $\sigma|_{\mathcal{L}}$ being upper triangular and with positive diagonal entries.

Finally, fix any $\gamma = (\sigma, b, w) \in \mathrm{G}(\sigma)$, and consider its conjugation mapping C_γ, cf. Remark 1. The product $\mathbb{R} \times \mathcal{L}$ is C_γ-invariant, since $\mathcal{L}$ is σ-invariant, and the spectrum of the restriction $\mathrm{C}_\gamma|_{\mathbb{R} \times \mathcal{L}}$ is given by $(q^a)_{a \in Y}$, for the set Y in (38-iv). The existence of a lattice $\Sigma \subseteq \mathbb{R} \times \mathcal{L}$ which is mapped onto itself by C_γ is guaranteed whenever $q + q^{-1} \in \mathbb{Z}$ [20, Remark 2.1], so starting the argument with such a value of q allows us to apply Theorem 2.

The resulting compact quotients are locally homogeneous due to the choice of f in (42), but we have an infinite-dimensional freedom to deform it while keeping μ^+ and μ^- the same [20, Theorem A.1]. For this reason we obtain dilational but non locally-homogeneous compact quotients as well.

5 The bundle structure

All known compact rank-one ECS examples share a specific topological feature: they are all bundles over $\mathbb{S}^1$ in such a way that $\mathcal{D}^\perp$ appears as the vertical distribution. This is not accidental [17, Theorem A]:

Theorem 5. *Every non locally-homogeneous compact rank-one ECS manifold* (M, g) *for which the orthogonal distribution* $\mathcal{D}^\perp$ *is transversely orientable is the total space of a locally trivial fibration over* $\mathbb{S}^1$ *whose fibers are the leaves of* $\mathcal{D}^\perp$*. The conclusion remains valid in the locally homogeneous case if, in addition, one assumes that* $\mathcal{D}^\perp$ *has at least one compact leaf.*

Observe that this generalizes [13, Theorem B] from the Lorentzian signature, where the locally homogeneous case was ruled out (cf. Section 2). The proof of Theorem 5 however, uses different tools and clarifies the geometric role of $\mathcal{D}^\perp$. The precise differences between the new and old proofs in the Lorentzian case are explained in the Appendix of [17].

The central concept used in what follows is what we call the *dichotomy property* [21] for a codimension-one foliation $\mathcal{V}$ in a smooth manifold M, which has two alternatives (NC) and (AC) imposed on its compact leaves. Namely, $\mathcal{V}$ has the dichotomy property if every compact leaf L of $\mathcal{V}$ has a neighborhood U in M such that the leaves of $\mathcal{V}$ intersecting $U \smallsetminus L$ are either:

- (NC) all noncompact, or
- (AC) all compact, and some neighborhood of L in M saturated by compact leaves of $\mathcal{V}$ may be diffeomorphically identified with the product $\mathbb{R} \times L$ in such a way that $\mathcal{V}$ corresponds to the foliation $\{\{s\} \times L\}_{s \in \mathbb{R}}$.

The reason why we care about this property is that if M is compact, $\mathcal{V}$ is transversely orientable and has the dichotomy property, and there is one compact leaf of $\mathcal{V}$ satisfying (AC), then there is a locally trivial fibration $M \to \mathbb{S}^1$ whose fibers are the leaves of $\mathcal{V}$ [17, Theorem 4.1] — the main argument justifying it goes back to Reeb.

In view of the above, there are two main steps to be carried out:

$$\begin{aligned} &\text{(i) showing that } \mathcal{D}^\perp \text{ (when transversely orientable) has the dichotomy property, and} \\ &\text{(ii) establishing the existence of a compact leaf of } \mathcal{D}^\perp \text{ satisfying (AC).} \end{aligned} \tag{43}$$

While (43-i) does not rely on compactness of M, we could only achieve (43-ii) outside of the locally homogeneous case.

Let $(M, \mathtt{g})$ be a compact rank-one ECS manifold. Replacing $(M, \mathtt{g})$ with a suitable isometric double covering if needed, we may assume that $\mathcal{D}^\perp$ is transversely orientable. The crucial fact needed to conclude (43-i) is that whenever L is a compact leaf of $\mathcal{D}^\perp$, some neighborhood U of L in M can be diffeomorphically identified with a neighborhood U' of the zero section L in the line bundle $\mathcal{D}^*_L$ (the dual bundle of $\mathcal{D}$ restricted to L) in such a way that the distribution $\mathcal{D}^\perp$ on U corresponds to the restriction to U' of the horizontal distribution of the natural flat connection induced in $\mathcal{D}^*_L$ [17, Theorem 10.1]. Once this in place, the dichotomy property for $\mathcal{D}^\perp$ follows, with options (AC) and (NC) corresponding to whether the holonomy group of the natural flat connection in $\mathcal{D}^*_L$ is finite or infinite, respectively.

The argument for (43-ii) is more elaborate, and we need to consider cohomology classes of continuous 1-forms. Namely, a continuous 1-form is called *closed* if it is locally given by the differential of a C^1 function, in which case it indeed gives rise to a cohomology class (that is itself trivial if and only if a global C^1 anti-derivative exists). The cohomology space $H^1_{\mathrm{dR}}(M)$ obtained in such a way can be identified with $\mathrm{Hom}(\pi_1(M), \mathbb{R})$, as usual.

Fix the universal covering $(\widetilde{M}, \widetilde{\mathtt{g}})$ of $(M, \mathtt{g})$, as well as a smooth function $t\colon \widetilde{M} \to \mathbb{R}$ whose parallel gradient spans $\mathcal{D}$, cf. (2) and (3). Writing $M = \widetilde{M}/\Gamma$ for some group $\Gamma \cong \pi_1(M)$ (see Section 2) and introducing the space $\mathcal{F}$ of all $\chi \in \mathrm{C}^0(\widetilde{M})$ for which $\chi\,\mathrm{d}t$ is closed and Γ-invariant, we may consider the linear operator $P\colon \mathcal{F} \to H^1_{\mathrm{dR}}(M)$ given by $P\chi = [\chi\,\mathrm{d}t]$. Here, by $[\chi\,\mathrm{d}t]$ we mean the cohomology class of the 1-form in M whose pullback under the universal covering projection is $\chi\,\mathrm{d}t$. Using this operator P, we may show that if $(M, \mathtt{g})$ is not locally homogeneous,

$$\text{there is a nonconstant } \mu \in \mathrm{C}^1(M) \text{ which is constant along } \mathcal{D}^\perp, \tag{44}$$

cf. [17, Theorem 9.1]. More precisely, either $\dim \mathcal{F} < \infty$ and $(M, \mathtt{g})$ is locally homogeneous, or $\dim \mathcal{F} = \infty$ and μ in (44) exists — the latter case is justified by choosing $\chi \in \mathrm{Ker}\,P \smallsetminus \{0\}$ and letting $\mu \in \mathrm{C}^1(M)$ be an anti-derivative of the (now exact) 1-form in M induced by $\chi\,\mathrm{d}t$. The first case is ruled out due to set-theoretical reasons [17, Lemma 3.3] ultimately implying that $f(t) =$

$\varepsilon(t-b)^{-2}$ for some $b \in \mathbb{R}$ and $\varepsilon = \pm 1$, for the function f characterized by $\mathrm{Ric} = (2-n)f(t)\,\mathrm{d}t \otimes \mathrm{d}t$ in $\widetilde{M}$. (Such formula for f implies local homogeneity of $(M, \mathtt{g})$, by [17, Theorem 7.3].)

With μ as in (44) in hands, the conclusion of (43-ii) follows from Sard's theorem: the image of μ in $\mathbb{R}$ contains an open interval of regular values of μ; any connected component of a level set $\mu^{-1}(c)$, with c in a such open interval, is a compact leaf of $\mathcal{D}^{\perp}$ satisfying (AC). It is worth noting that while Sard's theorem usually applies for a C^k function from an n-manifold into an m-manifold, where $k \geq \max\{n-m+1, 1\}$, compactness of μ together with μ being locally a function of the real-valued function t (which is without critical points) allows us to apply Sard with $n = 1$ as opposed to $n \geq 4$, cf. [17, Remark 9.2].

6 Genericity and classification results

The known classification results for compact rank-one ECS manifolds, which we briefly discuss[1] in this final section, rely on a certain notion of *genericity*, explained next.

Namely, if $(V, \langle\cdot,\cdot\rangle)$ is a pseudo-Euclidean vector space and $A\colon V \to V$ is traceless and self-adjoint, we say that A is *generic* if only finitely many isometries of $(V, \langle\cdot,\cdot\rangle)$ commute with A. For example, all diagonalizable operators with mutually distinct eigenvalues are generic. As shown in [19, Section 4], every nonzero A is generic when $\dim V = 2$, and generic endomorphisms form an open and dense subset of the space of all traceless and self-adjoint operators. Still at the linear algebra level, the algebraic structure of nilpotent generic operators is known [18, Theorem 5.1] and, in this case, for each $q \in (0, \infty)$ there are only two isometries $C, -C$ of $(V, \langle\cdot,\cdot\rangle)$ such that $CAC^{-1} = q^2 A$ [18, Corollary 5.3].

Proceeding, a rank-one ECS manifold $(M, \mathtt{g})$ is *generic* if the operator A in (6) is generic in the sense described in the previous paragraph. When $(M, \mathtt{g})$ is a standard ECS plane wave, this is the same as requiring the operator A used to define it as in Section 3 to be generic. In particular, note that every four-dimensional rank-one ECS manifold is generic.

One of the main reasons genericity plays a central role in obtaining classifications results is given in [19, Corollary D]:

Theorem 6. *The universal covering of a generic compact rank-one ECS manifold is globally isometric to a standard ECS plane wave.*

This result had been proved in the Lorentzian case by Schliebner, in the preprint [38], without the genericity assumption. With the aid of an impossible combinatorial structure [18, Section 8], similar in spirit to the $\mathbb{Z}$-spectral systems mentioned in Section 4.4, we show that in the generic case, the image of (4-ii) is not infinite cyclic, leading to [18, Theorem C]:

Theorem 7. *A generic compact rank-one ECS manifold is either translational, or locally homogeneous.*

[1] Most proof outlines are omitted due to space limitations.

In other words, a generic compact rank-one ECS manifold is dilational if and only if it is locally homogeneous. The locally homogeneous alternative itself, in turn, had to be ruled out with a different array of arguments, relying on particular features of the isometry group of a homogeneous standard ECS plane wave [19, Theorem E]:

Theorem 8. *All generic compact rank-one ECS manifolds are translational, complete, and not locally homogeneous.*

While Theorem 8 improves Theorem 7, it does not replace it: the latter is crucially used to prove the former.

We conclude with one important consequence of Theorem 8. If $(M, \mathtt{g})$ is a compact four-dimensional rank-one ECS manifold then, being generic, it must be translational and have a standard ECS plane wave as its universal covering. Replacing $(M, \mathtt{g})$ with a suitable isometric finite covering, we may assume that (4-i) becomes a homomorphism $\mathit{\Gamma} \ni \gamma \mapsto p \in \mathbb{R}$ (whose image is infinite cyclic), and that $\mathit{\Gamma}$ acts trivially on $(V, \langle\cdot,\cdot\rangle)$. It follows that $\mathit{\Gamma} \subseteq \mathrm{G}(\sigma)$ for some $\sigma \in \mathrm{S}$ of the form $\sigma = (1, p, \mathrm{Id}_V)$. With a direct adaptation of [15, Lemma 8.1], we may apply Theorem 2 to obtain our last result [19, Corollary F]:

Corollary 2. *There are no four-dimensional compact rank-one ECS manifolds.*

References

1. A. B. Aazami, R. Ream: *Almost Kähler metrics and pp-wave spacetimes.* Lett. Math. Phys. **112** (2022), no. 4, art. 84, 17 pp.
2. D. V. Alekseevsky, A. G. Galaev: *Two-symmetric Lorentzian manifolds.* J. Geom. Phys **61** (2011), 2331–2340.
3. B. Araneda: *Parallel spinors, pp-waves, and gravitational perturbations.* Classical Quant. Grav. **40** (2023), art. 025006, 21 pp.
4. A. L. Besse: *Einstein manifolds.* Classics in Mathematics. Springer-Verlag, Berlin, 2008. Reprint of the 1987 edition.
5. M. Cahen, Y. Kerbrat: *Transformations conformes des espaces symétriques pseudo-riemanniens.* Ann. Mat. Pura Appl. **132** (1982), no. 4, 275–289.
6. E. Calviño-Louzao, E. García-Río, J. Seoane-Bascoy, R. Vásquez-Lorenzo: *Three-dimensional conformally symmetric manifolds.* Ann. Mat. Pura Appl. **193** (2014), no. 4, 1661–1670.
7. E. Calviño-Louzao, E. García-Río, M. E. Vázquez-Abal, R. Vásquez-Lorenzo: *Geometric properties of generalized symmetric spaces.* Proc. Roy. Soc. Edinburgh Sec. A **145** (2015), 47–71.
8. I. P. Costa e Silva, J. L. Flores, J. Herrera: *Rigidity of geodesic completeness in the Brinkmann class of gravitational wave spacetimes.* Adv. Theor. Math. Phys. **22** (2018), no. 1, 25–45.
9. A. Derdzinski: *On conformally symmetric Ricci-recurrent manifolds with Abelian fundamental groups.* Tensor (N. S.) **34** (1980), no. 1, 21–29.
10. A. Derdzinski, W. Roter: *On conformally symmetric manifolds with metrics of indices* 0 *and* 1. Tensor (N. S.) **31** (1977), no. 3, 255–259.

11. A. Derdzinski, W. Roter: *Global properties of indefinite metrics with parallel Weyl tensor*. Pure and Applied Differential Geometry – PADGE 2007, eds. F. Dillen and I. Van de Woestyne, Berichte aus der Mathematik, Shaker Verlag, Aachen, 2007, 63–72.
12. A. Derdzinski, W. Roter: *Projectively flat surfaces, null parallel distributions, and conformally symmetric manifolds*. Tohoku Math. J., vol. **58** (2008), no. 4, 565–602. doi:10.2748/tmj/1199649875.
13. A. Derdzinski, W. Roter: *On compact manifolds admitting indefinite metrics with parallel Weyl tensor*. J. Geom. Phys. **58** (2008), no. 9, 1137–1147.
14. A. Derdzinski, W. Roter: *The local structure of conformally symmetric manifolds*. Bull. Belg. Math. Soc. Simon Stevin **16** (2009), no. 1, 117–128. doi:2010.36045/bbms/1235574196.
15. A. Derdzinski, W. Roter: *Compact pseudo-Riemannian manifolds with parallel Weyl tensor*. Ann. Glob. Anal. Geom. **37** (2010), no. 1, 73–90. doi:10.1007/s10455-009-9173-9
16. A. Derdzinski, I. Terek: *New examples of compact Weyl-parallel manifolds*. Monatsh. Math. **203** (2024), no. 4, 859–871. doi:10.1007/s00605-023-01908-0.
17. A. Derdzisnki, I. Terek: *The topology of compact rank-one ECS manifolds*. Proc. Edinb. Math. Soc. **66** (2023), no. 3, 789–809. doi:10.1017/S0013091523000408.
18. A. Derdzinski, I. Terek: *Rank-one ECS manifolds of dilational type*. Port. Math. **81** (2024), no. 1/2, 69–96. doi:10.4171/PM/2110.
19. A. Derdzinski, I. Terek: *The metric structure of compact rank-one ECS manifolds*. Ann. Glob. Anal. Geom. **64** (2023), no. 4, art. 24. doi:10.1007/s10455-023-09929-6.
20. A. Derdzinski, I. Terek: *Compact locally homogeneous manifolds with parallel Weyl tensor*. To appear in Adv. Geom. Preprint available at https://arxiv.org/abs/2306.01600
21. A. Derdzinski: *Compact Weyl-parallel manifolds*. Romanian Journal of Mathematics and Computer Science **13** (2023), no. 2, 1–5. Special Issue – Proceedings of the International Conference Riemannian Geometry and Applications – RIGA 2023.
22. A. Derdzinski, I. Terek: *Corrections of minor misstatementes in several papers on ECS manifolds*. e-print arXiv:2404.09766.
23. R. Deszcz, M. Hotloś: *On a certain subclass of pseudo-symmetric manifolds*. Publ. Math. Debrecen **53** (1998), 29–48.
24. R. Deszcz, M. Głogowska, M. Hotloś, M. Petrović-Torgašev, G. Zafindratafa: *A note on some generalized curvature tensor*. Int. Electron. J. Geom. **16** (2023), 379–397.
25. R. Deszcz, M. Głogowska, M. Hotloś, G. Zafindratafa: *On some curvature conditions of pseudosymmetry type*. Period. Math. Hungar. **70** (2015), 153–170.
26. J. Ehlers, W. Kundt: *Exact solutions of the gravitational field equations*. Gravitation: An introduction to current research, pp. 49–101. Wiley, New York, 1962.
27. J. L. Flores, M. Sánchez: *The Ehlers-Kundt conjecture about gravitational waves and dynamical systems*. J. Differ. Equations **268** (2020), no. 12, 7507–7534.
28. M. Hanounah, I. Kath, L. Mehidi, A. Zeghib: *Topology and dynamics of compact plane waves*, 2023. Preprint available at https://arxiv.org/pdf/2309.14955.pdf.
29. M. Hanounah, L. Mehidi, A. Zeghib: *On homogeneous plane waves*, 2023. Preprint available at https://arxiv.org/pdf/2311.07459.pdf.
30. M. Lejmi, X. S. Shen: *Canonical almost-Káhler metrics dual to general plane-fronted wave Lorentzian metrics*. Math. Z. **303**, no. 4, art. 94, 15 pp.
31. M. Hotloś: *On conformally symmetric warped products*. Ann. Acad. Pedagog. Crac. Stud. Math. **4** (2004), 75–85.

32. T. Leistner, D. Schliebner: *Completeness of compact Lorentz manifolds with abelian holonomy*. Math. Ann. **364** (2016), no. 3–4, 1469–1503. doi:10.1007/s00208-015-1270-4.
33. C. A. Mantica, Y. J. Suh: *Conformally symmetric manifolds and quasi conformally recurrent Riemannian manifolds*. Balkan J. Geom. Appl. **16** (2011), no. 1, 66–77.
34. L. Mehidi, A. Zeghib: *On completeness and dynamics of compact Brinkmann spacetimes*, 2022. Preprint available at https://arxiv.org/pdf/2205.07243.pdf.
35. Z. Olszak: *On conformally recurrent manifolds I: Special distributions*. Zesz. Nauk. Politech. Śl., Mat.-Fiz. **68** (1993), 213–225.
36. C. Roche, A. B. Aazami, C. Cederbaum: *Exact parallel waves in general relativity*. Gen. Relativ. Gravit. **55** (2023), no. 2, art. 40, 59 pp.
37. W. Roter: *On conformally symmetric Ricci-recurrent spaces*. Colloq. Math. **31** (1974), 87–96.
38. D. Schliebner: *On the Full Holonomy of Lorentzian Manifolds with Parallel Weyl Tensor*, 2012. Preprint available at https://arxiv.org/pdf/1204.5907.pdf.
39. Y. J. Suh, J.-H. Kwon, H. Y. Yang: *Conformally symmetric semi-Riemannian manifolds*. J. Geom. Phys. **56** (2006), 875–901.
40. I. Terek: *Conformal flatness of compact three-dimensional Cotton-parallel manifolds*. Proc. Amer. Math. Soc. **152** (2024), no. 2, 797–800.