MATH2177 RECITATION DIARY

This is a diary for the recitation classes offered for the **MATH2177 - Mathematical Topics for Engineers** course offered at OSU, on the Spring 2022 term. No additional material or exercises will be added here, as it is meant to be a reasonably faithful reflection of what happens in class (although I cannot promise I won't add a remark or footnote here or there, elaborating further on things I particularly find interesting). Since it is unlikely that all three recitations will be 100% equal all the time, you may occasionally see an exercise that was not discussed in class (but this only means that it may have been discussed in one of the other sections). Most of the exercises and problems discussed will be taken from reference [1], as expected. For corrections or inquiries, please write to terekcouto.1@osu.edu.

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1 January 11th

In this class we will deal with *several variables* (where "several" will usually be two or three), such as

$$f(x,y) = x^2 + y, \quad f(x,y,z) = xe^y + \sin z, \quad f(x,y,z,w) = e^{xy}z^2 \cos w,$$

as opposed to the functions $f(x)$ of a single variable studied in previous classes. The main program then was to find the critical points of a given function and study their *local nature* (that is, whether they're local maxima, local minima, or saddle/inflection points). The main tools to do so, in turn, were derivatives. For functions of several variables, we have *partial derivatives*

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial z}, \quad \frac{\partial f}{\partial w}, \ldots$$

These are computed from the usual rules from single-variable calculus (product rule, quotient rule, chain rule), by freezing (hence treating as constants) all the variables except for the one on which the differentiation process is acting.

The mantra repeated throughout the first semester of calculus was "the derivative is the slope of the tangent line". If we're dealing with a function of, say, two variables instead, the picture goes like this: the graph of f is a surface on three-dimensional space (henceforth denoted by $\mathbb{R}^3$), and the numerical value $f(x,y)$ is understood as the height of a point whose first two coordinates are x and y. Fix a vector v on the plane (to be called $\mathbb{R}^2$ from now on), starting at the point (x,y), and draw a vertical plane Π passing through the point (x,y) and containing the direction determined by v. The plane Π intersects the surface $z = f(x,y)$ along a curve, completely contained in Π, which may be thought of the graph of a function (more precisely, given by the relation $g(t) = f((x,y) + tv)$). The derivative $g'(0)$ is the slope of the tangent line to the intersection curve, in the plane Π, and it is called the *directional derivative of f in the direction v, at the point (x,y)*, and it is denoted by $(\partial f/\partial v)(x,y)$.

Instead of thinking about directional derivatives along infinitely many directions, we recall the relation

$$\frac{\partial f}{\partial v}(x,y) = \nabla f(x,y) \cdot v,$$

where $\cdot$ stands for the *dot product* of vectors, and $\nabla f(x,y)$ is the gradient of f at the point (x,y). More generally:

Definition 1

Let f be a differentiable function of the variables $x_1, \ldots, x_n$. The **gradient** of f is the vector (field) ∇f defined by

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right).$$

The gradient is a mathematical device to gather all information about the first-order derivatives of a function, into a single object: namely, a vector. Knowledge of the

gradient is indeed knowledge about all directional derivatives of the function, due to the dot product relation (which remains true in all dimensions).

Recall that, in single-variable calculus, a number a is a critical point of a function f if $f'(a) = 0$. Picturing the graph of f, this condition is to be expected, as the tangent line to the graph of f at the candidates to local max/min is horizontal, the derivative is the slope of the tangent line, and the slope of a horizontal line is zero.

With this in place:

Definition 2

Let f be a differentiable function of the variables $x_1, \ldots, x_n$. Then $(a_1, \ldots, a_n)$ is a **critical point** of f if $\nabla f(a_1, \ldots, a_n) = (0, \ldots, 0)$.

In other words, a critical point is one for which the gradient vector is the zero vector. In two variables, a point (a, b) is a critical point of f if $\nabla f(a, b) = (0, 0)$. Meaning that we replace the single equation $f'(a) = 0$ with the more elaborate system $\nabla f(a_1, \ldots, a_n) = (0, \ldots, 0)$ (one equation for each variable). Or yet, such system is obtained by setting all the partial derivatives of f equal to zero (which by the dot product relation is the same as requiring all directional derivatives to be zero).

Example 1

Find the critical points of the function $f(x, y) = 1 + x^2 + y^2$.

The graph of this function is called a **paraboloid**, and it is obtained by rotating the parabola $y = 1 + x^2$ around the z-axis. Geometrically, it should be clear that there is only one critical point, $(0, 0)$, and that is is a global minimum for the function. Sometimes it is not so simple to get the answer so quickly by using geometric intuition. Doing it directly, we compute that $\nabla f(x, y) = (2x, 2y)$, and this equals $(0, 0)$ if and only if $(x, y) = (0, 0)$. Observe that the $(0, 0)$'s indicated in different colors play different roles!

Example 2

Find the critical points of the function $f(x, y) = (3x - 2)^2 + (y - 4)^2$.

The graph of this function is very similar to the one in the previous example. This can be made clearer by letting $u = 3x - 2$ and $v = y - 4$, so that (abusing notation) $f(u, v) = u^2 + v^2$. This means that the graph of the function considered here is a paraboloid, up to this change of variables (which amounts to an offset and a stretching of the x-axis). Geometrically, we see that the critical point is described by $u = v = 0$, which means that $x = 2/3$ and $y = 4$. Thus there is only one critical point $(2/3, 4)$, which is a global minimum for the function. Changing variables like this will be useful not only to try and gain geometric intuition for situations like this (where what you have in front of you is similar to something you have already seen before, but not quite equal to it), but it will also be a very important tool when dealing with change of variables for computing double integrals.

Doing it directly, however, we have that $\nabla f(x,y) = (6(3x-2), 2(y-4))$, and this equals $(0,0)$ if and only if $(x,y) = (2/3, 4)$. Compare the choice of colors here with the one made in the previous example.

Example 3

Functions may not have critical points at all! Consider $f(x,y,z) = g(x,y) + z$, where $g(x,y)$ is the most horrible expression you can come up with. Then we have that $\nabla f = (*, *, 1)$. As the last component of ∇f is never zero, then ∇f can never be the zero vector, and so f has no critical points.

Once critical points have been found, the next task is to decide their nature, i.e., whether they are local max/min or saddle/inflection points. In the situation where we had a critical point a of a function f of a single-variable, we knew that:

- If $f''(a) > 0$, then a is a local minimum ($\cup$)
- If $f''(a) < 0$, then a is a local maximum ($\cap$)
- If $f''(a) = 0$, the test is inconclusive[1].

Let's focus on the case where f is a function of two variables instead. The first derivative became the gradient vector. The second derivative must correspond to what comes after a vector: a matrix.

Definition 3

Let f be a twice-differentiable function of the variables x and y. The **Hessian matrix** of f is defined by

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix},$$

where we use the shorthands

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{xy} = f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}.$$

We also let $D = f_{xx}f_{yy} - f_{xy}^2$ be the **determinant** of H.

[1] A very common mistake is to think that $f''(a) = 0$ means that a is a saddle/inflection point. This is not true. Case in point: $f(x) = x^4$ and $a = 0$ ($f(0) = f'(0) = f''(0) = f'''(0) = 0$ but $f^{(4)}(0) > 0$ and a is a global minimum). If $f''(a) = 0$ but $f'''(a) \neq 0$, then a is a saddle point. If $f'''(a) = 0$, then we must look at the fourth derivative $f^{(4)}(a)$: if positive, then a is a local minimum, and if negative, a is a local maximum. If $f^{(4)}(a) = 0$, we must look at the fifth derivative. If $f^{(5)}(a) \neq 0$, then a is a saddle point. If $f^{(5)}(a) = 0$, we must look at the sixth derivative $f^{(6)}(a)$: if positive, then a is a local minimum, and if negative, a is a local maximum. This procedure repeats, alternating between even-order derivatives and odd-order derivatives, until a conclusion is obtained. The "proof" of this fact relies on the Taylor polynomial of f. More about this on office hours if anyone is interested.

Observe that the expression given for D already takes into account the equality $f_{xy} = f_{yx}$, which in words says that taking the derivative with respect to x first, and then y, produces the same result as doing things in the reverse order, namely, differentiating with respect to y first, and then x. This is sometimes called the Clairaut-Schwarz theorem[2]. Another observation which may be very useful for computing Hessians quickly is that, writing $\nabla f = (f_x, f_y)$, the first row of H is the gradient of the first component f_x, while the second row of H is the gradient of the second component f_y.

With this in place, we have the second derivative test for functions of two variables:

Theorem 1

Let f be a twice-differentiable function of the variables x and y, and let (a, b) be a critical point of f. Then:

- If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then (a,b) is a local minimum.
- If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then (a,b) is a local minimum.
- If $D(a,b) < 0$, then (a,b) is a saddle point.
- If $D(a,b) = 0$, the test is inconclusive.

Remark.

- The second item may have been stated with the condition $f_{yy}(a,b) < 0$ instead of $f_{xx}(a,b) < 0$. Both are correct; their equivalence is seen by applying one version of the criterion to the function $f(y,x)$ with the order of variables switched, noting that the off-diagonal elements of H, as well as D, remain both unchanged.
- There is a version of the second derivative test for functions with more than two variables. The number of conditions to be considered increases with the number of variables. Justifying it requires more Linear Algebra than what we have available now. The keyword is "Sylvester's Criterion" (for positivity of matrices).

Example 4

Find the critical points of the function $f(x,y) = 4 + 2x^2 + 3y^2$ and classify them.

Before anything else, convince yourself with geometric intutition (based on previous examples) that $(0,0)$ is the only critical point, and that it is a global minimum. Now let's propely solve it. First, note that $\nabla f(x,y) = (4x, 6y)$, and this equals $(0,0)$ if and only if $(x,y) = (0,0)$. Next, we have that

$$H(x,y) = \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix} \implies H(0,0) = \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}.$$

[2]Whose formal assumption is that f is twice-differentiable and all the second order derivatives are continuous (or, more generally, defined on all points of an open set).

Hence $D(0,0) = 24 > 0$ and $f_{xx}(0,0) = 4 > 0$ together say that $(0,0)$ is a local minimum of f. A local minimum, when the unique critical point, is automatically a global minimum.

It is important to note that the implication arrow indicated in red, while it might seem like a completely unnecessary step, is crucial: dealing with more complicated functions, it is perfectly possible (and expected!) that x and y will survive in the expression for $H(x,y)$. In other words, $H(x,y)$ should not be expected to be a constant matrix. The Hessian matrix only gives you relevant information when evaluated at a critical point. When evaluated at other random points, it gives you no relevant information whatsoever. **Plugging $(x,y) = (0,0)$ is an important step which should not be forgotten or taken for granted.**

Example 5

Find the critical points of the function $f(x,y) = -4x^2 + 8y^2 - 3$ and classify them.

Compute $\nabla f(x,y) = (-8x, 16y)$. Setting $\nabla f(x,y) = (0,0)$ immediately gives that $(x,y) = (0,0)$, and this is the unique critical point. Then we have that

$$H(x,y) = \begin{pmatrix} -8 & 0 \\ 0 & 16 \end{pmatrix} \Longrightarrow H(0,0) = \begin{pmatrix} -8 & 0 \\ 0 & 16 \end{pmatrix},$$

so $D(0,0) < 0$ says that $(0,0)$ is a saddle point.

2 January 18th

We continue with the discussion on maxima and minima of functions of two variables. Most of the time, the region R to be considered will be closed and bounded (those are called "compact", in short), and so may be thought of as having two regions: an interior $\text{int}(R)$, and a boundary ∂R. **Candidates** to global max/min in $\text{int}(R)$ are nothing more than the critical points of f which turn out to land there. Observe that the second derivative test is only good for determining the local behavior of critical points, not the global one. So, in the end of the day, one must still list all candidates and compute the value of the function on each of them.

The question still remains of how to study the boundary ∂R. Very frequently, it may be described as an equation of the form $g(x,y) = 0$, where g is nice enough. In this case, we have a specific technique to use:

Theorem 2 (Lagrange multipliers)

Let f and g be differentiable functions of the variables $x_1, \ldots, x_n$, with continuous partial derivatives. Assume that $\nabla g(x_1, \ldots, x_n)$ is not the zero vector whenever $g(x_1, \ldots, x_n) = 0$. If $(a_1, \ldots, a_n)$ is the global maximum or minimum of f restricted to the set $\{(x_1, \ldots, x_n) \mid g(x_1, \ldots, x_n) = 0\}$, there is a real number λ such that

$$\nabla f(a_1, \ldots, a_n) = \lambda \nabla g(a_1, \ldots, a_n),$$

i.e., the gradients of f and g are proportional in such point.

Back to the problem at hand, the strategy will be to solve the system

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y), \\ g(x,y) = 0. \end{cases}$$

The condition $g(x,y) = 0$ there is a crucial part of the problem, and it is always used in practice. Without it, one could find points *not on the given curve* for which the gradients of f and g are proportional, but we do not care about those. On the other hand, knowing the value of the Lagrange multiplier λ is not crucial. Sometimes it is convenient to solve for it as an intermediate step in finding x and y. If one manages to find all the candidates to max/min without solving for λ, there is nothing wrong.

Let's see examples.

Example 6

Find the global maximum and global minimum values of the function

$$f(x,y) = x^2 + y^2 - 2y + 1$$

on the region $R = \{(x,y) \mid x^2 + y^2 \leq 4\}$.

We'll always organize our work in two parts. Observe that R describes a closed disk with center in $(0,0)$ and radius 2.

(a) Candidates on $\operatorname{int}(R)$: the interior consists of the points (x, y) satisfying the relation $x^2 + y^2 < 4$. By replacing the inequality $\leq$ with the strict inequality $<$, we are forgetting about the boundary circle and considering only the open disk bounded by it. We compute the gradient of f as $\nabla f(x, y) = (2x, 2y - 2)$. Setting this equal to $(0, 0)$, we obtain $(x, y) = (0, 1)$. **Is the point $(0, 1)$ in the interior of R?** Yes. The reason why we ask ourselves this is because f is defined everywhere (in particular, its domain is larger than R), so a priori it could be that critical points found here lie outside R. If this were to happen, such critical points would have to be ignored.

(b) Candidates on ∂R: we use Lagrange multipliers, letting $g(x, y) = x^2 + y^2 - 4$. In this case, $\nabla g(x, y) = (2x, 2y)$ cannot be the zero vector whenever we have $x^2 + y^2 = 4$, which means that the assumptions for using Lagrange multipliers are satisfied. Thus, we have that

$$\begin{cases} \nabla f(x, y) = \lambda g(x, y) \\ g(x, y) = 0 \end{cases} \implies \begin{cases} 2x = 2\lambda x \\ 2y - 2 = 2\lambda y \\ x^2 + y^2 = 4 \end{cases}$$

Consider the first equation $2x = 2\lambda x$. We would like to cancel x on both sides, but this step cannot be made if $x = 0$. Thus, we have cases to analyze.

- Case 1: if $x \neq 0$. Then $\lambda = 1$, and substituting this onto the second equation gives $2y - 2 = 2y$, and thus $-2 = 0$. This is clearly nonsense, which says that Case 1 does not happen, and so we get no candidates here.
- Case 2: if $x = 0$. In this case, we only have to solve for y, and then the third equation reads $0^2 + y^2 = 4$, so that $y = 2$ or $y = -2$. This case gave us two candidates, $(0, 2)$ and $(0, -2)$.

Conclusion:

Candidates	Values
$(0, 1)$	0
$(0, 2)$	1
$(0, -2)$	9

Hence the global maximum is 9, realized at $(0, -2)$, while the global minimum is 0, realized at $(0, 1)$.

Remark. Sometimes, we can avoid Lagrange multipliers. For example, to analyze the boundary in the previous problem, we could consider $\mathbf{r}(t) = (2\cos t, 2\sin t)$, defined

for all t (in fact, $0 \leq t \leq 2\pi$ covers the circle). Then $f(\mathbf{r}(t)) = 5 - 4\sin t$. The maximum value of this expression is 9, for $t = 3\pi/2$, and we have $\mathbf{r}(3\pi/2) = (0, -2)$. The minimum along the curve is 1, for $t = \pi/2$, with $\mathbf{r}(\pi/2) = (0, 2)$, but this is not the global minimum of f (because the critical point in $\text{int}(R)$ gives a lower value).

Example 7

Find the global maximum and global minimum values of the function

$$f(x, y) = 2x^2 - 4x + 3y^2 + 2$$

on the region $R = \{(x, y) \mid (x-1)^2 + y^2 \leq 1\}$.

Note that R is a closed disk with center at $(1, 0)$ and radius 2.

(a) Candidates on $\text{int}(R)$: the interior consists of the points (x, y) satisfying the relation $(x-1)^2 + y^2 < 1$. The gradient of f is $\nabla f(x, y) = (4x - 4, 6y)$. Setting this equal to $(0, 0)$, we obtain $(x, y) = (1, 0)$. **Is the point** $(1, 0)$ **in the interior of** R**?** Yes (it's the center of the disk).

(b) Candidates on ∂R: we'll use Lagrange multipliers again, this time letting $g(x, y) = (x-1)^2 + y^2 - 1$. In this case, $\nabla g(x, y) = (2(x-1), 2y)$ cannot be the zero vector whenever we have $(x-1)^2 + y^2 = 1$, which means that the assumptions for using Lagrange multipliers are satisfied. Thus, we have that

$$\begin{cases} \nabla f(x, y) = \lambda g(x, y) \\ g(x, y) = 0 \end{cases} \implies \begin{cases} 4x - 4 = 2\lambda(x-1) \\ 6y = 2\lambda y \\ (x-1)^2 + y^2 = 1 \end{cases}$$

Consider the second equation $6y = 2\lambda y$. We would like to cancel y on both sides, but this step cannot be made if $y = 0$. Thus, we have cases to analyze.

- Case 1: if $y \neq 0$. Then $\lambda = 3$, and substituting this onto the first equation gives $4x - 4 = 6(x-1)$, from which we obtain $x = 1$. With this, the third equation reads $(1-1)^2 + y^2 = 1$, so $y = 1$ or $y = -1$. This case gave us two candidates $(1, 1)$ and $(1, -1)$.
- Case 2: if $y = 0$. In this case, we only have to solve for x, and then the third equation reads $(x-1)^2 + 0^2 = 1$, so that $x - 1 = \pm 1$ and hence $x = 0$ or $x = 2$. This case gave us two candidates, $(2, 0)$ and $(-2, 0)$.

Let's organize everything we have obtained so far in a table again:

Candidates	Values
$(1,0)$	0
$(2,0)$	2
$(0,0)$	2
$(1,1)$	3
$(1,-1)$	3

Hence, the global maximum is 3, realized at $(1,1)$ and $(1,-1)$, while the global minimum is 0, realized at $(1,0)$. It is ok that the global maximum was reached in more than one point (think of the extreme example where f is constant).

Example 8

Find the global maximum and global minimum values of the function

$$f(x,y) = x^2 + y^2 - 2x - 2y$$

on the region R bounded by the triangle of vertices at $(0,0)$, $(2,0)$ and $(0,2)$.

The boundary ∂R has three sides:

$$\begin{aligned} A &= \{(0,y) \mid 0 \leq y \leq 2\} \\ B &= \{(x,0) \mid 0 \leq x \leq 2\} \\ C &= \{(x,y) \mid y = -x+2 \text{ and } 0 \leq x \leq 2\} \end{aligned}$$

We proceed with our analysis as before:

(a) Candidates on $\operatorname{int}(R)$: this time we have $\nabla f(x,y) = (2x-2, 2y-2)$, so setting this equal to $(0,0)$ gives $(x,y) = (1,1)$. **Is the point $(1,1)$ in the interior of R?** No. In fact, $(1,1)$ lies on side C, and so it is not considered a candidate as far as $\operatorname{int}(R)$ is concerned.

(b) Candidates on ∂R: we'll study each of the sides A, B and C separately.

- Side A: evaluating f alongside Side A, we're led to consider the composition $g(y) = f(0,y) = y^2 - 2y$, defined on the interval $[0,2]$. The candidates here will be the points corresponding to the endpoints of the interval $[0,2]$, and critical points of g inside the open interval $]0,2[$. The

graph of g is a parabola which is concave up, and so $g'(y) = 2y - 2$ leads to $y = 1$. Side A thus gives us the candidates $(0,0)$, $(0,2)$ and $(0,1)$.

- Side B: the function f satisfies the symmetry $f(x,y) = f(y,x)$, and Side B is obtained from Side A by replacing the roles of x and y. We obtain, without calculations, the candidates $(0,0)$ (repeated), $(2,0)$ and $(1,0)$.
- Side C: evaluating f alongside Side C, we're led to consider the composition $h(x) = f(x, -x+2) = 2x(x-2)$ (after simplifying it), defined on $[0,2]$. As before the endpoints of $[0,2]$ gives us the candidates $(2,0)$ and $(0,2)$ (both repeated), and $h'(x) = 0$ yields $x = 1$, meaning that $(1,1)$ is the last candidate (previously ignored on the discussion about $\text{int}(R)$, but now included for the right reasons).

Summarizing it:

Candidates	Values
$(0,0)$	0
$(2,0)$	0
$(0,2)$	0
$(0,1)$	-1
$(1,0)$	-1
$(1,1)$	-2

Hence, the global maximum is 0, realized at $(0,0)$, $(2,0)$ and $(0,2)$, while the global minimum is -2, realized at $(1,1)$.

We also observe that defining $L(x,y,\lambda) = f(x,y) - \lambda g(x,y)$ (here, we think of "applying g as a penalty to f"; L is called the Lagrangian function for the optimization problem), the condition $\nabla L(x,y,\lambda) = (0,0,0)$ is equivalent to the system

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y), \\ g(x,y) = 0. \end{cases}$$

we have been working with so far. This is just an alternative way to think about it (which may help you not forget how important $g(x,y) = 0$ is). However, **do not try to apply the second derivative test to** L as a shortcut to decide whether the candidates you have found are global maxima or minima (again, the second derivative test regards only the local nature of critical points on the interior of the considered region).

3 January 25th

First, let's go over the most challenging problem on HW1.

Exercise

Find the critical points of the function $f(x,y) = e^{x^2y^2-2xy^2+y^2}$.

We compute the gradient of f as

$$\nabla f(x,y) = \left((2xy^2-2y^2)e^{x^2y^2-2xy^2+y^2}, (2x^2y-4xy+2y)e^{x^2y^2-2xy^2+y^2}\right).$$

Since $e^{x^2y^2-2xy^2+y^2} \neq 0$ no matter the values of x and y, the critical point condition $\nabla f(x,y) = (0,0)$ is equivalent to the system

$$\begin{cases} 2xy^2-2y^2 = 0 \\ 2x^2y-4xy+2y = 0. \end{cases}$$

The first equation can be factored as $2(x-1)y^2 = 0$. Again, we know that if a product of two factors equals zero, then *at least* one of the factors must be zero, but we don't know which one. Therefore we must analyze cases.

- Case 1: $x-1=0$. Then $x=1$. Substituting this into the second equation of the system gives $2y-4y+2y=0$, which is always satisfied. This means that for each real number y, the point $(1,y)$ is a critical point of f.
- Case 2: $y=0$. Substituting this into the second equation gives $0-0+0=0$, which is always satisfied. This means that for each real number x, the point $(x,0)$ is a critical point of f.

Conclusion: the set of critical points of f is the union of the lines in the plane whose equations are $x=1$ and $y=0$.

Now, we move on to double integrals. The main difficulty here is to deal with bounds of integration when setting up iterated integrals.

Example 9

Compute $\iint_R x\sec^2(xy)\,dA$, *where* $R = \{(x,y) \mid 0 \leq x \leq \pi/3 \text{ and } 0 \leq y \leq 1\}$.

Here, dA stands for the **infinitesimal area element**, which in rectangular coordinates is just given by $dA = dx\,dy$. The region R is simply a rectangle, as the bounds for x and y are all constants. Do not be mislead to think that R is a sector of a circle just because there's π there: R is not being described in polar coordinates,

there's no θ anywhere. One could set up the iterated integrals as

$$\int_0^1 \int_0^{\pi/3} x\sec^2(xy)\,\mathrm{d}x\,\mathrm{d}y \quad \text{or} \quad \int_0^{\pi/3}\int_0^1 x\sec^2(xy)\,\mathrm{d}y\,\mathrm{d}x.$$

Fubini's Theorem says that the order you choose does not matter, you will obtain the same result regardless of the choice made. Now, it could very well happen that one choice of order leads to a much easier computation than the other. In this case, the first option would require an unpleasant integration by parts, while the second one requires a simple u-substitution. Making $u = xy$, so $\mathrm{d}u = x\,\mathrm{d}y$ (as x is a constant from the perspective of y), we have that

$$\int x\sec^2(xy)\,\mathrm{d}y = \int \sec^2 u\,\mathrm{d}u = \tan u = \tan(xy).$$

We don't bother with the constant of integration here because we're dealing with definite integrals, so it would dissapear anyway. Thus

$$\begin{aligned}\int_0^{\pi/3}\int_0^1 x\sec^2(xy)\,\mathrm{d}y\,\mathrm{d}x &= \int_0^{\pi/3} \tan(xy)\Big|_{y=0}^{y=1}\mathrm{d}x\\ &= \int_0^{\pi/3}\tan x\,\mathrm{d}x\\ &= \ln|\sec x|\Big|_0^{\pi/3}\\ &= \ln 2 - \ln 0\\ &= \ln 2.\end{aligned}$$

Next, let's look at more general regions which are not rectangles.

Example 10

Set up iterated integrals for a generic continuous function $f(x,y)$ over the region R given in the picture.

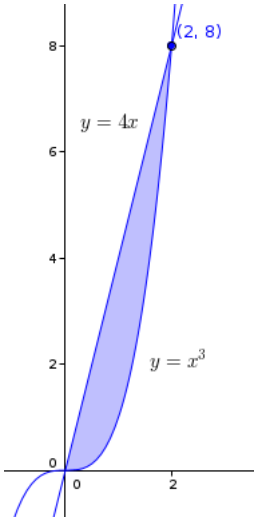
As usual, there are two orders we can set up.

- $\mathrm{d}x\,\mathrm{d}y$. Fix one value of the outermost variable, y. What is the range for the other variable x, as the horizontal line passing through y cuts the given region, from left to right? Here, we must express x as a function of y, so there's some small work to be done. The lower bound for x is $x = y/4$, coming from the line equation, and the upper bound is $x = \sqrt[3]{y}$, from the cubic equation. The variable y, in turn, goes from 0 to 8. Thus

$$\iint_R f(x,y)\,\mathrm{d}A = \int_0^8 \int_{y/4}^{\sqrt[3]{y}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

- $\mathrm{d}y\,\mathrm{d}x$. Fix one value of the outermost variable, x. What is the range for the other variable y, as the vertical line passing through x cuts the given region, upwards? Here, we must express y as a function of x, so there no work to be done. The lower bound for y is x^3, coming from the cubic equation, and the upper bound is $4x$, from the line equation. The variable x, in turn, goes from 0 to 2. Thus

$$\iint_R f(x,y)\,\mathrm{d}A = \int_0^2 \int_{x^3}^{4x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$



Here's another one:

Example 11

Set up iterated integrals for a generic continuous function $f(x,y)$ over the region R given in the picture.

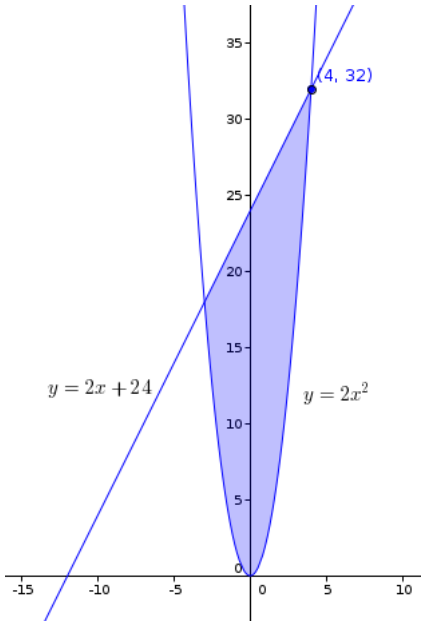
One more time, there are two orders we can set up. The exercise did not give us the coordinates for the left intersection point between the graphs, but we need it to know the full bounds for x. To find it, we consider $2x^2 = 2x + 24$, which is readily simplified to $x^2 - x - 12 = 0$. We already know that one of the solutions is $x = 4$. Due to the coefficient 12, the other one is -3 or 3, but it clearly cannot be the latter. Hence, the coordinates of the remaining intersection point are $(-3, 18)$ (where 18 is obtained by plugging $x = -3$ into either $y = 2x^2$ or $y = 2x + 24$). Now, let's study what happens with both orders of integration:

- $\mathrm{d}y\,\mathrm{d}x$: Fix one value of the outermost variable, x. What is the range for the other variable y, as the vertical line passing through x cuts the given region, upwards? Here, we must express y as a function of x, so there no work to be done. The lower bound for y is $2x^2$, coming from the quadratic equation, and the upper bound is $2x + 24$, from the line equation. The variable x, in

turn, goes from -3 to 4. Thus

$$\iint_R f(x,y)\,\mathrm{d}A = \int_{-3}^{4}\int_{2x^2}^{2x+24} f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

- $\mathrm{d}x\,\mathrm{d}y$: This time, we're forced to break the region into two pieces, as once a value for the outermost variable y is fixed, the lower bound for x cannot be written as a single formula as a function of y, due to the "break" at the point $(-3, 18)$. We know that if $R = R_1 \cup R_2$ with $R_1 \cap R_2 = \varnothing$, then the double integral of f over R equals the sum[a] of the double integrals over R_1 and R_2. Let's say that R_1 is the part of R which lies inside the strip $0 \leq y \leq 18$, and R_2 is the one inside the strip $18 < y \leq 32$.



Then we have that

$$\begin{aligned}\iint_R f(x,y)\,\mathrm{d}A &= \iint_{R_1} f(x,y)\,\mathrm{d}A + \iint_{R_2} f(x,y)\,\mathrm{d}A\\ &= \int_0^{18}\int_{-\sqrt{y/2}}^{\sqrt{y/2}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_{18}^{32}\int_{(y-24)/2}^{\sqrt{y/2}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Namely, the "upper bound" for x, once y is fixed, is always $\sqrt{y/2}$, but the lower bound depends on whether $0 \leq y \leq 18$ or $18 < y \leq 32$: in the former case, it is $-\sqrt{y/2}$, and in the latter case it is $(y-24)/2$ (obtained from solving for x in terms of y in $y = 2x + 24$).

[a] This is a two-variable version of the general rule $\int_a^c f(t)\,\mathrm{d}t + \int_c^b f(t)\,\mathrm{d}t = \int_a^b f(t)\,\mathrm{d}t$ for single-variable integrals.

By now, you should be convinced that a convenient choice of order of integration

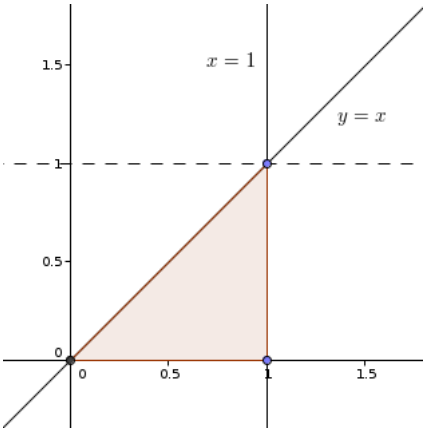
is crucial to making things simpler (getting the feeling for which choice is best takes some practice and experience). There are situations, however, where one choice simply makes the problem impossible, and we're forced to switch the order.

Example 12

Compute $\displaystyle\int_0^1 \int_y^1 e^{x^2}\,dx\,dy$.

The function $f(x) = e^{x^2}$ has no elementary anti-derivative, in the sense that its indefinite integral cannot be expressed in terms of well-known functions (such as polynomials, rational functions, exponentials, logarithms, and trigonometric functions). Knowing whether a given function of a single-variable has an elementary anti-derivative or not is not a simple task (keywords: Risch's Algorithm, and Differential Galois Theory). We will not concern ourselves with this. The extra tool we have in this case, is precisely to change the order of integration.

To draw the region of integration, one general strategy is: first recognize that the outermost bounds for y are 0 and 1, so whatever we draw will be inside the region where $0 \leq y \leq 1$. As for the innermost bounds, draw the curves described by the bounds, $x = y$ and $x = 1$. Namely, they're the usual diagonal, and a vertical line.



If the innermost original bounds were from 0 to y, the region of integration would be the upper triangle, as opposed to the lower one (as the picture indicates). Now:

$$\begin{aligned}\int_0^1 \int_y^1 e^{x^2}\,dx\,dy &= \int_0^1 \int_0^x e^{x^2}\,dy\,dx \\ &= \int_0^1 x e^{x^2}\,dx \\ &= \frac{e^{x^2}}{2}\bigg|_0^1 = \frac{e-1}{2}.\end{aligned}$$

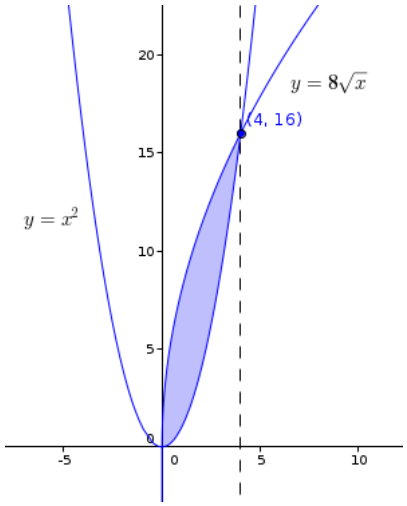
The x factor produced by realizing the integral with respect to y first saves the day.

Being able to sketch regions given algebraically is an important skill. Here's more practice:

Example 13

Sketch $R = \{(x,y) \mid 0 \leq x \leq 4$ *and* $x^2 \leq y \leq 8\sqrt{x}\}$ *and set the iterated integral of a generic continuous function* $f(x,y)$ *over* R *in the order* $\mathrm{d}y\,\mathrm{d}x$.

We immediately know that whatever we draw will remain inside the vertical strip $0 \leq x \leq 4$. As for $x^2 \leq y \leq 8\sqrt{x}$, forget for one moment that we're dealing with inequalities, and draw the bounds $y = x^2$ and $y = 8\sqrt{x}$ instead. Recall that the graph of $\sqrt{x}$ is obtained by reflecting the graph of x^2 about the diagonal line $y = x$, and that 8 is just a vertical stretching factor (made by design to make $(4,16)$ the rightmost intersection of the two curves).



Fixed x, the lower bound for y is x^2 and the upper bound is $8\sqrt{x}$. As x itself ranges from 0 to 4, we simply have that

$$\iint_R f(x,y)\,\mathrm{d}A = \int_0^4 \int_{x^2}^{8\sqrt{x}} f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

While sketching the regions of integration is always helpful, it is also possible to switch the order of integration without making any pictures (although I do not recommend doing so).

Example 14

Reverse the order of integration in $\displaystyle\int_0^1 \int_1^{\mathrm{e}^y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$.

Let's do it algebraically, starting from $0 \leq y \leq 1$ and $1 \leq x \leq \mathrm{e}^y$, and rewriting

them in an equivalent form which makes the bounds for x constant (as the region of integration is not a rectangle, we will pay the price: the bounds for y will be functions of x instead of constants as well).

Since $y \leq 1$, we have $e^y \leq e^1 = e$. So, we already have $1 \leq x \leq e$. As for the bounds for y, applying ln (which is an increasing function and thus preserves inequalities) to $x \leq e^y$, we obtain $\ln x \leq y \leq 1$ (the latter inequality given initially). Conclusion:

$$\int_0^1 \int_1^{e^y} f(x,y)\,dx\,dy = \int_1^e \int_{\ln x}^1 f(x,y)\,dy\,dx.$$

4 February 1st

Let's start solving the most challenging problem from HW2, about Lagrange multipliers.

Exercise

Find the maximum and minimum values of $f(x,y) = x^2 + y^2$ along the curve described by the equation $2x^2 + 3xy + y^2 = 7$.

Observe that the given constraint curve is an ellipse centered at the origin and that f is just the "squared distance to the origin". Thus, from geometric considerations, we already know that we'll obtain two points realizing the global maximum and two points realizing the global minimum (namely, the "vertices" of the ellipse. All the work to follow boils down to finding the coordinates of such points. Also, the problem asks us to optimize f along the ellipse, so what happens on the "interior" (i.e., on inside the region bounded by the ellipse) does not matter.

Let $g(x,y) = 2x^2 + 3xy + y^2 - 7$, and compute $\nabla g(x,y) = (4x + 3y, 3x + 4y)$. This equals the zero vector only when $x = y = 0$, but $(0,0)$ does not lie on the ellipse. So, we're allowed to proceed with Lagrange multipliers. We set up the Lagrange system:

$$\begin{cases} \nabla f(x,y) = \lambda g(x,y) \\ g(x,y) = 0 \end{cases} \implies \begin{cases} 2x = \lambda(4x + 3y) \\ 2y = \lambda(3x + 4y) \\ 2x^2 + 3xy + 2y^2 = 7 \end{cases}$$

There are several ways to attack this system, and no single correct way to do it. We'll exploit a certain symmetry and let nature do the rest of the job. Multiply the first equation by y and the second one by x (so the both of the new left sides become $2xy$), to conclude that $\lambda y(4x + 3y) = \lambda x(3x + 4y)$. The next thing we want to do is to simplify λ on both sides, but this can only be done if $\lambda \neq 0$. If we had $\lambda = 0$, the first two original equations give $x = y = 0$, but again $(0,0)$ does not lie on the ellipse, so that this case is impossible. With the knowledge that $\lambda \neq 0$, it now follows that (after distributing) $4xy + 3y^2 = 3x^2 + 4xy$. Cancelling $4xy$, we have that $3y^2 = 3x^2$, so $y = \pm x$.

- Case 1: $y = x$. In the third equation, we have $7x^2 = 7$, so $x = \pm 1$. This case gave us two candidates, $(1,1)$ and $(-1,-1)$.
- Case 2: $y = -x$. In the third equation, we have $x^2 = 7$, so $x = \pm\sqrt{7}$. This case gave us two candidates, $(\sqrt{7}, -\sqrt{7})$ and $(-\sqrt{7}, \sqrt{7})$.

Let's summarize everything in a table, as usual.

Candidates	Values
$(1,1)$	2
$(-1,-1)$	2
$(\sqrt{7},-\sqrt{7})$	14
$(-\sqrt{7},\sqrt{7})$	14

Hence, the global minimum is 2, realized at $(1,1)$ and $(-1,-1)$, while the global maximum is 14, realized at $(\sqrt{7},-\sqrt{7})$ and $(-\sqrt{7},\sqrt{7})$.

Now, we move on to change of variables in multiple integrals. **This is the multivariable version of the u-substitution, taking into account the region of integration and all variables involved at the same time.** The big idea is to find new coordinates (u,v) and a transformation $(x,y) = T(u,v)$ that makes the region R on the xy-plane correspond to a simpler figure (hopefully a rectangle) on the uv-plane. There is more to it, though: in the single-variable case, letting $x = f(u)$, we had $\mathrm{d}x = f'(u)\,\mathrm{d}u$. Point in case being that $\mathrm{d}x$ does not equal $\mathrm{d}u$, and we had a correction factor (namely, $f'(u)$). The same will happen on the multivariable case and again we must pay the price. We may express the correct adjustment in two equivalent ways:

$$\mathrm{d}x\,\mathrm{d}y = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v \quad \text{or} \quad \mathrm{d}u\,\mathrm{d}v = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| \mathrm{d}x\,\mathrm{d}y,$$

where

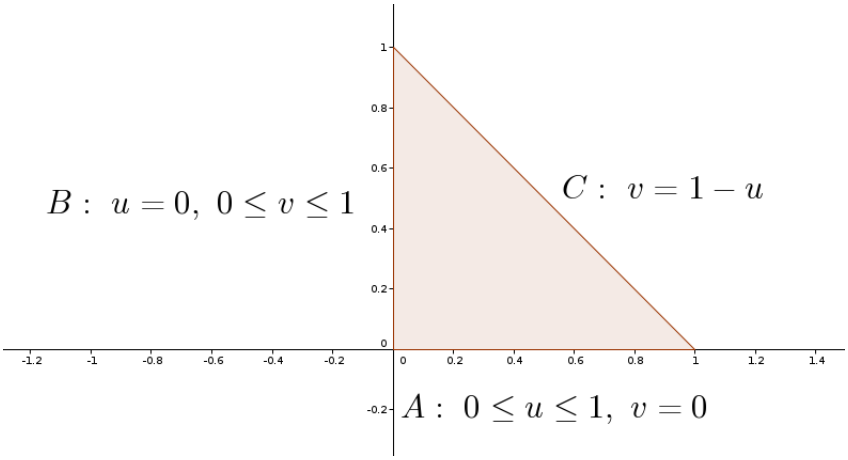
$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \quad \text{and} \quad \frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

with shorthand notation for partial derivatives. They are called the **Jacobians** of the change of variables.

Example 15

Make sketches of S and $R = T[S]$, where $S = \{(u,v) \mid 0 \leq v \leq 1-u$ and $0 \leq u \leq 1\}$ and T: $x = u$, $y = v^2$.

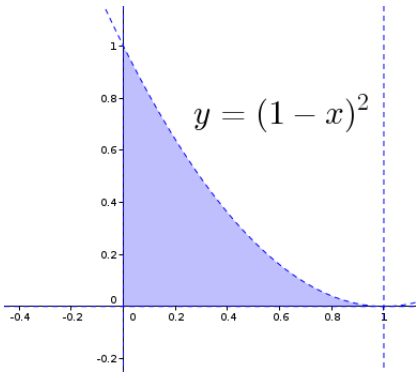
Another way to describe T is by just writing $(x,y) = T(u,v) = (u,v^2)$. This conveys the same information as the first way of writing it, and it is just a matter of taste. To sketch S on the uv-plane, it suffices to draw the line $v = 1-u$ (whose slope is -1 and v-intercept is 1; hence the u-intercept is 1).



With this in place, the general strategy to describe R goes as follows: since T is a continuous function, it will take the boundary of S, which is a closed curve in the uv-plane, to a closed curve on the xy-plane. The interior of S will be mapped (due to continuity of T) to the interior of the region bounded by this image closed curve on the xy-plane. What happens with each side?

- Side A: $(x,y) = T(u,0) = (u,0)$, and $0 \le x \le 1$ as $u = x$ and $0 \le u \le 1$. This means that every point of side A is fixed by T.
- Side B: $(x,y) = T(0,v) = (0,v^2)$, and if v ranges from 0 to 1, so does v^2. This means that the image of side B under T equals again side B itself, but the geometric difference between what happened with side A, is that all points in the interior of side B actually got affected by T.
- Side C: $(x,y) = T(u,1-u) = (u,(1-u)^2)$, so what does the graph of the function $y = (1-x)^2$, as x ranges from 0 to 1, look like? This is the image of side C under T.

Hence, the region R on the xy-plane looks like:



Before proceeding, let's register a useful shortcut (for specific situations):

Lemma

Let f be a continuous function on the rectangle $[a,b] \times [c,d]$, and assume that variables may be separated, i.e., that $f(x,y) = g(x)h(y)$, where g is a continuous function on $[a,b]$ and h is a continuous function on $[c,d]$. Then

$$\iint\limits_{[a,b]\times[c,d]} f(x,y)\,\mathrm{d}A = \left(\int_a^b g(x)\,\mathrm{d}x\right)\left(\int_c^d h(y)\,\mathrm{d}y\right).$$

In other words, **when the region of integration is a rectangle and variables can be separated on the integrand**, the integral of a product equals the product of the integrals.

Proof: Just compute

$$\begin{aligned}\iint\limits_{[a,b]\times[c,d]} f(x,y)\,\mathrm{d}A &= \int_a^b \int_c^d g(x)h(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_a^b g(x)\left(\int_c^d h(y)\,\mathrm{d}y\right)\mathrm{d}x \\ &= \left(\int_a^b g(x)\,\mathrm{d}x\right)\left(\int_c^d h(y)\,\mathrm{d}y\right),\end{aligned}$$

where in the first equalilty we used Fubini's Theorem, on the second one we pulled $g(x)$ out of the inner integral relative to y, and on the third equal sign we pulled out the **number** (and not function!) $\int_c^d h(y)\,\mathrm{d}y$ out of the outer integral relative to x. □

Let's do one very complete example, emphasizing each step to be carried out.

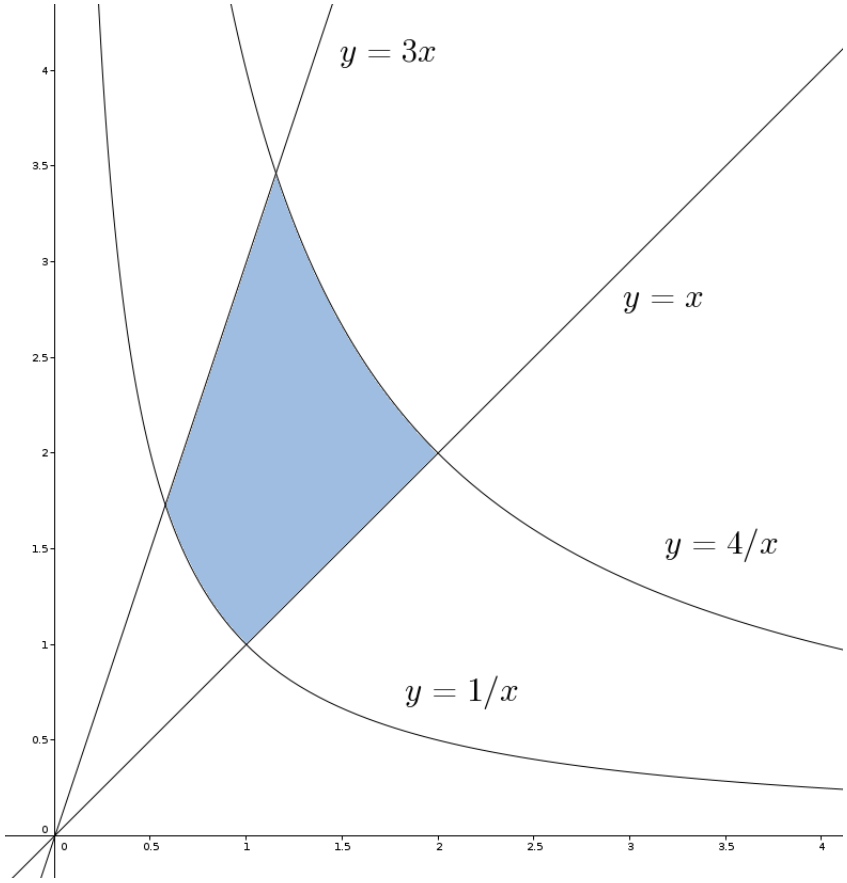
Example 16

Compute the integral

$$\iint\limits_R \mathrm{e}^{xy}\,\mathrm{d}A,$$

where R is the region in the first quadrant bounded by the hyperbolas $y = 1/x$ and $y = 4/x$, and the lines $y = x$ and $y = 3x$.

- Step 1: sketch the region R. You just have to remember that the graph of $y = 1/x$ is a branch of a hyperbola and that the 4 in $y = 4/x$ is just a rescaling factor.



- Step 2: rewrite the bounds of R in the form "something" = "constant" and hopefully read the new variables and their bounds from there. Namely, we have that

$$\begin{array}{rcl} y = 1/x & \to & xy = 1 \\ y = 4/x & \to & xy = 4 \\ y = x & \to & y/x = 1 \\ y = 3x & \to & y/x = 3 \end{array}$$

  This suggests letting $u = xy$ and $v = y/x$. Immediately, the new bounds are $1 \leq u \leq 4$ and $1 \leq v \leq 3$. Who is u and who is v is not relevant here: the rectangle in the uv-plane corresponding to R will just come out rotated, and the negative sign you get in the Jacobian determinant disappears because of the absolute value present when we write $\mathrm{d}u\,\mathrm{d}v$ in terms of $\mathrm{d}x\,\mathrm{d}y$ or vice-versa.

- Step 3: compute the Jacobian determinant. As we have u and v in terms of x and y, it's easier to begin with

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} y & x \\ -y/x^2 & 1/x \end{pmatrix} = \frac{y}{x} - x\left(-\frac{y}{x^2}\right) = 2\frac{y}{x} = 2v.$$

As $v \geq 0$, we have $|2v| = 2v$, so

$$\mathrm{d}u\,\mathrm{d}v = 2v\,\mathrm{d}x\,\mathrm{d}y \implies \mathrm{d}x\,\mathrm{d}y = \frac{1}{2v}\,\mathrm{d}u\,\mathrm{d}v.$$

- Step 4: plug everything into the original integral and solve it.

$$\iint_R \mathrm{e}^{xy}\,\mathrm{d}A = \int_1^3 \int_1^4 \frac{\mathrm{e}^u}{2v}\,\mathrm{d}u\,\mathrm{d}v = \left(\int_1^3 \frac{1}{2v}\,\mathrm{d}v\right)\left(\int_1^4 \mathrm{e}^u\,\mathrm{d}u\right) = \frac{\ln 3}{2}(\mathrm{e}^4 - \mathrm{e}).$$

Finally, let's briefly talk about polar coordinates. What you need to know is:

- $x = r\cos\theta$;
- $y = r\sin\theta$;
- $x^2 + y^2 = r^2$;
- $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$.

Remembering this, you should be able to solve essentially every problem involving polar coordinates. We'll start with a fun example (which you should see at least once in your life):

Example 17

We know from single-variable calculus that the integral

$$\int \mathrm{e}^{-x^2}\,\mathrm{d}x$$

cannot be solved, in the sense that there is no elementary anti-derivative for e^{-x^2}. If you don't remember this or don't believe me, set up a timer on your phone for, say, 15 minutes (but no longer!) and try to solve it yourself. Failure builds up the character. So, let's take the impossible and make it worse. Consider

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x.$$

Indulging the lack of self-love of yours truly, let's not stop here and square it:

$$\left(\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x\right)^2.$$

Now, the fun begins.

$$\left(\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x\right)^2 = \left(\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x\right)\left(\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x\right)$$

$$= \left(\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x\right)\left(\int_{-\infty}^{+\infty} \mathrm{e}^{-y^2}\,\mathrm{d}y\right)$$
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y$$

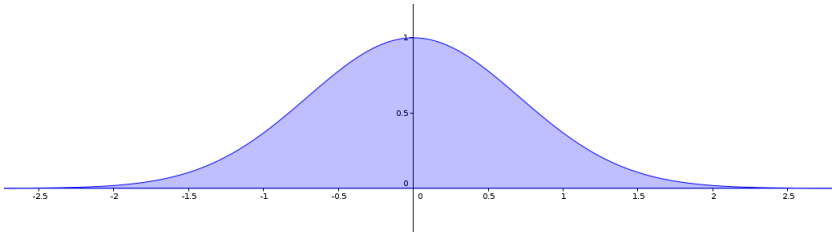
Performing a change of variables to polar coordinates, we continue:

$$\begin{aligned}\left(\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x\right)^2 &= \int_0^{2\pi}\int_0^{+\infty} \mathrm{e}^{-r^2} r\,\mathrm{d}r\,\mathrm{d}\theta\\ &\overset{(*)}{=} \left(\int_0^{2\pi}\mathrm{d}\theta\right)\left(\int_0^{+\infty} r\mathrm{e}^{-r^2}\,\mathrm{d}r\right)\\ &= 2\pi\left(-\frac{1}{2}\mathrm{e}^{-r^2}\right)\Big|_0^{+\infty}\\ &= \pi(0-(-1))\\ &= \pi,\end{aligned}$$

where on $(*)$ we used an obvious variant of the lemma regarding the integral of a product over a rectangle. Note how the correction factor r coming from the Jacobian saved the day. We conclude (as the original definite integral is positive to begin with) that

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}.$$

This integral appears in statistics, when studying random variables $X \sim \mathcal{N}(\mu, \sigma^2)$ with normal probability distribution (here, μ is the mean and σ^2 is the variance); the graph of $f(x) = \mathrm{e}^{-x^2}$ is the "bell curve" you might be already familiar with:



We have effectively shown that the area under the bell curve equals $\sqrt{\pi}$.

We'll conclude with the setup for one of the problems assigned for HW3:

Exercise

Find the volume bounded by the surfaces $z = 2 - x^2 - y^2$ and $z = x^2 + y^2$.

The volume will clearly be

$$\iint_R \left((2 - x^2 - y^2) - (x^2 + y^2)\right)\mathrm{d}A = 2\iint_R (1 - x^2 - y^2)\,\mathrm{d}A,$$

following the old philosophy of "top minus bottom" for computing volumes between surfaces. The question that remains, though, is what is the region R on the plane we should integrate over. Draw a picture to see what the situation looks like. The strategy is to find the "shadow" cast by the intersection of the surfaces on the plane. Setting $2 - x^2 - y^2 = x^2 + y^2$ leads to $x^2 + y^2 = 1$, and so the shadow is $R: \ x^2 + y^2 \leq 1$, a circle centered at the origin with radius 1. A circle begs for polar coordinates. Hence

$$2\iint_R (1 - x^2 - y^2)\,\mathrm{d}A = \int_0^{2\pi}\int_0^1 (1 - r^2)r\,\mathrm{d}r\,\mathrm{d}\theta.$$

Make sure to understand why the bounds are what they are. You take it from here.

5 February 8th

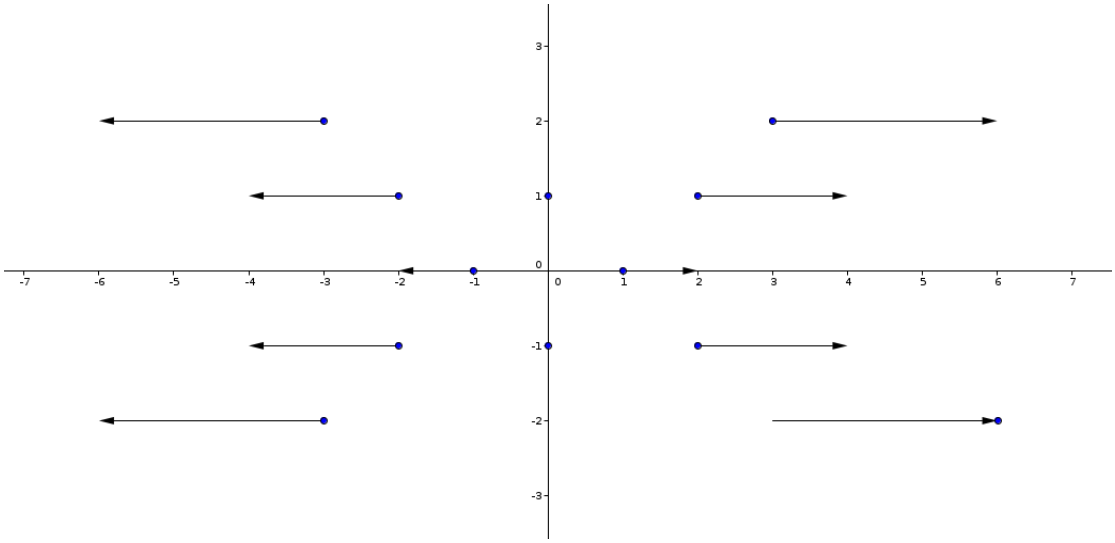
We'll proceed to talk about vector fields and line integrals.

A vector field on a region of the plane or space associates with each point, a vector starting from said point. Everything you already know about vectors can be applied to vector fields as well, pointwise. Namely, one can add and subtract two vector fields and take their dot or cross product (the latter in three dimensions), one can multiply a vector field by a function to obtain a new vector field, one can compute the magnitude of a vector field and, at each point, a vector field is characterized by its magnitude and direction.

Example 18

Sketch the vector field on the plane given by $\boldsymbol{F}(x,y) = (x,0)$.

At each point (x,y), the vector $\mathbf{F}(x,y)$ is horizontal, its magnitude is given by $\|\mathbf{F}(x,y)\| = |x|$; the vectors point to the right if $x > 0$, and to the left if $x < 0$. Each point in the y-axis receives the zero vector, which is to say, "no arrow" there.



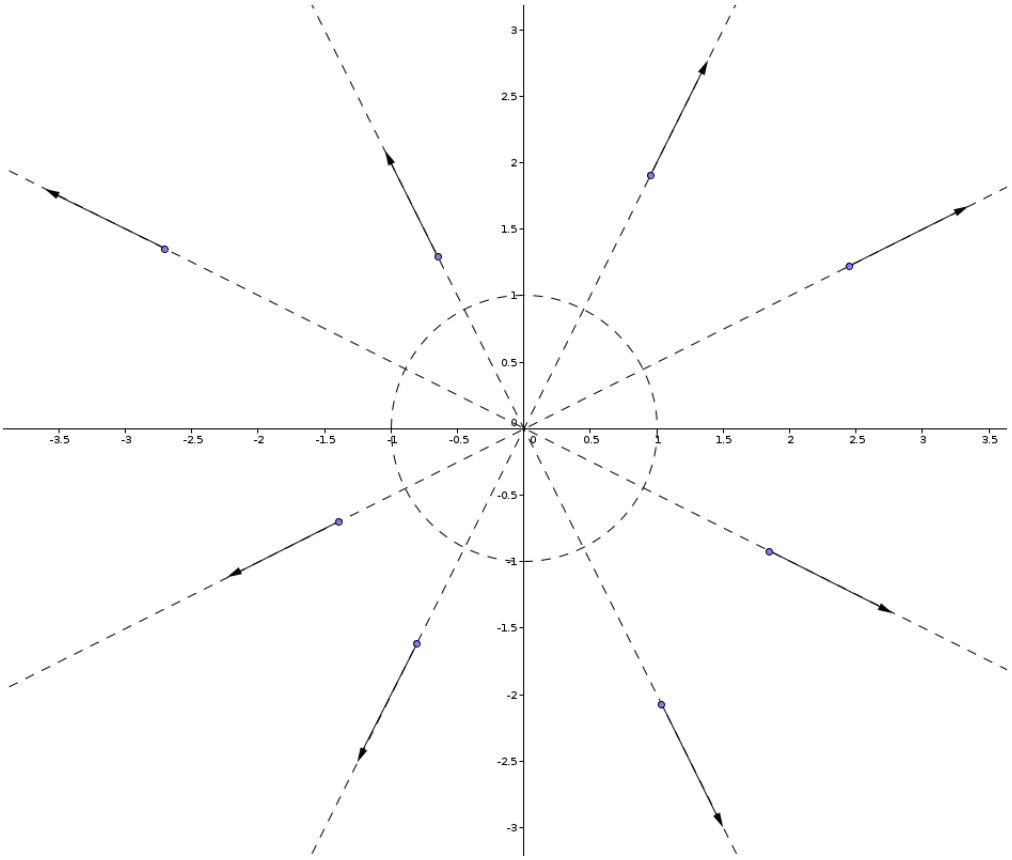
Example 19

Sketch the vector field on the plane with the origin removed, given by

$$\boldsymbol{F}(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right).$$

We just need to understand the magnitude and direction of $\boldsymbol{F}$. A direct calculation shows that $\|\boldsymbol{F}(x,y)\| = 1$ for all (x,y), so our field consists of unit vectors. As for the direction, we note that $\boldsymbol{F}(x,y)$ is always a (function) multiple of the position field (x,y). Vector fields with this property are called **radial**, and have a very special geometric visualization: the vector $\boldsymbol{F}(x,y)$ always lies in the direction

of the ray joining $(0,0)$ to (x,y).

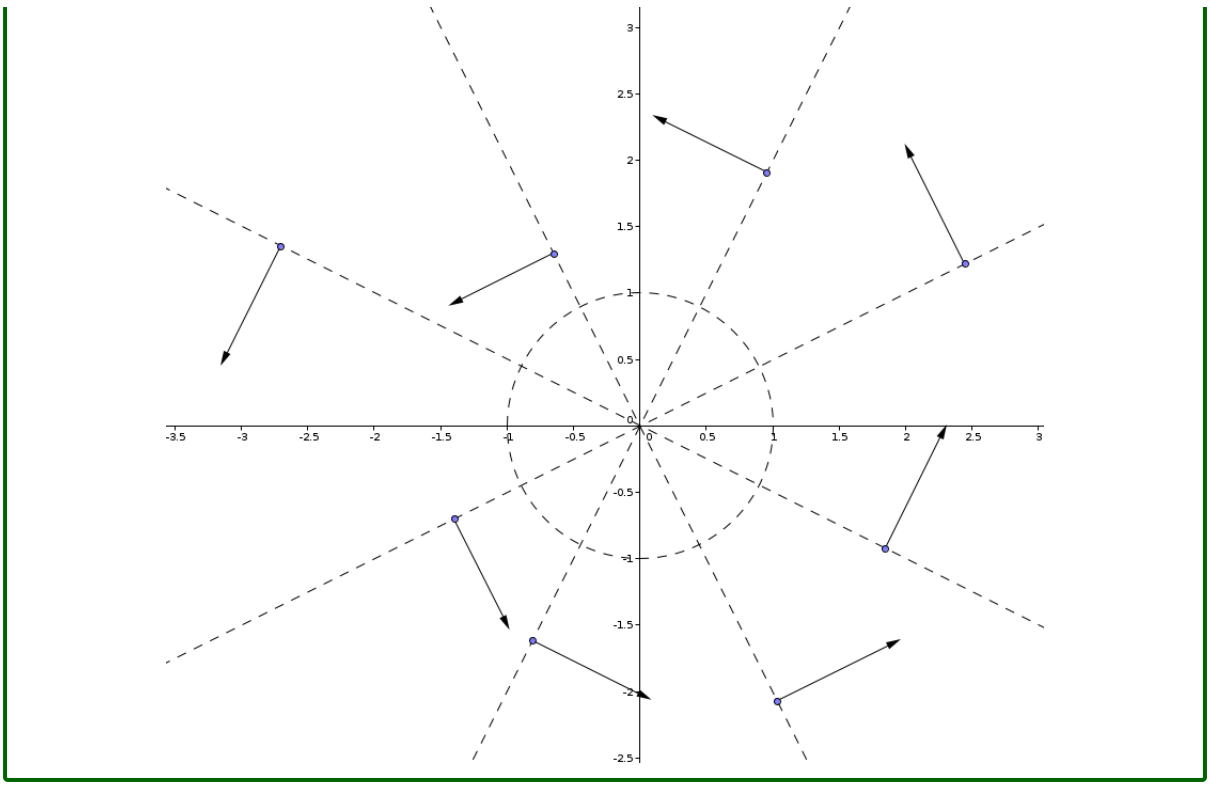


Example 20

Sketch the vector field on the plane with the origin removed, given by

$$\boldsymbol{F}(x,y) = \left(\frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}} \right).$$

This is a small modification of the previous example. We again have that the magnitude of $\boldsymbol{F}$ is just $\|\boldsymbol{F}(x,y)\| = 1$, so $\boldsymbol{F}$ consists of unit vectors as well. This field is clearly not radial, but note that the dot product between this field and the one from the previous example equals zero, which means that at each point, they're orthogonal. This means that at each (x,y), the vector $\boldsymbol{F}(x,y)$ lies in the direction orthogonal to the ray joining $(0,0)$ to (x,y).



Gradients of functions are also important examples of vector fields. Line integrals of such fields are particularly easy to compute, as there's a version of the Fundamental Theorem of Calculus for line integrals lurking in the background.

In any case, before moving to further examples, let's understand what line integrals are really about. Formally, they can be defined as certain limits of Riemann-like sums. What you need to keep in mind, however, is that there are two types of line integrals: scalar line integrals of functions, and line integrals of vector fields.

- Scalar case: $\int_C f\,\mathrm{d}s$. For curves in the plane, $\int_C f\,\mathrm{d}s$ equals the area of the "curtain" determined by the curve C inside the domain of f, and the graph surface $z = f(x,y)$ in space. "Stretching" the curve C like a string, the curve C can be thought of an "axis", and the integral computes the area of a graph over this axis. So, scalar line integrals are very similar to single-variable integrals, in this sense. When $f = 1$, the integral $\int_C \mathrm{d}s$ gives simply the length of the curve C. If $f > 0$ is regarded as a mass density, then the integral $\int_C f\mathrm{d}s$ is the total mass of the string C. In particular, with line integrals we can compute the center of mass/gravity of a string with non-uniform mass distribution, as the point

$$\left(\frac{\int_C xf\,\mathrm{d}s}{\int_C f\,\mathrm{d}s}, \frac{\int_C yf\,\mathrm{d}s}{\int_C f\,\mathrm{d}s}, \frac{\int_C zf\,\mathrm{d}s}{\int_C f\,\mathrm{d}s}\right).$$

- Vector case: $\int_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}$. Let's think of basic mechanics: if a *constant* force F moves a block (with a certain mass) along a linear path, with displacement d, the work realized by the force to perform this action is $W = Fd$. The assumptions that the force F is constant and that the displacement happens along a linear path are

very restrictive. Say that the initial position of the block is at $x = a$ and that the final one is at $x = b$, with $a < b$. Then $W = \int_a^b F\,\mathrm{d}x$ (the constant force may be pulled out of the integral, while $\int_a^b \mathrm{d}x = b - a = d$). This suggests that the work done by a (generally non-constant) force field $\boldsymbol{F}$ on space to move a block from an initial position to a final position, along some now curvilinear path C, should simply be $W = \int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$. The generalization was

$$\int_a^b \to \int_C, \quad F \to \boldsymbol{F}, \quad \mathrm{d}x \to \mathrm{d}\boldsymbol{r}.$$

We still need to understand what the notations for line integrals mean. First, we observe that the dot $\cdot$ in $\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$ is actually a dot product, between the vector field $\boldsymbol{F}$ and $\mathrm{d}\boldsymbol{r}$, the "infinitesimal tangent vector to C". We write $\mathrm{d}\boldsymbol{r} = (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$. Why does it make sense to form a vector with those differentials? If there's any justice in the world, we should be able to write $\mathrm{d}\boldsymbol{r} = \boldsymbol{r}'(t)\,\mathrm{d}t$. But then

$$\mathrm{d}\boldsymbol{r} = \boldsymbol{r}'(t)\,\mathrm{d}t = (x'(t), y'(t), z'(t))\,\mathrm{d}t = (x'(t)\,\mathrm{d}t, y'(t)\,\mathrm{d}t, z'(t)\,\mathrm{d}t) = (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z).$$

As for the remaining differential $\mathrm{d}s$, a similar reasoning goes. Writing $\mathrm{d}s = s'(t)\,\mathrm{d}t$, it remains to understand what $s'(t)$, or more generally $s(t)$, means. This turns out to be a standard notation for the arclength function of a curve. Namely, given a parametrization $\boldsymbol{r}(t)$ of C on some interval $[a, b]$, the integral

$$s(t) = \int_a^t \|\boldsymbol{r}'(\tau)\|\,\mathrm{d}\tau$$

computes the arclength of C from the initial point $\boldsymbol{r}(a)$ to the chosen instant $\boldsymbol{r}(t)$ (in particular, $s(b)$ is the full length of C). The Fundamental Theorem of Calculus says that $s'(t) = \|\boldsymbol{r}'(t)\|\,\mathrm{d}t$. We now have our **dictionary of differentials** completed:

$$\begin{aligned}
\mathrm{d}\boldsymbol{r} &= (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)\\
\mathrm{d}s &= \|\boldsymbol{r}'(t)\|\,\mathrm{d}t\\
\mathrm{d}x &= x'(t)\,\mathrm{d}t\\
\mathrm{d}y &= y'(t)\,\mathrm{d}t\\
\mathrm{d}z &= z'(t)\,\mathrm{d}t
\end{aligned}$$

As for how to ahead and compute line integrals, generally, we can organize ourselves with steps, as in Example 16 (p. 22).

Step 0: If dealing with a vector line integral $\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$, writing the components of $\boldsymbol{F}$ as (P, Q, R), evaluate the dot product

$$\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_C (P, Q, R) \cdot (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z) = \int_C P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z.$$

Step 1: Find a parametrization $\mathrm{r}(t)$, $a \leq t \leq b$, for C. Sometimes, the problem gives $\boldsymbol{r}(t)$ to you. Other times, you have to figure it out yourself (but it's generally not too difficult).

Step 2: Set up all the differentials appearing in the problem as something times $\mathrm{d}t$, by using the little dictionary above.

Step 3: Plug $x = x(t)$, $y = y(t)$, $z = z(t)$ into the given integral, as well as all the differentials you have set up in Step 2. You have now obtained a single-variable integral $\int_a^b \cdots \mathrm{d}t$. Solve it.

Of course, everything here was described for functions and fields in space. When working in two dimensions only, just ignore the variable z and the third components of everything.

Example 21

Compute the integral

$$\oint_C xy\,\mathrm{d}s,$$

where C is the unit circle in the plane with parametrization $\boldsymbol{r}(t) = (\cos t, \sin t)$, defined on $0 \leq t \leq 2\pi$.

First, we note that the circle on the integral, as in $\oint_C$ as opposed to $\int_C$, is **just a reminder** that the curve C on which we integrate over is closed. It makes absolutely no difference on how we'll solve it. Step 0 is unneeded, it's a scalar line integral. Step 1 is also unneeded, as the problem gave us $\boldsymbol{r}(t)$. For Step 2, we have that $\boldsymbol{r}'(t) = (-\sin t, \cos t)$, so $\|\boldsymbol{r}'(t)\| = 1$ for all t. This means that $\mathrm{d}s = \mathrm{d}t$. Finally, Step 3:

$$\oint_C xy\,\mathrm{d}s = \int_0^{2\pi} \cos t \sin t\,\mathrm{d}t = \frac{1}{2}\int_0^{2\pi} \sin(2t)\,\mathrm{d}t = 0.$$

Example 22

Compute the integral

$$\int_C (x+y)\,\mathrm{d}x,$$

where C is the upper half-circle (centered at the origin) with radius 2, oriented counterclockwise.

This is not really a scalar line integral, but a vector line integral in disguise, with Step 0 already performed. One may see it as $\int_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}$ with the field $\boldsymbol{F}$ given by $\boldsymbol{F}(x,y) = (x+y, 0)$. In any case, for Step 1, we may take $\boldsymbol{r}(t) = (2\cos t, 2\sin t)$, with $0 \leq t \leq \pi$. For Step 2, there's only one differential to be considered, $\mathrm{d}x$: since $x = 2\cos t$, we have $\mathrm{d}x = -2\sin t\,\mathrm{d}t$. Now, Step 3 becomes just

$$\int_C (x+y)\,\mathrm{d}x = \int_0^{\pi} (2\cos t + 2\sin t)(-2\sin t\,\mathrm{d}t)$$

$$= -4\int_0^{\pi} (\cos t \sin t + \sin^2 t)\,\mathrm{d}t$$
$$= -2\pi.$$

Example 23

Compute the integral

$$\int_C (xy)^{1/3}, \mathrm{d}s,$$

where C is the arc of parabola $y = x^2$, $0 \leq x \leq 1$.

This is a scalar line integral, so Step 0 is unneeded. For Step 1, we observe that graphs of functions always admit natural parametrizations: generally, if we have $y = f(x)$ with $a \leq x \leq b$, and we want to write $\boldsymbol{r}(t) = (\cdots, \cdots)$, the second entry must necessarily be obtained by applying f to the first one! So we don't need to really think about it, and the only question that remains is how to parametrize the interval $a \leq x \leq b$. Obviously, one goes for the simplest thing possible, $x = t$, and $a \leq t \leq b$. So $\boldsymbol{r}(t) = (t, f(t))$ does the trick. In our case, we take $\boldsymbol{r}(t) = (t, t^2)$ with $0 \leq t \leq 1$. Moving on to Step 2, we have that $\boldsymbol{r}'(t) = (1, 2t)$, so $\|\boldsymbol{r}'(t)\| = \sqrt{1+4t^2}$, and hence $\mathrm{d}s = \sqrt{1+4t^2}\,\mathrm{d}t$. Finally, Step 3 boils down to

$$\begin{aligned}\int_C (xy)^{1/3}\,\mathrm{d}s &= \int_0^1 (tt^2)^{1/3}\sqrt{1+4t^2}\,\mathrm{d}t \\ &= \int_0^1 t\sqrt{1+4t^2}\,\mathrm{d}t \\ &= \int_1^5 \sqrt{u}\,\frac{\mathrm{d}u}{8} \\ &= \frac{1}{8}\left(\frac{2}{3}u^{3/2}\right)\Bigg|_1^5 \\ &= \frac{5\sqrt{5}-1}{12},\end{aligned}$$

after doing the substitution $u = 1 + 4t^2$.

Example 24

Compute $\int_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}$, where $\boldsymbol{F}(x,y) = (x,y)$ and C is parametrized by $\boldsymbol{r}(t) = (4t, t^2)$, with $0 \leq t \leq 1$.

We're dealing with a vector line integral, so let's do Step 0:

$$\int_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \int_C (x,y)\cdot(\mathrm{d}x, \mathrm{d}y) = \int_C x\,\mathrm{d}x + y\,\mathrm{d}y.$$

There's no need to do Step 1, as the problem gave us $\boldsymbol{r}(t)$. As for Step 2, the only differentials appearing here[a] are $\mathrm{d}x$ and $\mathrm{d}y$. So we just note that $x = 4t$ and $y = t^2$

immediately give us that $\mathrm{d}x = 4\,\mathrm{d}t$ and $\mathrm{d}y = 2t\,\mathrm{d}t$. With this in place, Step 3 is straightforward:

$$\begin{aligned}\int_C x\,\mathrm{d}x + y\,\mathrm{d}y &= \int_0^1 4t(4\,\mathrm{d}t + t^2(2t\,\mathrm{d}t)\\ &= \int_0^1 (16t + 2t^2)\,\mathrm{d}t\\ &= 8 + \frac{2}{3}\\ &= \frac{26}{3}.\end{aligned}$$

[a] If a vector field in three dimensions has zero as one of its components, the corresponding differential will simply not appear: the zero component kills it with the dot product.

Example 25

Compute $\int_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}$, where $\boldsymbol{F}(x,y) = (-y,x)$ and C is any radial line segment.

Geometrically, $\boldsymbol{F}$ is always orthogonal to the line segment, as the latter is radial (revisit Example 20, p. 28). There is no component of $\boldsymbol{F}$ in the direction of $\mathrm{d}\boldsymbol{r}$ (recall the relation between dot products and projections). The integral is zero.

Remark. If C is a curve on plane or space, and $\boldsymbol{T}$ is a chosen unit tangent field to C, we may consider both

$$\int_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} \quad \text{and} \quad \int_C \boldsymbol{F}\cdot\boldsymbol{T}\,\mathrm{d}s.$$

They are morally the same thing. However, one can only say that $\mathrm{d}\boldsymbol{r} = \boldsymbol{T}\,\mathrm{d}s$ if $\boldsymbol{T}$ points in the same direction as the velocity vector $\boldsymbol{r}'(t)$ of the particular parametrization chosen for C. Reversing the orientation of the parametrization does not change the result of the first integral, but the new parametrization may no longer be "compatible" with the field $\boldsymbol{T}$, fixed *a priori*. The bottom line here is that the two integrals in display are **not interchangeable** (unless one is extremely careful with orientations), despite being so similar.

6 February 15th

We solved the first midterm in detail and talked about how things are going so far in this course.

7 February 22nd

We move on to the second part of this course: linear systems and matrices. The big idea here is to, given a linear system, convert it to a matrix, cast this matrix into a simpler form, and draw conclusions about the original system from the simplified matrix.

More precisely, we will consider matrices in **reduced row echelon form (RREF)**, which look like this:

$$\begin{bmatrix} 1 & 0 & * & 0 & 0 & * & * \\ 0 & 1 & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The **pivots**, which are the first non-zero entries in each row, must all be equal to 1. On each column where a pivot 1 appears, all entries other than the pivot itself must be zero. The pivots must form a "staircase" shape (hence the name "echelon"). The word "reduced", in turn, refers to the fact that all entries in a given column with a pivot, above the pivot, are zero. If a row consists only of zeros, it must be on the bottom of the matrix.

To cast a given matrix into RREF, the following **elementary operations** are allowed:

(1) switch two rows;

(2) multiply any row by a non-zero number;

(3) add to any row a multiple of another row.

There are several ways of solving linear systems (one of them being "Cramer's method", for example), but dealing with matrices in RREF is preferred from a computational viewpoint for being more efficient, in the sense that solving a linear system using this algorithm is what takes a computer the fewer number of operations to do. Moreover, RREF is the answer to the natural question "could I do something else to make the matrix simpler?" or "am I missing something?". If the matrix is in RREF, the answer is "no": you did everything possible to simplify the matrix. And the reason why we'll only stick with the elementary operations above is because they make sure that the matrix in RREF obtained in the end does represent a system equivalent to the original one.

Example 26

Decide whether the following matrix is in RREF and, if not, put it in RREF.

$$\begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 2 & -2 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The pivots of this matrix are, respectively, 1, 2 and 1. Since we have a pivot which is not equal to 1, the matrix is not in RREF. Each step we carry out has

a specific objective, and whatever happens with the rest of the matrix, happens. Our first goal is to turn that pivot 2 into a 1, and for that we may divide the whole second row by 2 (this is an allowed elementary operation):

$$R_2 := \frac{1}{2}R_2 \qquad \sim \qquad \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & -1 & -3/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note how we're borrowing notation from computer science to keep track of the operations performed. This is very important to do as it improves organization and readability. You should take this seriously on HW assignments and exams.

In lecture, it was mentioned that there's also something called **echelon form** (without the word "reduced"), but that you shouldn't worry about it, as RREF is much more important. While this is true, it is convenient to know what a matrix in "echelon form" is: it should have the same "staircase" shape as in RREF with all pivots being 1 and rows of zeros being in the bottom, but elements above a pivot don't need to be zero. So, the matrix is now in echelon form, but not RREF. Recognizing when a matrix is in echelon form but not RREF tells you that you're halfway through the process; it's a checkpoint: all that's left to do now is to take out the trash above the pivots. This is done systematically, again from left to right, but now from bottom to top. The next goal should be to eliminate the 2 in the first row of the matrix.

$$R_1 := R_1 - 2R_2 \qquad \sim \qquad \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -3/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since on the third column there is no pivot, nothing can be done about the non-zero entries there. We move on to clean up what's above the pivot on the fourth column. As both R_1 and R_2 will interact with R_3, but not with each other, we can do two steps at once:

$$\begin{matrix} R_1 := R_1 - R_3 \\ R_2 := R_2 + \frac{3}{2}R_3 \end{matrix} \qquad \sim \qquad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We have obtained the RREF.

Let's now see how things work if we start with a system.

Example 27

Solve, if possible, the following system:

$$\begin{cases} x_1 + x_2 \qquad\qquad - x_5 = 1 \\ \qquad x_2 + 2x_3 + x_4 + 3x_5 = 1 \\ x_1 \qquad - x_3 + x_4 + x_5 = 0 \end{cases}$$

Converting a system to a matrix is always the easiest thing to do: just read off the coefficients from each variable on each equation, and place them into rows.

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 & 3 & 1 \\ 1 & 0 & -1 & 1 & 1 & 0 \end{array}\right]$$

This is called an **augmented matrix** because the last column takes into account the right-hand side of the equations in the original system. This is important to consider, as doing operations between rows is morally the same as doing operations between equations in the system (with the obvious advantage that we won't keep writing variable names x_1, x_2, etc., all the time), and when doing operations between equations, those operations happen on the right-hand side of the equations involved too.

That being said, the matrix is obviously not in RREF; we don't even have the "staircase" shape. The first 1 in the first row, however, is the pivot, and so we need to eliminate all entries below it. This can be achieved as follows:

$$R_3 := R_3 - R_1 \qquad \sim \qquad \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 & 3 & 1 \\ 0 & -1 & -1 & 1 & 2 & -1 \end{array}\right].$$

We have cleaned up everything below the first pivot, so we move on to the next. The pivot in the second row is the 1 appearing in the second column as well, so we need to eliminate the -1 below it. This is done by:

$$R_3 := R_3 + R_2 \qquad \sim \qquad \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 & 5 & 0 \end{array}\right].$$

Observe that this -1 could also have been eliminated via $R_3 := R_3 + R_1$, but this operation would produce a 1 in the bottom left corner of the matrix! The reason why this happened is because we're trying to make an operation involving a row whose pivot has already had everything cleaned up below it. The moral of the story here is that each step to be carried has a laser-like focus, a single goal, and if you do an operation which undoes something that should have been already ok by that point, you have done something wrong. As a matter of fact, the matrix is now in echelon form, but not RREF. We must proceed to clean up the spaces above the pivots, going from left to right, as usual. Above the pivot in the second row, we have a 1, which is the next target:

$$R_1 := R_1 - R_2 \qquad \sim \qquad \left[\begin{array}{ccccc|c} 1 & 0 & -2 & -1 & -4 & 0 \\ 0 & 1 & 2 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 & 5 & 0 \end{array}\right].$$

Observe that this operation did not destroy the "staircase" shape because we have only started to clean up spaces above the pivots once the matrix was already in

echelon form. We move on to the next pivot, on the third row: we must clean up the -2 and 2 above it. As in the previous example (and this turns out to be a general phenomenon), since both R_1 and R_2 will interact with R_3, but not with each other, we can do the two steps at once:

$$\begin{matrix} R_1 := R_1 + 2R_3 \\ R_2 := R_2 - 2R_3 \end{matrix} \quad \sim \quad \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 3 & 6 & 0 \\ 0 & 1 & 0 & -3 & -7 & 1 \\ 0 & 0 & 1 & 2 & 5 & 0 \end{array}\right].$$

The matrix is now in RREF, and it corresponds to the system

$$\begin{cases} x_1 \qquad\quad + 3x_4 + 6x_5 = 0 \\ \quad x_2 \qquad - 3x_4 - 7x_5 = 1 \\ \qquad\quad x_3 + 2x_4 + 5x_5 = 0, \end{cases}$$

which is equivalent to the original one (that is, this system and the first one have the same solution set). The variables x_4 and x_5 (corresponding to the columns in RREF which could not be cleaned up – as we didn't have pivots to use) are **free**, in the sense that the remaining variables x_1, x_2 and x_3 may be written in terms of x_4 and x_5. We may change notation, say, to $t_1 = x_4$ and $t_2 = x_5$, effectively **parametrizing the solution set of the system**, and writing

$$S = \{(-3t_1 - 6t_2, 3t_1 + 7t_2 + 1, -2t_1 - 5t_2, t_1, t_2) \in \mathbb{R}^5 \mid t_1, t_2 \in \mathbb{R}\}.$$

The solution set S is a 2-dimensional plane in $\mathbb{R}^5$, not passing through the origin $(0,0,0,0,0)$ of $\mathbb{R}^5$. Every point of such plane corresponds to a solution of the system. For example, choosing $t_1 = 1$ and $t_2 = 2$, we obtain the solution $(-15, 18, -12, 1, 2)$ of the original system. This is the same as saying that plugging

$$x_1 = -15, \quad x_2 = 18, \quad x_3 = -12, \quad x_4 = 1 \quad \text{and} \quad x_5 = 2$$

on the equations of the original system, the right-hand sides come out to be 1, 1, and 0. Every time you have even a single free variable, the system has infinitely many solutions. These variables are called "free" because you're free to choose values to substitute into it, thus generating different solutions to the system. If there are infinitely many solutions (in fact, **uncountably many**, in a very precise sense), you are not supposed to try and list them one by one. This is why it is important to understand how free variables work and how to describe your solution set in a "parametric form": it will carry all the information you need in a sucint way.

Here's one last example to hopefully help you think outside the box:

Example 28

Find all the values of α and β between 0 and 2π such that:

$$\begin{cases} 2\cos\alpha + 4\sin\beta = 3 \\ 3\cos\alpha - 5\sin\beta = -1 \end{cases}$$

If your first urge is to complain that this system is not linear in the variables α and β, you are absolutely right. It is linear, however, in the variables $\cos\alpha$ and $\sin\beta$, which means that we can use our knowledge about linear systems to attack this problem, and then follow up with trigonometry. Compare with the situation where you have $\sin^2\theta - 2\sin\theta + 1 = 0$ and are asked to solve the equation for θ: surely it is not a quadratic equation on the variable θ, but it is quadratic on the variable $\sin\theta$, so the quadratic formula applies to solve for $\sin\theta$ directly. What is happening here is morally a "substitution", but learning to recognize those situations "abstractly" is very helpful when trying to solve some problems. If, for psychological reasons, you need to cast these things into an actual linear system (or the single equation into a legitimate quadratic equation), just set $x_1 = \cos\alpha$ and $x_2 = \sin\beta$ (or $x = \sin\theta$, respectively). The point of this discussion is to let go of the training wheels and do this naturally, avoiding explicit substitutions. Convert the system for $\cos\alpha$ and $\sin\beta$ into a matrix:

$$\left[\begin{array}{cc|c} 2 & 4 & 3 \\ 3 & -5 & -1 \end{array}\right].$$

This is not in RREF, nor echelon form to begin with. The first pivot, in the first row, is 2 (and not 1). Fix it:

$$R_1 := \frac{1}{2}R_1 \qquad \sim \qquad \left[\begin{array}{cc|c} 1 & 2 & 3/2 \\ 3 & -5 & -1 \end{array}\right].$$

Next, we clean up the trash below this pivot:

$$R_2 := R_2 - 3R_1 \qquad \sim \qquad \left[\begin{array}{cc|c} 1 & 2 & 3/2 \\ 0 & -11 & -11/2 \end{array}\right].$$

Next step, next pivot: the current pivot on the second row is -11, so to make it 1, we divide the second row by -11:

$$R_2 := -\frac{1}{11}R_2 \qquad \sim \qquad \left[\begin{array}{cc|c} 1 & 2 & 3/2 \\ 0 & 1 & 1/2 \end{array}\right].$$

Now, the matrix is in echelon form, but not RREF. We proceed to eliminate the 2 above the second pivot:

$$R_1 := R_1 - 2R_2 \qquad \sim \qquad \left[\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \end{array}\right].$$

Finally, we have obtained the RREF. This means that $\cos\alpha = 1/2$ and $\sin\beta = 1/2$. These two equations for α and β are independent, so solutions of the original system are obtained by combining all possible solutions for each equation (on the interval $[0, 2\pi]$, as imposed): the solutions (α, β) are

$$\left(\frac{\pi}{3}, \frac{\pi}{6}\right), \quad \left(\frac{\pi}{3}, \frac{5\pi}{6}\right), \quad \left(\frac{5\pi}{3}, \frac{\pi}{6}\right) \quad \text{and} \quad \left(\frac{5\pi}{3}, \frac{5\pi}{6}\right).$$

One last sanity-check: a general fact about linear systems is that they have either zero solutions, one solution, or infinitely many. Here, we have obtained four solutions, which seems like an apparent contradiction. It is alright because, again, the original system is not linear for the variables α and β, but linear for $\cos\alpha$ and $\sin\beta$ instead. For the latter variables, we indeed have obtained a single solution, namely, $(1/2, 1/2)$.

8 March 1st

We start with a quick review on matrix algebra. Namely, the question is "what can we do with matrices?". We can:

- add two matrices **of the same size**, as in
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix},$$
entrywise.
- multiply a matrix (of any size) by a real number, as in
$$3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 0 & 6 \end{bmatrix},$$
again entrywise.
- take **transposes** of matrices, which means that rows become columns and vice-versa, as in
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \implies A^{\top} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$
- multiply a $n \times m$ matrix A by a $m \times k$ matrix B, to obtain a $n \times k$ matrix AB, whose (i, j)-entry equals the dot product between the i-th row of A and the j-th column of B. Idea: $(n \times m)(m \times k) = n \times k$.

Here's a reason about why matrix multiplication is like that: it all comes back to the big idea of representing a given linear system with a matrix, studying the matrix instead, and drawing conclusions about the original system from, for example, the reduced row echelon form of the obtained matrix. More precisely, think of the silliest case possible, where we have only one equation and one variable: $ax = b$. If we want to study linear systems (say, with n variables and m equations) with sort of the same notation, $A\boldsymbol{x} = \boldsymbol{b}$, we need to make sense of what does it mean to multiply the $m \times n$ matrix A with the $n \times 1$ column vector $\boldsymbol{x}$, to obtain the $m \times 1$ column vector $\boldsymbol{b}$. Write the system explicitly:
$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n = b_2 \\ \qquad\qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$$
The entries of the product $A\boldsymbol{x}$ should be the entries of $\boldsymbol{b}$, but the system itself gives the expression fot the entries of $\boldsymbol{b}$ in terms of A and $\boldsymbol{x}$. The definition
$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix}$$

is thus **forced** upon us, in the sense that if we want to make sense of the system in the form $A\boldsymbol{x} = \boldsymbol{b}$, there is only one possible choice for what the vector $A\boldsymbol{x}$ must be. To multiply A by a second matrix B which is $n \times k$, one applies the above for each column of B — and this is the matrix multiplication you have first seen in class. In particular, this should justify why the number of columns of A must equal the number of rows of B for this product to make sense.

Of course, while the calculation shows that this unpleasant definition of matrix multiplication is what we really need to study linear systems properly, it still leaves room for one to wonder why the obvious first guess for what matrix multiplication should be doesn't work as well. Namely, why can't we just multiply two matrices entrywise, like we do for addition? You might be surprised to hear that this is actually a thing, called the **Hadamard product** of matrices (there's even a wikipedia page about it). The first (and fatal) flaw of this, for our purposes, is that this Hadamard product can only be computed between matrices of the same size, making us lose all hope of using this as a tool to help us study linear systems whose number of equations is not equal to the number of variables involved. The second flaw is ultimately a consequence of the following result:

Theorem 3

Let A be a $n \times n$ square matrix. The following conditions are equivalent:

(i) A is non-singular.

(ii) the inverse A^{-1} exists.

(iii) $\det A \neq 0$.

(iv) A has full rank.

The determinant function (whatever it is for square matrices whose size is bigger than 2) is a useful tool to detect whether a given square matrix is non-singular or not, and we have that $\det(AB) = \det(A)\det(B)$ whenever A and B are square matrices of the same size. This nice algebraic relation (which has the consequence that, for example, the product of two non-singular matrices is non-singular), fails if one replaces the usual matrix multiplication with the Hadamard product. There are other deeper reasons why standard matrix multiplication is what it is, but explaining them would require more Linear Algebra than what we have available to use right now.

Here are some pitfalls to avoid, regarding matrix multiplication:

- matrix multiplication is **non-commutative**, i.e., AB is not equal to BA, in general; one way to convince yourself quickly of this is that unless both A and B are square matrices of the same size, in general only one of the products AB or BA is well-defined, while the other is not, so it doesn't even make sense to compare them (and to make it worse, there are examples of square matrices A and B for which $AB \neq BA$). This means that while one does $(a+b)^2 = a^2 + 2ab + b^2$ for real numbers a and b, one must write $(A+B)^2 = A^2 + AB + BA + B^2$ in full, with no further simplifications available.

- one cannot "cancel" non-zero matrices like real numbers: if $AB = AC$ and $A \neq 0$, we **cannot conclude** that $B = C$. In fact, this only works provided that A is non-singular!
- the identity matrix (which plays the role of 1 for real numbers) is not a the matrix full of 1's, but instead the matrix with 1's in the diagonal and zeros everywhere else (the former is an "identity matrix" for the Hadamard product, not for standard matrix multiplication).

And as far as transposition goes, observe that if $\boldsymbol{u}$ and $\boldsymbol{v}$ are column vectors of the same size, then $\boldsymbol{u}^\top \boldsymbol{v}$ equals the dot product between $\boldsymbol{u}$ and $\boldsymbol{v}$. Doing something similar for matrices as opposed to vectors provides a way to compute the "dot product" between two matrices; we have no need to pursue this further in this class. This is also a good chance to register some properties of the transposition operation:

Proposition 1

Let A and B be a $n \times m$ matrices, C be a $m \times k$ matrix, and $c \in \mathbb{R}$ be a real number. Then:

(i) $(A+B)^\top - A^\top + B^\top$.

(ii) $(cA)^\top = cA^\top$.

(iii) $(A^\top)^\top = A$.

(iv) $(AC)^\top = C^\top A^\top$.

In short, transposition is a linear operation which when applied twice returns the original matrix, and the transpose of a product is the product of the transposes **in the reverse order** (note that $A^\top C^\top$ may even not be well-defined). Let's put all of this together in a concrete example:

Example 29

Let A be a 2×2 matrix, and consider

$$B = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \quad \textit{and} \quad C = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}.$$

(a) If $A^\top + B = C$, what is A?

(b) If $A^\top B = C$, what is A?

A lazy strategy is to just substitute numerical values after no further simplification may be done with matrix algebra. To solve for A in (a), we start by subtracting B on both sides, so $A^\top = C - B$, then take transposes to obtain $(A^\top)^\top = (C-B)^\top$,

which reduces to $A = C^\top - B^\top$. So

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}^\top - \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}^\top = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}.$$

As for item (b), things become more tricky. When looking at $A^\top B = C$, our first urge is to "divide both sides by B", to solve for $A^\top$. However, even if $B \neq 0$ it's not guaranteed we can "divide by B"! But since $\det B = 1 \neq 0$, we know that B is non-singular, and so B^{-1} does exist. Next, we still have to be careful with "dividing by B" (which you should now read as "multiplying by B^{-1}"), as matrix multiplication is non-commutative. This means that multplying both sides of $A^\top B = C$ by B^{-1} through the left or through the right makes a crucial difference. As the goal is to "cancel B", this multiplication should happen through the right, leading to $A^\top = CB^{-1}$ (as opposed to $B^{-1}A^\top B = B^{-1}C$, whose left side is messy). Thus, we have that $A = (CB^{-1})^\top$. It remains to compute B^{-1}. We'll use the:

Gauss-Jordan inversion: *If B is a non-singular square matrix, we set up an augmented matrix $[B|I]$, where I is the identity matrix of the same size as B (it's similar to when we form an augmented matrix to study a linear system, but we augment B with an entire identity matrix on the right, as opposed to a single column vector). Perform row operations to put B into RREF. As B is assumed non-singular, the RREF of B turns out to be... I. So we have something like $[I|*]$, where $*$ is some mess that was obtained from doing the row operations (aimed to bring B into RREF) to I. This $*$ is exactly B^{-1}.*

We now apply Gauss-Jordan inversion to B, setting up the augmented matrix

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right].$$

So:

$$R_2 := R_2 - R_1 \qquad \sim \qquad \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right].$$

Then

$$R_1 := R_1 - 3R_2 \qquad \sim \qquad \left[\begin{array}{cc|cc} 1 & 0 & 4 & -3 \\ 0 & 1 & -1 & 1 \end{array}\right].$$

The conclusion is that

$$B^{-1} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}.$$

With this in place, we go back to $A = (CB^{-1})^\top$, and compute

$$A = \left(\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}\right)^\top = \begin{bmatrix} 5 & -3 \\ 11 & -7 \end{bmatrix}^\top = \begin{bmatrix} 5 & 11 \\ -3 & -7 \end{bmatrix}.$$

One could alternatively use that $A = (CB^{-1})^\top = (B^{-1})^\top C^\top = (B^\top)^{-1}C^\top$ (i.e., the inverse of a transpose is the transpose of the inverse), apply Gauss-Jordan to find the inverse of $B^\top$, and multiply the result by $C^\top$ through the right — this yields the same result.

One last notion we should be comfortable with is:

Definition 4

Let $\boldsymbol{u}_1, \dots, \boldsymbol{u}_k \in \mathbb{R}^n$ be vectors. We say that they are **linearly independent** if whenever $a_1\boldsymbol{u}_1 + \cdots + a_k\boldsymbol{u}_k = \boldsymbol{0}$ (here, $a_1, \dots, a_k$ are real numbers), we must necessarily have $a_1 = \cdots = a_k = 0$. If they are not linearly independent, we call them **linearly dependent**.

Briefly, saying that a collection of vectors is linearly independent is saying that none of the vectors considered is a linear combination of the others. Similarly, saying that a collection of vectors is linearly dependent is saying that there is at least one of them which may be expressed as a linear combination of the others.

Example 30

In $\mathbb{R}^2$, consider the vectors $\boldsymbol{u}_1 = (1,0)$, $\boldsymbol{u}_2 = (0,1)$, and $\boldsymbol{u}_3 = (1,1)$. We claim that:

- $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are linearly independent. Indeed, write a generic linear combination and set it equal to the zero vector: $a\boldsymbol{u}_1 + b\boldsymbol{u}_2 = \boldsymbol{0}$. This reads, in full, as $a(1,0) + b(0,1) = (0,0)$, and so $(a,b) = (0,0)$ gives $a = b = 0$. Thus, $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are linearly dependent.
- $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ are linearly dependent. It suffices to note that $\boldsymbol{u}_3 = \boldsymbol{u}_1 + \boldsymbol{u}_2$ is a linear combination of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ (namely, the coefficients are all 1).

Linear independence is a notion of "uniqueness" in disguise, because if a given vector may be written as a linear combination of a collection of linearly independent vectors, such linear combination is unique, while if the collection of vectors used to write the linear combination is linearly dependent, there may be more of one possible way to write the combination (which creates ambiguity when trying to talk about "the coordinates of the vector relative to the collection").

Consider $(2,3) \in \mathbb{R}^2$. We have that $(2,3) = 2(1,0) + 3(0,1)$, and this is the only way to express $(2,3)$ as a combination of $(1,0)$ and $(0,1)$, in the sense that if you were to write $(2,3) = a(1,0) + b(0,1)$, you would **inevitably** arrive at $a = 2$ and $b = 3$. Throwing in the third vector $(1,1)$ to create a linearly dependent collection, we have that

$$(2,3) = 2(1,0) + 3(0,1) + 0(1,1) = 0(1,0) + 1(0,1) + 2(1,1),$$

and so on. The combination is not unique. Moral of the story: we like when vectors are linearly independent.

We conclude with a slightly more elaborate example:

Example 31

Consider the (column) vectors

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \quad \boldsymbol{u}_3 = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}, \quad \boldsymbol{u}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

(a) *Are $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, and $\boldsymbol{u}_3$ linearly independent?*

(b) *Are $\boldsymbol{u}_2$, $\boldsymbol{u}_3$, and $\boldsymbol{u}_4$ linearly independent?*

We start with (a). Here's the shortcut we will use now and forever. Just put the vectors into the columns of a matrix and decide whether it has full rank or not. If it has full rank, the vectors are linearly independent. If you try to put it into echelon form and get a row of zeros, the vectors are linearly dependent. Why is it so simple? Because this shortcut amounts to skipping a couple of steps in the following procedure: set up a linear combination $a_1\boldsymbol{u}_1 + a_2\boldsymbol{u}_2 + a_3\boldsymbol{u}_3 = \boldsymbol{0}$, and plug in the concrete expressions for $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$. We obtain a homogeneous linear system for the variables a_1, a_2 and a_3. If the matrix representing the left side of this system (such matrix is obtained precisely by placing the given vectors into columns) has full rank, the system has only the trivial solution, which forces $a_1 = a_2 = a_3 = 0$, giving linear independence as desired. If the matrix doesn't have full rank, there are non-trivial solutions for a_1, a_2 and a_3, which give us a non-trivial linear combination of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ resulting in $\boldsymbol{0}$ (so $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ are linearly dependent).

Applying this, we consider the matrix

$$[\boldsymbol{u}_1|\boldsymbol{u}_2|\boldsymbol{u}_3] = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 4 \\ -1 & -3 & 3 \end{bmatrix}.$$

Then

$$\begin{matrix} R_2 := R_2 - 2R_1 \\ R_3 := R_3 + R_1 \end{matrix} \quad \sim \quad \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 6 \\ 0 & -1 & 2 \end{bmatrix}.$$

Now

$$R_2 := R_2 - 3R_13 \quad \sim \quad \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{bmatrix}.$$

having a row full of zeros says that $\boldsymbol{u}_1, \boldsymbol{u}_2$ and $\boldsymbol{u}_3$ are linearly **dependent**.

For item (b), the same idea works. To make the calculations a bit easier, we note that the order of the vectors is completely irrelevant in the definition of linear independence (which makes perfect sense, as the rank of a matrix is not affected

by permuting columns), so we may just as well place $\boldsymbol{u}_4$ as the first vector and set up

$$[\boldsymbol{u}_4|\boldsymbol{u}_2|\boldsymbol{u}_3] = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 4 \\ 0 & -3 & 3 \end{bmatrix}.$$

Then

$$R_2 = R_2 - R_1 \qquad \sim \qquad \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & -3 \\ 0 & -3 & 3 \end{bmatrix},$$

and next

$$R_3 := R_3 - 3R_2 \qquad \sim \qquad \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & -3 \\ 0 & 0 & 12 \end{bmatrix}.$$

This last matrix is in echelon form and there are no zero pivots. This means that the matrix has full rank, and so $\boldsymbol{u}_2, \boldsymbol{u}_3$ and $\boldsymbol{u}_4$ are linearly **independent**.

9 March 8th

We solved the practice sheet for Midterm 2 (solutions are available on Carmen).

10 March 15th

Spring Break. No class.

11 March 22nd

We now start with the third part of this course: **Ordinary Differential Equations** ("ODE"s, for short). The idea is simple: we have equations involving derivatives, but instead of solving for a number or vector, we solve for a function instead. Consider a simple example: $y'(t) = 2t$, and imagine you want to solve for y. Integrating, we obtain that $y(t) = t^2 + c$ for some constant of integration $c \in \mathbb{R}$. Keeping track of such constants is now crucial (and perhaps the reason why you kept losing points in Calculus 1 for forgetting the $+c$; it was all a prelude to this moment), for different choices of c lead to different solutions. As we have infinitely many choices of c, we see that even the simplest differential equation will have infinitely many solutions.

The adjective "ordinary" refers to the fact that all the functions involved are functions of a single variable, and that there are no partial derivatives of anything in play. If this were to be the case, we would be dealing with **Partial Differential Equations** ("PDE"s, for short) instead.

In any case, the point remains that solving ODEs can be very hard and one usually resorts to softwares or numerical methods to understand the behavior of solutions to an ODE. We will focus on very specific types of ODEs which we can indeed solve. To understand when this is the case, some vocabulary is useful. Let's always organize our equations by placing in the left side all the terms involving y, and on the right side all the terms not involving y.

- **order:** the order of a differential equation is the highest derivative that appears.
- **linearity:** a differential equation is linear if its left side is a linear combination (with function coefficients) of y, y', y'', etc.
- **homogeneity:** a differential equation is homogeneous if its right side equals zero.

We care about this because: the higher the order, the harder the equation should be to solve; we like linear things better than non-linear things, and homogeneous equations are generally easier to deal with (because we like zeros).

Example 32

Classify the differential equation $y'' - 4y' + 2y = 10t^2$.

It's a second order equation (because the term with the highest derivative is y'', it is linear because the left side is a linear combination of y, y' and y'' (with coefficients 1, -4 and 2, respectively), and it is non-homogeneous as the right side does not equal zero.

The square in $10t^2$ is irrelevant for classifying linearity of the equation, as the square operation is not being performed on the function variable y. Similarly, if we had something like $ty'' - 4(\sin t)y' + 2e^t y = 10t^2$ instead, it would still be second order linear non-homogeneous, because the left side is still a linear combination of y, y' and y'' with **function** coefficients (namely, t, $-4\sin t$ and $2e^t$).

Example 33

Classify the differential equation $y' - 2y^3 = -4t$.

It's a first order equation (because the term with the highest derivative is y'. It is non-linear because of the y^3 term (the non-linear operation of taking a cube is being applied to the function variable y). It is non-homogeneous for hopefully obvious reasons.

To proceed, we pose a question: how to check whether a function is a solution of a given ODE? Just like the situation we had with linear systems, the answer is what you're probably guessing: just plug the function into the left side of the ODE and see what happens. Here's an example:

Example 34

Verify whether $y = 3e^{2t} - 5e^{-2t}$ *is a solution of* $y'' - 4y = 0$.

It's a direct computation:

$$\begin{aligned} y'' - 4y &= (3e^{2t} - 5e^{-2t})'' - 4(3e^{2t} - 5e^{-2t}) \\ &= (6e^{2t} + 10e^{-2t})' - 12e^{2t} + 20e^{-2t} \\ &= 12e^{2t} - 20e^{-2t} - 12e^{2t} + 20e^{-2t} \\ &= 0. \end{aligned}$$

So we conclude that **yes**, the given function is a solution of the ODE.

One problem remains. If the proposed solution were not given to us, how would we find it? Again, this is in general very hard to do. So, we will focus on **second order linear homogeneous ODEs, with constant coefficients**.

Theorem 4

Consider the differential equation $ay'' + by' + cy = 0$, with $a, b, c \in \mathbb{R}$, and $a \neq 0$. If r_1 and r_2 are the solutions of the **characteristic equation** $ar^2 + br + c = 0$, then:

(i) if r_1 and r_2 are both real and distinct, the general solution of the given ODE is $y = c_1e^{r_1t} + c_2e^{r_2t}$, with $c_1, c_2 \in \mathbb{R}$.

(ii) if $r := r_1 = r_2$ is a real double root, the general solution of the given ODE is $y = c_1e^{rt} + c_2te^{rt}$, with $c_1, c_2 \in \mathbb{R}$.

(iii) if r_1 and r_2 are complex (and hence conjugate to each other[a]), the general solution of the given ODE is $y = c_1e^{\alpha t}\cos(\beta t) + c_2e^{\alpha t}\sin(\beta t)$, with $c_1, c_2 \in \mathbb{R}$, where $r_1 = \alpha + i\beta$.

[a] Complex roots of a real polynomial always come in conjugate pairs.

Of course, memorizing the above is probably a waste of time. The idea is to **try** an exponential $y = e^{rt}$, and find the values of r for which this is indeed a solution. Namely, $y' = re^{rt}$ and $y'' = r^2e^{rt}$, so

$$0 = ay'' + by' + cy = ar^2e^{rt} + bre^{rt} + ce^{rt} = (ar^2 + br + c)e^{rt} \implies ar^2 + br + c = 0,$$

as $e^{rt} \neq 0$ may be cancelled.

Working case by case from here on saves you brain power. On the following examples, we will simultaneously explore this idea, as well as the fact that once suitable initial conditions have been imposed, the solution to the so-called **Initial Value Problem** ("IVP", for short) becomes unique.

Example 35 (Two distinct real roots)

(a) *Find the general solution of* $y'' - 3y' - 18y = 0$.

(b) *Find the unique solution with initial conditions* $y(0) = 0$ *and* $y'(0) = 4$.

For item (a), we start setting up the characteristic equation $r^2 - 3r - 18 = 0$. It may be factored as $(r - 6)(r + 3) = 0$, which says that the characteristic roots are $r_1 = 6$ and $r_2 = -3$. Therefore, we know that $y_1 = e^{6t}$ and $y_2 = e^{-3t}$ are two solutions. They are **linearly independent** because one is not a (real) multiple of the other. However, **since the given ODE is linear and homogeneous, the dimension of the space of solutions equals the order of the equation**. This says that taking linear combinations of y_1 and y_2 does, in fact, produce all solutions of this ODE. In other words, the general solution is

$$y = c_1e^{6t} + c_2e^{-3t}, \text{ with } c_1, c_2 \in \mathbb{R}.$$

For item (b), imposing **two** initial conditions (at the same point) allows us to solve for the **two** coefficients c_1 and c_2. The relations $y(0) = 0$ and $y'(0) = 4$ becomes the linear system

$$\begin{cases} c_1 + c_2 = 0 \\ 6c_1 - 3c_2 = 4 \end{cases} \implies c_1 = \frac{4}{9} \quad \text{and} \quad c_2 = -\frac{4}{9}.$$

The solution of the given IVP is

$$y = \frac{4}{9}e^{6t} - \frac{4}{9}e^{-3t}.$$

Let's see next what happens in the case where two complex conjugate roots appear. Two additional facts are crucial to understand this case:

Theorem 5 (Euler's Formula)

For any real number θ, we have $e^{i\theta} = \cos\theta + i\sin\theta$.

One can prove this by playing around with Taylor series and using that the radii of convergence of all functions involved are infinite. The second fact, which seems worthy to justify now, is:

Proposition 2

Real and imaginary parts of a **complex solution** to a **real** linear homogeneous ODE are **real solutions**.

Proof: Write your differential equation as $L(y) = 0$, where L is a linear differential operator. Say, in Example 35 we would have $L(y) = y'' - 3y' - 18y$. In any case, assume that $y_1 + \mathrm{i}y_2$ is a complex solution, i.e., y_1 and y_2 are real functions such that $L(y_1 + \mathrm{i}y_2) = 0$. Since the ODE is real and linear, we have the relation $L(y_1 + \mathrm{i}y_2) = L(y_1) + \mathrm{i}L(y_2)$. Now, a complex number equals zero if and only if both real and imaginary parts are also equal to zero, so we conclude that $L(y_1) = L(y_2) = 0$, which is to say precisely that y_1 and y_2 are solutions as well. □

The justificative above suggests that solving linear ODEs really amounts to computing the kernel/nullspace of a certain operator. We move on (phrasing things in a lazier way, but with the same content as in Example 35):

Example 36 (Complex conjugate roots)

Solve the IVP:

$$\begin{cases} y'' + 9y = 0 \\ y(0) = 8, \quad y'(0) = -8 \end{cases}$$

The strategy will always be the same: first find the general solution of the differential equation alone, and then use the given initial conditions to solve for c_1 and c_2. The characteristic equation is simply $r^2 + 9 = 0$, whose roots are $\pm 3\mathrm{i}$. We may just focus on one of them, say $3\mathrm{i}$. This says that $\mathrm{e}^{3\mathrm{i}t}$ is a complex solution. By Euler's Formula,

$$\mathrm{e}^{3\mathrm{i}t} = \cos(3t) + \mathrm{i}\sin(3t),$$

so Proposition 2 says that $y_1 = \cos(3t)$ and $y_2 = \sin(3t)$ are real solutions. They're clearly linearly independent, so the general solution of the differential equation is

$$y = c_1\cos(3t) + c_2\sin(3t), \quad c_1, c_2 \in \mathbb{R}.$$

With this in place, we move on to impose the initial conditions. Computing the derivative as $y' = -3c_1\sin(3t) + 3c_2\cos(3t)$, we see that $y(0) = 8$ and $y'(0) = -8$ together give us

$$c_1 = 8 \quad \text{and} \quad 3c_2 = -8,$$

so the unique solution to the IVP is

$$y = 8\cos(3t) - \frac{8}{3}\sin(3t).$$

There is one last case to study.

Example 37 (Real double root)

Solve the IVP:

$$\begin{cases} y'' - 2y' + y = 0 \\ y(0) = 4, \quad y'(0) = 0 \end{cases}$$

The characteristic equation of the given ODE in this case is just $r^2 - 2r + 1 = 0$, so that $(r-1)^2 = 0$ says that $r = 1$ is a real double root. Hence $y_1 = \mathrm{e}^t$ is one solution, but since the equation has order 2, we need a second solution y_2, linearly independent from y_1, to span the whole solution space via linear combinations. Indeed, repeating y_1 and writing $c_1\mathrm{e}^t + c_2\mathrm{e}^t$ just leads to $(c_1 + c_2)\mathrm{e}^t$, but $c_1 + c_2$ is as arbitrary as c_1 and c_2, so it really counts as one single degree of freedom. One attempt to create linear independence is to replace $c_1 + c_2$ with $c_1 + c_2 t$, which suggests that $y_2 = t\mathrm{e}^t$ works as a second solution independent from y_1 and that the general solution is

$$y = c_1\mathrm{e}^t + c_2 t\mathrm{e}^t, \quad c_1, c_2 \in \mathbb{R}.$$

This is indeed the case by Theorem 4. Imposing initial conditions works the same as the other cases, so we may compute $y' = c_1\mathrm{e}^t + c_2\mathrm{e}^t + c_2 t\mathrm{e}^t$ by the product rule. Thus, $y(0) = 4$ and $y'(0) = 0$ together read as the system

$$\begin{cases} c_1 = 4 \\ c_1 + c_2 = 0 \end{cases} \implies c_1 = 4 \quad \text{and} \quad c_2 = -4,$$

and the unique solution to the IVP is

$$y = 4\mathrm{e}^t - 4t\mathrm{e}^t.$$

As a last observation, this mechanism can be used to solve linear homogeneous ODEs with constant coefficients of **any order**, say

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0, \quad \text{with } a_1, \ldots, a_n \in \mathbb{R},\ a_n \neq 0,$$

provided one can solve the characteristic equation

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

instead. Once this equation has been completely factored, one builds the general solution from the factors obtained, using what has been discussed so far. Let's illustrate this with one more complicated last example.

Example 38 (Dealing with higher order)

Determine the general solution of a 5th order linear homogeneous ordinary differential equation whose characteristic equation is factored as

$$(r-2)(r-3)^2(r-(4+5\mathrm{i}))(r-(4-5\mathrm{i}))=0.$$

Let's understand each factor separately:

- The term $(r-2)$ provides $y_1 = \mathrm{e}^{2t}$.
- The term $(r-3)^2$ provides $y_2 = \mathrm{e}^{3t}$ and $y_3 = t\mathrm{e}^{3t}$.
- The term $(r-(4+5\mathrm{i}))$ provides the complex solution $\mathrm{e}^{(4+5\mathrm{i})t}$, so using Euler's formula to write

$$\mathrm{e}^{(4+5\mathrm{i})t} = \mathrm{e}^{4t}\mathrm{e}^{5\mathrm{i}t} = \mathrm{e}^{4t}(\cos(5t)+\mathrm{i}\sin(5t)) = \mathrm{e}^{4t}\cos(5t)+\mathrm{i}\mathrm{e}^{4t}\sin(5t)$$

gives us the real solutions $y_4 = \mathrm{e}^{4t}\cos(5t)$ and $y_5 = \mathrm{e}^{4t}\sin(5t)$.

We conclude that the general solution is

$$y = c_1\mathrm{e}^{2t}+c_2\mathrm{e}^{3t}+c_3t\mathrm{e}^{3t}+c_4\mathrm{e}^{4t}\cos(5t)+c_5\mathrm{e}^{4t}\sin(5t),$$

with $c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}$.

If you want an extra reference for these things, I particularly like Chapters 3 and 4 of [2].

12 March 29th

As a follow up from last class, we now move on to discuss **non-homogeneous** second order linear ODE's with constant coefficients. Namely, we consider

$$ay'' + by' + cy = f,$$

where $a, b, c \in \mathbb{R}$ with $a \neq 0$ (or else the equation would be of first order instead), but with the right side being equal to some arbitrary function f. Here's what we need to know about it:

Theorem 6

The general solution of $ay'' + by' + cy = f$ is $y = y_p + y_h$, where y_h is the general solution of the associated homogeneous equation $ay'' + by' + cy = 0$, and y_p is any particular solution of the original non-homogeneous equation.

In other words, to solve such a non-homogeneous equation, we first consider its homogeneous version and find y_h, by methods already studied. As for finding y_p, however, the answer is dissapointing: we have to guess it. Of course, by "guess" we mean a "reasonable" guess, in the sense that if f is a polynomial, trigonometric function, exponential, etc., we'll try to find y_p of the same type. This method of guessing **almost** always works. We will explore this in the next examples, and also see how could guessing could go wrong.

Example 39 (Polynomial non-homogeneous term)

Find the general solution of $y'' - 2y' - 15y = 3t^3 - 2t - 4$.

We will always proceed with two steps, first finding the general solution y_h of the associated homogeneous equation $y'' - 2y' - 15y = 0$. The characteristic equation is simply $r^2 - 2r - 15 = 0$, which may be factored as $(r-5)(r+3) = 0$. As this characteristic equation appeared by looking for the values of r for which e^{rt} was actually a solution of the homogeneous equation, we obtain two linearly independent solutions e^{5t} and e^{-3t}, so that $y_h = c_1 e^{5t} + c_2 e^{-3t}$, with $c_1, c_2 \in \mathbb{R}$. It remains to find y_p. As $3t^3 - 2t - 4$ is a polynomial of degree 3, we try to make $y_p = At^3 + Bt^2 + Ct + D$ a polynomial of degree 3 as well. The goal is to find A, B, C, and D, for which y_p is actually a solution of the original non-homogeneous equation. Plugging $y_p' = 3At^2 + 2Bt + C$ and $y_p'' = 6At + 2B$ into the original equation, and noting that $y_p'' - 2y_p' - 15y_p$ is a polynomial (as a linear combination of derivatives of a polynomial is a polynomial), we may compare coefficients in each degree to obtain the relations

$$\begin{cases} -15A & = 3 \\ -6A - 15B & = 0 \\ 6A - 4B - 15C & = -2 \\ 2B - 2C - 15D & = -4 \end{cases}$$

Now, you may use your vast knowledge of linear systems[a] to obtain

$$A = -\frac{1}{5}, \quad B = \frac{2}{25}, \quad C = \frac{4}{125}, \quad \text{and} \quad D = \frac{512}{1875}.$$

Hence we have that

$$y_p = -\frac{t^3}{5} + \frac{2t^2}{25} + \frac{4t}{125} + \frac{512}{1875}$$

and that the general solution of the original non-homogeneous equation is

$$y = \frac{t^3}{5} + \frac{2t^2}{25} + \frac{4t}{125} + \frac{512}{1875} + c_1 e^{5t} + c_2 e^{-3t},$$

with $c_1, c_2 \in \mathbb{R}$.

[a] ... or cheat like me and use a software ◡̈

Example 40 (Exponential non-homogeneous term)

Find the general solution of $y'' - 4y' - 32y = 6e^{-3t}$.

Let's start by finding the general solution y_h of the associated homogeneous equation $y'' - 4y' - 32y = 0$ first. Its characteristic equation is $r^2 - 4r - 32 = 0$, which may be factored as $(r-8)(r+4) = 0$. This implies, as usual, that we have $y_h = c_1 e^{8t} + c_2 e^{-4t}$, with $c_1, c_2 \in \mathbb{R}$. As the non-homogeneous term in the original equation is an exponential, we try $y_p = Ae^{-3t}$, and substitute it together with its derivatives $y_p' = -3Ae^{-3t}$ and $y_p'' = 9Ae^{-3t}$ into the original equation to find the value of A which makes y_p into a solution. We have that

$$9Ae^{-3t} - 4(-3Ae^{-3t}) - 32Ae^{-3t} = 6e^{-3t} \implies 9A + 12A - 32A = 6,$$

so $A = -6/11$, since e^{-3t} may be cancelled everywhere. Therefore, the particular solution is $y_p = (-6/11)e^{-3t}$, and the general solution of the original non-homogeneous equation is

$$y = -\frac{6}{11}e^{-3t} + c_1 e^{8t} + c_2 e^{-4t},$$

with $c_1, c_2 \in \mathbb{R}$.

Example 41 (Trigonometric non-homogeneous term)

Find the general solution of $y'' - y = 3\sin(2t)$.

We start considering the associated homogeneous equation $y'' - y = 0$, whose characteristic equation $r^2 - 1 = 0$ has roots 1 and -1, to obtain that its general solution is $y_h = c_1 e^t + c_2 e^{-t}$, with $c_1, c_2 \in \mathbb{R}$. Now, as the non-homogeneous term is trigonometric, we may try $y_p = A\sin(2t) + B\cos(2t)$. Of course one might expect that $B = 0$ as the right side of the equation doesn't have any y' term, but

for more complicated equations it is safer to keep both the sin and cos terms, as the derivative of each of them equals the other, and things balance themselves out (this is not a formal argument, but just a rough intuition). Anyway, we have that $y_p'' = -4A\sin(2t) - 4A\cos(2t)$, and thus

$$y_p'' - y_p = 3\sin(2t) \implies -5A\sin(2t) - 5B\cos(2t) = 3\sin(2t),$$

so that $A = -3/5$ and $B = 0$ (as expected). We conclude that the solution of the original non-homogeneous equation is

$$y = -\frac{3}{5}\sin(2t) + c_1 e^t + c_2 e^{-t},$$

with $c_1, c_2 \in \mathbb{R}$.

At this point, it is worth pointing out that if you were asked to solve an Initial Value Problem (IVP) with non-homogeneous ODE, you would have to, on top of everything already done, use the given initial conditions to find c_1 and c_2 (by solving a linear system). Here's a tricky example:

Example 42 (Resonance phenomenon)

Find the general solution of $y'' + y = 3\cos t$.

To find y_h, we consider $y'' + y = 0$ instead, whose characteristic equation is $r^2 + 1 = 0$, with roots $r = \pm i$. This says that e^{it} is a complex solution, but we would like to have two linearly independent real solutions instead. They are obtained by using Euler's formula to write $e^{it} = \cos t + i\sin t$, and using that the real and imaginary parts will be real solutions (as the equation is linear; revisit Proposition 2, p. 53). So, $y_h = c_1\cos t + c_2\sin t$, with $c_1, c_2 \in \mathbb{R}$. As for y_p, the obvious guess is $y_p = A\sin t + B\cos t$. Plugging this into the original non-homogeneous equation leads to a disaster: $0 = 3\cos t$, which is complete nonsense.

This disaster, however, is known as **resonance**. Namely, the reason why the initial guess has failed is because the right hand side (which in this case is $3\cos t$) was already a solution of the associated homogeneous equation (choose $c_1 = 0$ and $c_2 = 3$ in the formula for y_h). In vague terms, this creates an "artificial double-root effect" on the homogeneous equation, so in the same way that when r was a double root of the characteristic equation $ar^2 + br + c = 0$ we had to, in addition to the solution e^{rt}, consider a second solution te^{rt}, we multiply our old guess for y_p by t and try again.

Take two: let $y_p = At\sin t + Bt\cos t$. Plugging this into the original non-homogeneous equation leads (after 30 seconds or so of calculations) to

$$2A\cos t - 2B\sin t = 3\cos t \implies A = \frac{3}{2} \quad \text{and} \quad B = 0.$$

It was to be expected that terms with $t\cos t$ and $t\sin t$ should completely dissapear: the introduction of the extra factors of t was just meant to kill the resonance,

but the original right side did not have any such terms. In any case, we conclude that the general solution of the original non-homogeneous equation is

$$y = \frac{3t}{2}\sin t + c_1 \cos t + c_2 \sin t,$$

with $c_1, c_2 \in \mathbb{R}$.

Remark (Nightmare fuel)**.** It's possible for resonance to persist even after multiplying our initial guess by t. Imagine that instead of $y'' + y$, we had something like scary like

$$y^{(4)} + 2y'' + y = 3\cos t,$$

such that $3\cos t$ and $3t\cos t$ are both solutions of the associated homogeneous equation (verify it yourself, or try to understand how I reverse engineered it). The point is that the guess $At\sin t + Bt\sin t$ would also fail! Since crying is not an option, we go one step further and try $y_p = At^2\sin t + Bt^2\cos t$ instead. Exercise for the masochists: check that this works. So, what's the moral of the story? Keep multiplying your guesses by t until the resonance goes away. It is probably a terrible idea to try to identify immediately whether the given equation has resonance, and try to find the smallest power of t you need to use to kill it. Do not let this cause you any paranoia: try what you want to try, and if it doesn't work, just multiply by t.

Example 43

Find the general solution of $y''' - y' = \mathrm{e}^{2t} + \mathrm{e}^{3t}$.

Higher order linear ODEs with constant coefficients behave just like second order equations, with the only additional difficulty being actually solving the characteristic equation. In other words, there's nothing special about order 2, we just focus mostly on it for simplicity. The characteristic equation of $y''' - y' = 0$ is just $r^3 - r = 0$, which is factored as $r(r-1)(r+1) = 0$, so that $y_h = c_1 + c_2\mathrm{e}^t + c_3\mathrm{e}^{-t}$, with $c_1, c_2, c_3 \in \mathbb{R}$. Note that c_1 appears alone because it corresponds to $r = 0$ and $\mathrm{e}^{0t} = 1$. As for finding y_p, we try $y_p = A\mathrm{e}^{2t} + B\mathrm{e}^{3t}$, combining the separate guesses for each non-homogeneous term. Substituting this into the original equation leads to

$$6A\mathrm{e}^{2t} + 24B\mathrm{e}^{3t} = \mathrm{e}^{2t} + \mathrm{e}^{3t} \implies A = \frac{1}{6} \quad \text{and} \quad B = \frac{1}{24}.$$

We conclude that the general solution of the original non-homogeneous equation is

$$y = \frac{1}{6}\mathrm{e}^{2t} + \frac{1}{24}\mathrm{e}^{3t} + c_1 + c_2\mathrm{e}^t + c_3\mathrm{e}^{-t},$$

with $c_1, c_2, c_3 \in \mathbb{R}$.

13 April 5th

We did a review on everything that was discussed about ordinary differential equations so far, answering students' questions at random.

14 April 12th

Now, we move on to **Partial Differential Equations** ("PDE"s, for short). They are considerably harder to deal with than everything we have seen so far. We will try to get a feeling for what is going on by exploring the strategy to solve two types of problems:

- The **heat flow problem**:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \beta \dfrac{\partial^2 u}{\partial x^2}(x,t), & 0 < x < L,\ t > 0 \\ u(0,t) = u(L,0) = 0 & \\ u(x,0) = f(x), & 0 < x < L \end{cases}$$

  Here's what this is trying to model: we have a wire of length L, placed in the x-axis, with endpoints at $x = 0$ and $x = L$, the value $u(x,t)$ is the temperature of the wire at the point x at time t, and the constant β is the heat diffusivity of the wire's material. The function $f(x)$ describes the initial temperature distribution along the wire, and the condition $u(0,t) = u(L,t) = 0$ says we want to keep the temperature of the endpoints of the wire constant, and equal to 0°C. The point is that knowing the diffusivity and the initial temperature distribution, we can predict how the temperature will be distributed in the future, provided we keep the temperature at the endpoints of the wire always constant.

- The **vibrating string problem**:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}(x,t), & 0 < x < L,\ t > 0 \\ u(0,t) = u(L,0) = 0 & \\ u(x,0) = f(x) & 0 < x < L \\ \dfrac{\partial u}{\partial t}(x,0) = g(x) & 0 < x < L \end{cases}$$

  Here, we again have a string of length at least equal to L, positioned in the plane as to have its endpoints at the x-axis, at coordinates $x = 0$ and $x = L$. The initial position of the string is described by the graph of $f(x)$, while $g(x)$ is the initial (vertical) velocity of the string, which is about to start oscillating. The condition $u(0,t) = u(L,0) = 0$ means that the endpoints of the string will remain fixed throughout all of the motion to happen. Then $u(x,t)$ is the position (more precisely, the height), at time t, of the point which started at $(x, f(x))$. Given all this information, we can predict the whole motion of the string, in the sense that given any instant of time t and the initial position of a point in the string, we'll know where such point is located at time t. The PDE itself may be regarded as Newton's force law (force on the right side equals the acceleration at the left side).

The strategy for solving both of these problems starts with a simple technique, called **separation of variables**. Namely, we write $u(x,t) = X(x)T(t)$ for some single-variable functions X and T, and see what the PDE says about X and T. The reason this will work well here is because the PDE will give us ODEs for X and T instead — and ODEs we are able to solve, to boot.

We will start considering a concrete heat flow problem.

Example 44

Solve the following heat flow problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = 3\dfrac{\partial^2 u}{\partial x^2}(x,t), & 0 < x < \pi,\ t > 0 \\ u(0,t) = u(\pi,0) = 0 & \\ u(x,0) = \sin x - 6\sin(4x), & 0 < x < \pi \end{cases}$$

We start making a separation of variables, $u(x,t) = X(x)T(t)$. The PDE itself becomes $X(x)T'(t) = 3X''(x)T(t)$. We then have that

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{3T(t)} = -\lambda \in \mathbb{R},$$

as the left side doesn't depend on t and the right side doesn't depend on x. The reason for the negative sign on λ is a matter of convenience (in fact, as to make plus signs appear next). In any case, we have the equations

$$X''(x) + \lambda X(x) = 0 \quad \text{and} \quad T'(t) + 3\lambda T(t) = 0,$$

and we'll use the first one to find out which values of λ may actually occur when doing this procedure. To do so, we also need to see what the initial condition $u(0,t) = u(\pi,t) = 0$ means in terms of X and T. But this is easy: it becomes just the initial condition $X(0) = X(\pi) = 0$. So:

- If $\lambda < 0$, then $X(x) = c_1 e^{\sqrt{-\lambda}t} + c_2 e^{-\sqrt{\lambda}t}$, so $X(0) = X(\pi) = 0$ gives us that $c_1 = c_2 = 0$, and so $X = 0$, which is no good.
- If $\lambda = 0$, then we have $X(x) = c_1 + c_2 x$, so $X(0) = X(\pi) = 0$ gives us that $c_1 = c_2 = 0$, and so $X = 0$, which is again no good.
- If $\lambda > 0$, then $X(x) = c_1\cos(\sqrt{\lambda}t) + c_2\sin(\sqrt{\lambda}t)$. Now $X(0) = 0$ means that $c_1 = 0$, while $X(\pi) = 0$ says that $\sin(\pi\sqrt{\lambda}) = 0$ (provided $c_2 \neq 0$). This can only occur if $\pi\sqrt{\lambda} = n\pi$ for some integer n, meaning that we must have that $\lambda = n^2$.

So, for each integer n, we have found a solution $X_n(x) = a_n \sin(nx)$, where a_n is a real number (we will keep relabeling constants as needed). With this in place,

what is the corresponding $T_n(t)$? Solving the first order ODE $T_n'(t) = -3n^2 T_n(t)$ leads to $T_n(t) = b_n e^{-3n^2 t}$ for some second real constant b_n. With this in place, setting $c_n = a_n b_n$, we have that

$$u_n(x,t) = X_n(t)T_n(t) = c_n e^{-3n^2 t}\sin(nx).$$

For each n, this function u_n satisfies almost everything required to solve the heat flow problem, except for the initial distribution condition $u(x,0) = \sin x - 6\sin(4x)$. To achieve this, we consider the series $u = \sum_{n\geq 1} u_n$ and find the c_n's to make this work (as the PDE itself and the endpoint conditions are homogeneous, adding u_n's will also result in something satisfying them). Namely, setting

$$\sum_{n\geq 1} c_n e^{-3n^2 t}\sin(nx)\bigg|_{t=0} = \sum_{n\geq 1} c_n \sin(nx) = \sin x - 6\sin(4x)$$

tells us that $c_1 = 1$ and $c_4 = -6$ does the trick (with $c_n = 0$ for every n not equal to 1 or 4). The desired solution is

$$u(x,t) = e^{-3t}\sin x - 6e^{-48t}\sin(4x).$$

Note that, just like in the above example, when looking at possibilities for λ we will always have that $\lambda \leq 0$ leads to $X = 0$. When solving concrete problems, one could directly jump to the case where $\lambda > 0$, but it is still instructive to eliminate $\lambda \leq 0$ to keep track of what's going on in each step of the solution. If $f(x)$ is not a combination of sines, finding c_n's may be tricky. Here's where Fourier series come in: almost all functions $f(x)$ may be expressed as a Fourier series, and once this is done, reading the coefficients c_n becomes much easier. More on this later.

Next, we move on to a vibrating string problem:

Example 45

Solve the following vibrating string problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) = 3\dfrac{\partial^2 u}{\partial x^2}(x,t), & 0 < x < \pi,\ t > 0 \\ u(0,t) = u(\pi,0) = 0 & \\ u(x,0) = 6\sin(2x) + 2\sin(6x) & 0 < x < \pi \\ \dfrac{\partial u}{\partial t}(x,0) = 11\sin(9x) - 14\sin(15x) & 0 < x < \pi \end{cases}$$

We start making a separation of variables, $u(x,t) = X(x)T(t)$. The PDE itself becomes $X(x)T''(t) = 3X''(x)T(t)$. We then have that

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{3T(t)} = -\lambda \in \mathbb{R},$$

as the left side doesn't depend on t and the right side doesn't depend on x. This leads to

$$X''(x) + \lambda X(x) = 0 \quad \text{and} \quad T''(t) + 3\lambda T(t) = 0,$$

while $u(0,t) = u(\pi,t) = 0$ implies that $X(0) = X(L) = 0$. As with the heat flow problem, which values of λ may occur? There's still nothing different, so let's copy and paste previous work:

- If $\lambda < 0$, then $X(x) = c_1 e^{\sqrt{-\lambda}t} + c_2 e^{-\sqrt{\lambda}t}$, so $X(0) = X(\pi) = 0$ gives us that $c_1 = c_2 = 0$, and so $X = 0$, which is no good.
- If $\lambda = 0$, then we have $X(x) = c_1 + c_2 x$, so $X(0) = X(\pi) = 0$ gives us that $c_1 = c_2 = 0$, and so $X = 0$, which is again no good.
- If $\lambda > 0$, then $X(x) = c_1 \cos(\sqrt{\lambda}t) + c_2 \sin(\sqrt{\lambda}t)$. Now $X(0) = 0$ means that $c_1 = 0$, while $X(\pi) = 0$ says that $\sin(\pi\sqrt{\lambda}) = 0$ (provided $c_2 \neq 0$). This can only occur if $\pi\sqrt{\lambda} = n\pi$ for some integer n, meaning that we must have that $\lambda = n^2$.

Hence, for each integer n, we have found a solution $X_n(x) = c_n \sin(nx)$, where c_n is a real number. Finding the corresponding $T_n(t)$, however, will go in a slightly different way, since the ODE for $T_n(t)$ has second order instead of first order. Now, solving $T_n(t) + 3n^2 T_n(t) = 0$ leads to the characteristic equation $r^2 + 3n^2 = 0$, so that $r = \pm in\sqrt{3}$. We conclude that $T_n(t) = c_{n,1}\cos(n\sqrt{3}t) + c_{n,2}\sin(n\sqrt{3}t)$, for some constants $c_1, c_2 \in \mathbb{R}$. It follows that

$$\begin{aligned} u_n(x,t) &= X_n(x)T_n(t) \\ &= a_n \sin(nx)\left(c_{n,1}\cos(n\sqrt{3}t) + c_{n,2}\sin(n\sqrt{3}t)\right) \\ &= a_n \sin(nx)\cos(n\sqrt{3}t) + b_n \sin(nx)\sin(n\sqrt{3}t), \end{aligned}$$

where we reassemble constants, is a solution of the original PDE, satisfying the initial endpoint condition $u(0,t) = u(\pi,t) = 0$, but not necessarily the initial position and velocity conditions. We then consider the series $u = \sum_{n\geq 1} u_n$, and try to find the values of a_n and b_n so everything works. Namely, if

$$u(x,t) = \sum_{n\geq 1} a_n \sin(nx)\cos(n\sqrt{3}t) + b_n \sin(nx)\sin(n\sqrt{3}t),$$

then

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n\geq 1} -n\sqrt{3}a_n \sin(nx)\sin(n\sqrt{3}t) + n\sqrt{3}b_n \sin(nx)\cos(n\sqrt{3}t),$$

so

$$u(x,0) = \sum_{n\geq 1} a_n \sin(nx) = 6\sin(2x) + 2\sin(6x)$$

says we must have $a_2 = 6$ and $a_6 = 2$, with $a_n = 0$ for every n not equal to 2 or 6, while

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n\geq 1} n\sqrt{3}b_n \sin(nx) = 11\sin(9x) - 14\sin(15x)$$

says that we must have $9\sqrt{3}b_9 = 11$ and $15\sqrt{3}b_{15} = -14$, so that $b_9 = 11\sqrt{3}/27$ and $b_{15} = -14\sqrt{3}/45$, while $b_n = 0$ for every n not equal to 9 or 15.

Putting all of this together, we have that the solution to the given vibrating string problem is

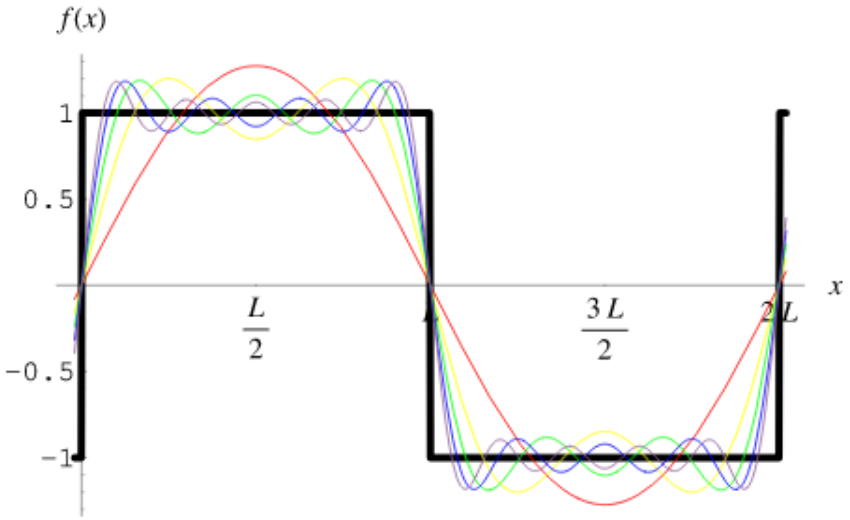
$$\begin{aligned} u(x,t) = {} & 6\sin(6x)\cos(2\sqrt{3}t) + 2\sin(6x)\cos(6\sqrt{3}t) \\ & + \frac{11\sqrt{3}}{27}\sin(9x)\sin(9\sqrt{3}t) - \frac{14\sqrt{3}}{45}\sin(15x)\sin(15\sqrt{3}t). \end{aligned}$$

15 April 19th

We finally explore a bit of Fourier series here. To get some first intuition, let's compare it with a type of series we have already studied before:

- **Taylor series:** its partial sums (Taylor polynomials) approximate a function, near a given point, with higher and higher degree polynomial expressions.
- **Fourier series:** its partial sum (trigonometric polynomials) approximate a function by superposing more and more "waves" of various frequencies.

Imagine something like the following picture[3]:



But how to actually compute them? The Fourier expansion[4] of a function $f(x)$ on a symmetric interval $[-L, L]$ is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n\geq 1}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right),$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \quad \text{and} \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x.$$

Here, the formula for a_n holds for $n = 0, 1, 2, \ldots$, while the formula for b_n holds for $n = 1, 2, \ldots$. There are many things to unpack here. First, there's no b_0 terms because trying to plug $n = 0$ into the formula for b_n gives just 0. Second, the coefficient a_0 plays a different role than the coefficients a_n for $n \geq 1$, and must always be addressed separately. One reason for this is that their "qualitative" behavior is very distinct. For example, the constant function 1 regarded as a wave has no amplitude, and its

[3] Taken from https://mathworld.wolfram.com/FourierSeriesSquareWave.html.

[4] "Fourier series" and "Fourier expansion" are used interchangeably. "Fourier transform" is something completely different, though.

usual antiderivative, x, is a polynomial, while $\cos(n\pi x/L)$ has varying amplitute and its usual antiderivative is trigonometric. The factor of 2 simply makes things work and is explained by general symmetry reasons we'll have opportunity to explore in detail soon. Lastly, when expressing the Fourier series of a function, one uses the approximate sign $\sim$ instead of an equality sign, as a reminder that convergence issues of Fourier series are more subtle than convergence issues for, say, Taylor series.

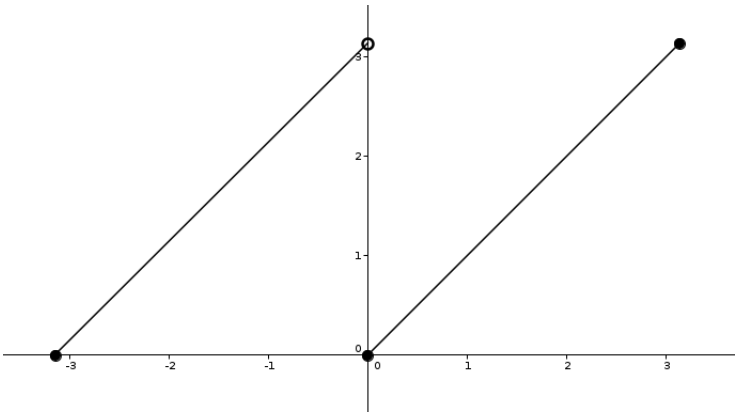
It's also convenient to note that while the argument $n\pi x/L$ doesn't look exactly friendly, in most problems we have to deal with, the number L will be an integer multiple of π, which makes things more tractable.

Example 46

Compute the Fourier expansion of the piecewise function f given by

$$f(x) = \begin{cases} x, & \text{if } 0 \leq x \leq \pi, \\ x + \pi, & \text{if } -\pi \leq x < 0. \end{cases}$$

We may start trying to get some intution for what the graph of this function looks like. See the next figure.



We must simply compute a_0, a_n, and b_n, and insert the results in a series. From the figure, we immediately have that

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\cdot\frac{\pi^2}{2}\cdot 2 = \pi,$$

as we know that integrals of positive functions compute areas under graphs, and we have two triangles with both base and height equal to π. As for a_n and b_n with $n \geq 1$, we must break the integral from $-\pi$ to π into two integrals, so we can actually use the concrete expressions for $f(x)$ (on each subinterval) given to us. For example:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x$$

$$
\begin{aligned}
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\cos(nx)\,\mathrm{d}x + \int_{0}^{\pi} f(x)\cos(nx)\,\mathrm{d}x\right)\\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} (x+\pi)\cos(nx)\,\mathrm{d}x + \int_{0}^{\pi} x\cos(nx)\,\mathrm{d}x\right)\\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} \pi\cos(nx)\,\mathrm{d}x + \cancel{\int_{-\pi}^{\pi} x\cos(nx)\,\mathrm{d}x}\right)\\
&= \int_{-\pi}^{0} \cos(nx)\,\mathrm{d}x\\
&= \frac{\sin(nx)}{n}\Bigg|_{-\pi}^{0}\\
&= 0.
\end{aligned}
$$

Again, breaking the original integral into two was needed so we could use the concrete expressions given for $f(x)$ on each interval. Being able to join things back together on the $\int_{-\pi}^{\pi} x\cos(nx)\,\mathrm{d}x$ was a convenient coincidence due to the fact that x appeared in both expressions defining $f(x)$. This integral is zero for symmetry reasons: **the integral of an odd function over a symmetric interval vanishes** (namely, x is odd and $\cos(nx)$ is even, so the product $x\cos(nx)$ is odd). Dealing with b_n's is similar, this time using that **the integral of an even function over a symmetric interval equals twice the integral over the right (or left) half of the interval**. We have that:

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x\\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\sin(nx)\,\mathrm{d}x + \int_{0}^{\pi} f(x)\sin(nx)\,\mathrm{d}x\right)\\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} (x+\pi)\sin(nx)\,\mathrm{d}x + \int_{0}^{\pi} x\sin(nx)\,\mathrm{d}x\right)\\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} \pi\sin(nx)\,\mathrm{d}x + \int_{-\pi}^{\pi} x\sin(nx)\,\mathrm{d}x\right)\\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} \pi\sin(nx)\,\mathrm{d}x + 2\int_{0}^{\pi} x\sin(nx)\,\mathrm{d}x\right)\\
&= \frac{1}{\pi}\left(-\frac{\pi}{n}\cos(nx)\Big|_{-\pi}^{0} + 2\left(-\frac{x}{n}\cos(nx)\Big|_{0}^{\pi} + \frac{1}{n}\int_{0}^{\pi}\cos(nx)\,\mathrm{d}x\right)\right)\\
&= \frac{1}{\pi}\left(-\frac{\pi}{n}(1-(-1)^n) + 2\left(-\frac{\pi}{n}(-1)^n + \cancel{\frac{\sin(nx)}{n^2}\Big|_{0}^{\pi}}\right)\right)\\
&= -\frac{1}{n}(1-(-1)^n + 2(-1)^n)\\
&= -\frac{(1+(-1)^n)}{n}
\end{aligned}
$$

by using that $\cos(n\pi) = (-1)^n$ for any integer n. In green, we have also used integration by parts with

$$u = x \qquad \mathrm{d}v = \sin(nx)\,\mathrm{d}x$$
$$v = -\frac{1}{n}\cos(nx) \qquad \mathrm{d}u = \mathrm{d}x$$

In any case, we have obtained the Fourier expansion

$$f(x) \sim \frac{\pi}{2} - \sum_{n\geq 1} \frac{1+(-1)^n}{n}\sin(nx) \quad \text{on } [-\pi,\pi],$$

as desired. Since $1+(-1)^n$ equals 0 when n is odd and 2 when n is even, we may set $n = 2k$ and rewrite our answer as

$$f(x) \sim \frac{\pi}{2} - \sum_{k\geq 1} \frac{1}{k}\sin(2kx) \quad \text{on } [-\pi,\pi],$$

after simplifying $2/(2k) = 1/k$.

In the above example, we were able to find the Fourier expansion of a function $f(x)$ which was not continuous (namely, the figure shows a jump discontinuity at $x = 0$). This is another important difference between Taylor series and Fourier series. For Taylor series, the function must have all derivatives existing at the chosen center point, while a Fourier series does not require the choice of a center point (although one could reasonably argue that the center in this case is 0) or derivatives to exist. The only thing we must be able to do is to compute the relevant integrals, but integrals are insentitive to a countable number of discontinuities.

When finding Fourier expansions, it is important to actually set up the series as the final answer. While, of course, the bulk of the work is computing a_0, a_n, and b_n, requiring the series as the final answer is more than just simple nagging. When trying to apply these techniques to solve heat flow problems (and vibrating string problems, although we won't get into this) and thinking of the **prototype solution**

$$u(x,t) = \sum_{n\geq 1} c_n \mathrm{e}^{-\beta(n\pi/L)^2 t}\sin\left(\frac{n\pi x}{L}\right),$$

we would like to compare the series

$$u(x,0) = \sum_{n\geq 1} c_n \sin\left(\frac{n\pi x}{L}\right)$$

with **another series** – this leads us to the conclusion that the c_n's must be the Fourier coefficients of the initial temperature distribution function.

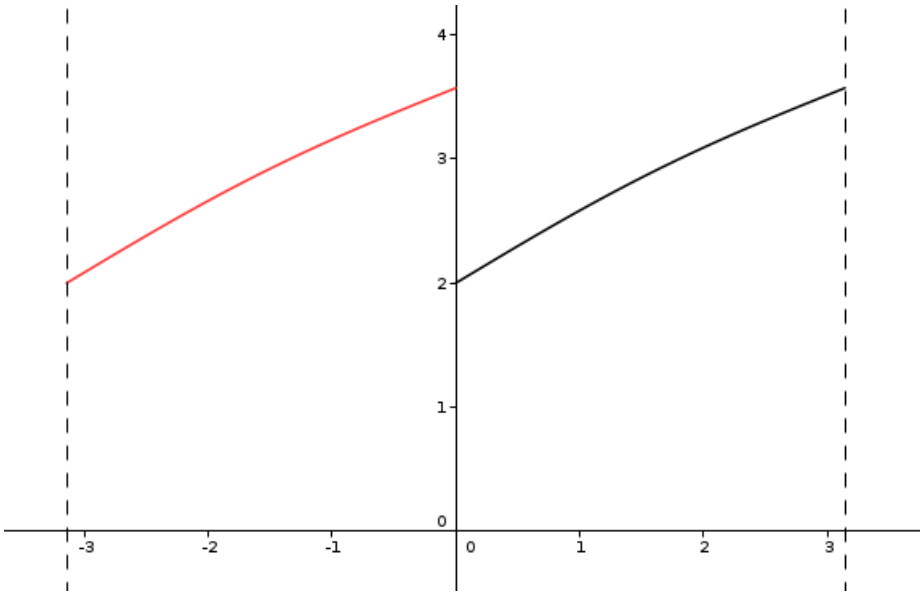
One issue, which is crucial to address here, is that the above series only has sine terms, while a generic Fourier series has both sine and cosine terms. Moreover, the Fourier expansions initially considered were happening on the symmetric interval $[-L, L]$, while heat flow problems happen on the domain $[0, L]\times[0,\infty)$ for (x,t).

Question: How do we deal with Fourier expansions when the given function $f(x)$ is only defined (or a priori considered) on the interval $[0, L]$, as opposed to $[-L, L]$?

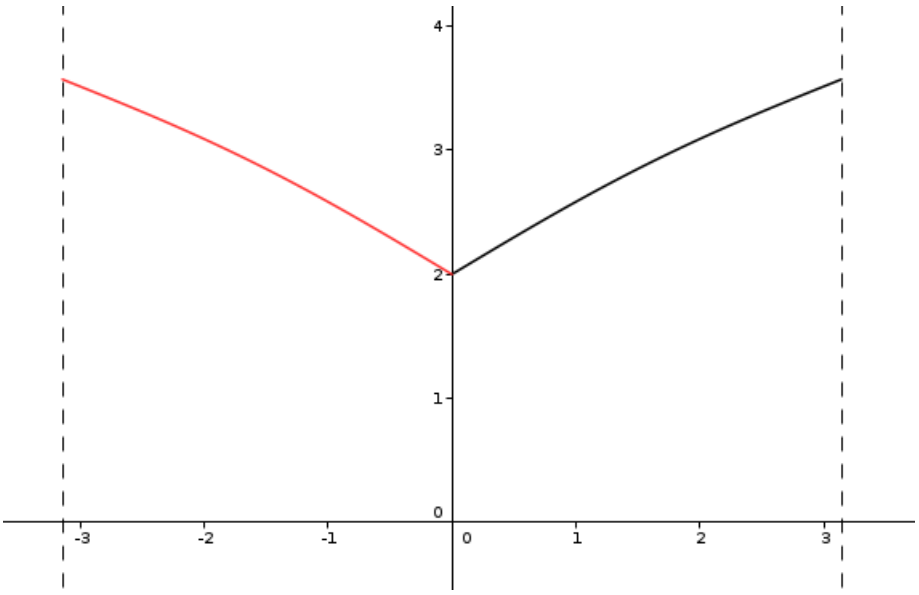
Answer: We first **extend** the function $f(x)$ to a function $\widetilde{f}(x)$ on $[-L, L]$, compute the Fourier expansion of $\widetilde{f}(x)$ on $[-L, L]$, and then restrict it back to the original interval $[0, L]$.

While this idea, taken at face value, is simple enough, the problem is that there is not a unique way to extend the function f. There are three main ways to do it:

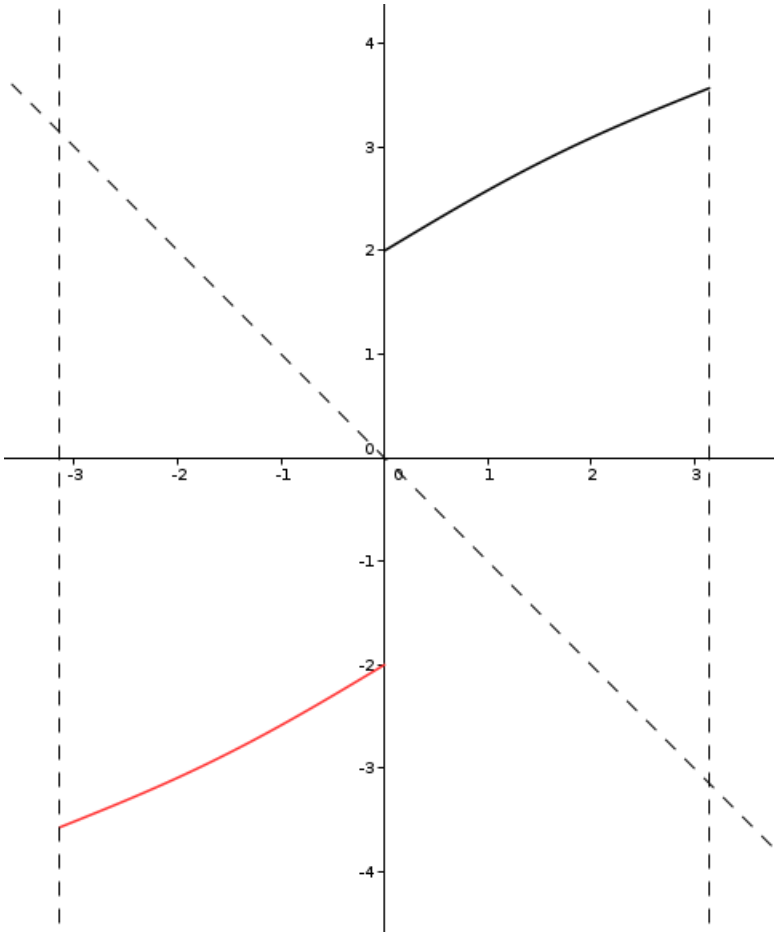
- **Periodic extension.** Copy and paste the function defined on $[0, L]$ to the interval $[-L, 0]$:



- **Even extension.** Flip it across the y-axis:



- **Odd extension.** Reflect the function across the origin:



This is a good point to observe that symmetries of a function reflect into symmetries of its Fourier series. More precisely, the Fourier series of an even function defined on a symmetric interval has no odd sine terms (i.e., $b_n = 0$), while the Fourier series of an odd function defined on a symmetric interval has no even cosine terms (i.e., $a_n = 0$). This gives rise to **Fourier sine series** and **Fourier cosine series** of a function defined on a half-interval $[0, L]$ (this is called a "half-expansion". This happens even if the function to be extended were to have a "natural" continuous extension to an even or odd function.

Different extensions serve different purposes. For example, for solving heat flow problems, we use odd extensions as we want the resulting Fourier series to consist only of sine terms, so we can easily read the coefficients c_n from the prototype solution. If the endpoint condition $u(0,t) = u(L,t) = 0$ were replaced with an endpoint condition $(\partial u/\partial t)(0,t) = (\partial u/\partial t)(L,t) = 0$ on the time derivative instead, we would have to use even extensions and Fourier cosine series, as the prototype solution would have been modified accordingly[5].

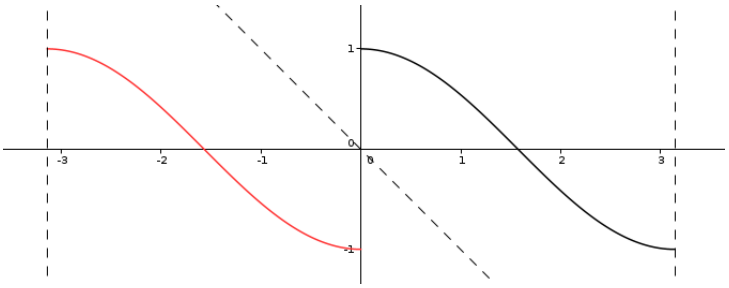
With this in place, let's see what happens with a concrete example:

[5] **Exercise:** what are the values of λ and solutions X to the equation $X''(x) + \lambda X(x) = 0$ subject to $X'(0) = X'(L) = 0$ instead of $X(0) = X(\pi) = 0$?

Example 47

Determine the Fourier sine-expansion of $\cos x$ *on* $[0, \pi]$.

Let's start by understanding what is the odd extension $\widetilde{\cos x}$ of $\cos x$ looks like (and noting that the fact that $\cos x$, **when considered on the full interval** $[-\pi, \pi]$ **to begin with**, was already even, is irrelevant):



In this particular case, the periodic extension and the odd extension agree. We already know that $a_n = 0$ for all n by symmetry reasons. As for the coefficients b_n, we have that

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \widetilde{\cos x}\sin(nx)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} \widetilde{\cos x}\sin(nx)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} \cos x \sin(nx)\,\mathrm{d}x,$$

since $\widetilde{\cos x}\sin(nx)$ is even (as the product of the two odd functions $\widetilde{\cos x}$ and $\sin(nx)$), and $\widetilde{\cos x} = \cos x$ for x in the right interval $[0, \pi]$ (this is what it means to say that $\widetilde{\cos x}$ is an **extension** of $\cos x$). Of course, when solving problems like this, you don't have to go and write the above step carrying the extension of the given function: go ahead and compute b_n (or a_n, for even extensions) with the extra coefficient of 2 and the integral being carried only over the "right" interval. The point of doing this step here is illustrating that while the textbook [1] presents different formulas for Fourier expansions, Fourier sine-expansions, and Fourier-cosine expansion, they're really the same thing, and no extra effort on memorizing things should be made.

In any case, it remains to compute this integral. For that, we must rely on product-to-sum trigonometric identities:

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} \cos x \sin(nx)\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}(\sin((n+1)x) + \sin((n-1)x))\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} \sin((n+1)x) + \sin((n-1)x)\,\mathrm{d}x \\
&= -\frac{1}{\pi}\left(\frac{\cos((n+1)x)}{n+1} + \frac{\cos((n-1)x)}{n-1}\right)\Bigg|_0^{\pi} \\
&= -\frac{1}{\pi}\left(\frac{(-1)^{n+1}-1}{n+1} + \frac{(-1)^{n-1}-1}{n-1}\right)
\end{aligned}$$

$$\overset{(*)}{=} \frac{1+(-1)^n}{\pi}\left(\frac{1}{n+1}+\frac{1}{n-1}\right)$$
$$= \frac{2n(1+(-1)^n)}{\pi(n^2-1)},$$

where in $(*)$ we have used that $(-1)^{n+1} = (-1)^{n-1}$ (as the powers differ by an even number) and distributed the negative sign to carry the simplification. This means that we have

$$\cos x \sim \sum_{n\geq 1} \frac{2(1+(-1)^n)n}{\pi(n^2-1)}\sin(nx) \quad \text{on } [0,\pi].$$

Noting that $1+(-1)^n$ is zero when n is odd and 2 when n is even, we may set $n = 2k$ and rewrite the answer as

$$\cos x \sim \sum_{k\geq 1} \frac{8k}{\pi(4k^2-1)}\sin(2k\pi) \quad \text{on } [0,\pi].$$

We conclude this course by seeing how things come full circle:

Example 48

Solve the following heat flow problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = 10\dfrac{\partial^2 u}{\partial x^2}(x,t), & 0<x<\pi,\ t>0 \\ u(0,t) = u(\pi,0) = 0, & \\ u(x,0) = \cos x, & 0<x<\pi \end{cases}$$

We start making a separation of variables, $u(x,t) = X(x)T(t)$. The PDE itself becomes $X(x)T'(t) = 10X''(x)T(t)$. We then have that

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{10T(t)} = -\lambda \in \mathbb{R},$$

as the left side doesn't depend on t and the right side doesn't depend on x. We obtain

$$X''(x) + \lambda X(x) = 0 \quad \text{and} \quad T'(t) + 3\lambda T(t) = 0,$$

and the endpoint conditions on u read $X(0) = X(\pi) = 0$. From here, it follows that $\lambda = n^2$ for some natural number $n \geq 1$, and $X_n(x) = a_n \sin(nx)$ for some real number a_n. Finding the corresponding $T_n(t)$, we solve the first order ODE $T_n'(t) = -10n^2 T_n(t)$: the solution is $T_n(t) = b_n e^{-10n^2 t}$ for some second real constant b_n. With this in place, setting $c_n = a_n b_n$, we have that

$$u_n(x,t) = X_n(t)T_n(t) = c_n e^{-10n^2 t}\sin(nx).$$

For each n, this function u_n satisfies almost everything required to solve the heat flow problem, except for the initial distribution condition $u(x,0) = \cos x$. To achieve this, we consider the series $u = \sum_{n\geq 1} u_n$ and find the c_n's to make this work. In other words, we have that

$$\sum_{n\geq 1} c_n e^{-10n^2 t}\sin(nx)\bigg|_{t=0} = \sum_{n\geq 1} c_n \sin(nx) = \cos x.$$

We have already seen that

$$\cos x \sim \sum_{n\geq 1} \frac{2(1+(-1)^n)n}{\pi(n^2-1)}\sin(nx) \quad \text{on } [0,\pi],$$

so that

$$c_n = \frac{2(1+(-1)^n)n}{\pi(n^2-1)}$$

for every $n \geq 1$, and we conclude that the solution to the given heat flow problem is

$$u(x,t) = \sum_{n\geq 1} \frac{2(1+(-1)^n)n}{\pi(n^2-1)} e^{-10n^2 t}\sin(nx).$$

Using again that $1+(-1)^n$ is zero when n is odd and 2 when n is even, we may set $n = 2k$ and rewrite the answer as

$$u(x,t) = \sum_{k\geq 1} \frac{8k}{\pi(4k^2-1)} e^{-40k^2 t}\sin(2k\pi).$$

"Maybe our paths will cross
when this universe folds in and makes another.
Maybe, at the point

when all that is, and all that's ever been,
collapses into everything else and is remade,
our paths will cross, however briefly, and

our terminus become a junction.
It may be a long shot. I will take it
and hope and trust our paths will cross again."

OLIVER TEARLY — "EPILOGUE"

References

[1] **Math 2177, Third Custom Edition for OSU**, Pearson. ISBN 10: 0-13-720383-7.

[2] Kreider, Kuller, Ostberg, Perkins; **An Introduction to Linear Analysis**, Addison-Wesley Publishing Company, Inc., Reading, Massachussets, U.S.A., 1966.