On compact Cotton-parallel three-manifolds

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These slides can also be found at

https://www.asc.ohio-state.edu/terekcouto.1/texts/JMM_slides_january2024.pdf

Conformal flatness

We start by recalling the notion of conformal flatness:

Definition

A pseudo-Riemannian manifold (M, g) is conformally flat if every $x \in M$ has an open neighborhood $U \subseteq M$ and a smooth function $\rho\colon U \to \mathbb{R}$ such that the manifold $(U, \mathrm{e}^{2\rho}\mathrm{g})$ is flat.

If the signature of g is (p, q), $p + q = n$, then the conformal class of g determines a reduction of the structure group of the frame bundle of M from $\mathrm{GL}(n)$ to $\mathrm{CO}(p, q)$. Conformal flatness of (M, g) amounts to integrability of this $\mathrm{CO}(p, q)$-structure.

When $\dim M \geq 4$, conformal flatness is controlled by the Weyl curvature tensor W: (M, g) is conformally flat if and only if $\mathrm{W} = 0$.

In the case where $\dim M = 3$, it is controlled by the Cotton tensor C instead: (M, g) is conformally flat if and only if $\mathrm{C} = 0$.

The Cotton tensor

But what is the Cotton tensor?

Definition

Let (M, g) be an n-dimensional pseudo-Riemannian manifold. Then:

(a) The Schouten tensor is P given by

$$\mathrm{P} = \mathrm{Ric} - \frac{\mathrm{scal}}{2(n-1)}\mathrm{g}.$$

(b) The Cotton tensor is C given by

$$\mathrm{C}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = (\nabla_{\mathbf{X}}\mathrm{P})(\mathbf{Y}, \mathbf{Z}) - (\nabla_{\mathbf{Y}}\mathrm{P})(\mathbf{X}, \mathbf{Z}),$$

for all vector fields $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathfrak{X}(M)$.

Note: P is the "$\mathrm{div} - \mathrm{d} \circ \mathrm{tr}$"-less part of Ric, and $\mathrm{C} = \mathrm{d}^{\nabla}\mathrm{P}$.

Properties of C

Here are the three properties characteristic to C:

- $\mathrm{C}(\mathbf{X},\mathbf{Y},\mathbf{Z})+\mathrm{C}(\mathbf{Y},\mathbf{X},\mathbf{Z})=0$; (clear)
- $\mathrm{C}(\mathbf{X},\mathbf{Y},\mathbf{Z})+\mathrm{C}(\mathbf{Y},\mathbf{Z},\mathbf{X})+\mathrm{C}(\mathbf{Z},\mathbf{X},\mathbf{Y})=0$; (6 terms cancel in pairs)
- $\mathrm{tr}_{\mathrm{g}}\big((\mathbf{X},\mathbf{Z})\mapsto\mathrm{C}(\mathbf{X},\mathbf{Y},\mathbf{Z})\big)=0$. ($\mathrm{div}\,\mathrm{P}=\mathrm{d}(\mathrm{tr}_{\mathrm{g}}\mathrm{P})$ in disguise)

With this in place, the condition we will discuss today, focusing on the three-dimensional case, is $\nabla\mathrm{C}=0$.

A three-dimensional pseudo-Riemannian manifold (M,g) is sometimes called essentially conformally symmetric if $\nabla\mathrm{C}=0$, but $\mathrm{C}\neq 0$.

We begin with an example:

Example (Conformally symmetric pp-wave manifold)

Given any smooth function $\mathfrak{a}\colon \mathbb{R} \to \mathbb{R}$, consider the Lorentzian manifold

$$(\widehat{M}, \widehat{\mathrm{g}}) = \big(\mathbb{R}^3, (x^3 + \mathfrak{a}(t)x)\mathrm{d}t^2 + \mathrm{d}t\,\mathrm{d}s + \mathrm{d}x^2\big).$$

Important facts about $(\widehat{M}, \widehat{\mathrm{g}})$:

- ∂_s is a null parallel field, spanning a rank-one distribution $\mathcal{D}$;
- $\mathrm{Ric} = -3x\,\mathrm{d}t \otimes \mathrm{d}t$, and the Ricci operator $-6x\,\mathrm{d}t \otimes \partial_s$ is $\mathcal{D}$-valued;
- $\mathrm{C} = 3(\mathrm{d}t \wedge \mathrm{d}x) \otimes \mathrm{d}t$, and so $\mathcal{D}_p = \{u \in T_p\widehat{M} \mid \mathrm{C}_p(u, \cdot, \cdot) = 0\}$.

Finally, $\mathrm{Iso}(\widehat{M}, \widehat{\mathrm{g}})$ is isomorphic to the subgroup of $\mathbb{Z}_2 \ltimes_\rho \mathbb{R}^2$ (where $\rho(-1) = -\mathrm{Id}_{\mathbb{R}^2}$) consisting of the triples (ε, p, r) with $\mathfrak{a}(\varepsilon t + p) = \mathfrak{a}(t)$, acting on $\widehat{M}$ via $(\varepsilon, p, r) \cdot (t, s, x) = (\varepsilon t + p, \varepsilon s + r, x)$.

There are no subgroups $\Gamma \leq \mathrm{Iso}(\widehat{M}, \widehat{\mathrm{g}})$ producing compact quotients $\widehat{M}/\Gamma$, as $(t, s, x) \mapsto x$ would induce a continuous unbounded function $\widehat{M}/\Gamma \to \mathbb{R}$.

Example (Conformally symmetric pp-wave manifold)

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The above example is locally universal [1]:

Theorem (García-Río et al., 2014)

Let (M, g) be a pseudo-Riemannian three-manifold satisfying $\nabla\mathrm{C} = 0$ and $\mathrm{C} \neq 0$. Then, reversing g if needed, any point in M has a neighborhood isometric to an open subset of $(\widehat{M}, \widehat{\mathrm{g}})$, for some suitable choice of $\mathfrak{a}$.

The structure theorem's proof sketch

Their argument consisted in three big steps:

Step 1: Proving that (M, g) must carry a null parallel distribution $\mathcal{D}$, by discussing the possibilities for the multiplicities of the eigenvalues of the Cotton operator (associated with the Cotton-York density, specific to $\dim M = 3$).

Step 2: Invoking Walker's Theorem about canonical coordinates adapted to degenerate parallel distributions on pseudo-Riemannian manifolds. For rank-one null parallel distributions on three-manifolds, we have

$$\mathrm{g} = \kappa(t, s, x)\,\mathrm{d}t^2 + \mathrm{d}t\,\mathrm{d}s + \mathrm{d}x^2.$$

Step 3: Solving the PDE for κ corresponding to the condition $\nabla \mathrm{C} = 0$ for the above metric, and then performing a sequence of suitable coordinate changes, it follows that

$$\mathrm{g} = (x^3 + \mathfrak{a}(t)x)\,\mathrm{d}t^2 + \mathrm{d}t\,\mathrm{d}s + \mathrm{d}x^2.$$

General conclusions

As a consequence of this local structure theorem,

everything that holds on $(\widehat{M}, \widehat{\mathrm{g}})$, holds locally on (M, g).

Here are some explicit conclusions:

- (M, g) must be Lorentzian (or anti-Lorentzian);
- the distinguished null parallel rank-one distribution $\mathcal{D}$ associated with (M, g) via C is explicitly given by $\mathcal{D}_x = \{u \in T_xM \mid \mathrm{C}_x(u, \cdot, \cdot) = 0\}$, for every $x \in M$.
- the connection induced by (M, g) in the distribution $\mathcal{D}$ is flat.
- the Ricci operator of (M, g) is $\mathcal{D}$-valued;
- the scalar curvature of (M, g) vanishes.

The algebraic structure of C

Inspired by the expression $\mathrm{C} = 3(\mathrm{d}t \wedge \mathrm{d}x) \otimes \mathrm{d}t$, valid in $(\widehat{M}, \widehat{\mathrm{g}})$, we have:

Lemma

Let $(V, \langle \cdot, \cdot \rangle)$ be a three-dimensional pseudo-Euclidean vector space, $\mathrm{C} \neq 0$ be a Cotton-like tensor on V, and $\mathcal{D} = \{u \in V \mid \mathrm{C}(u, \cdot, \cdot) = 0\}$. Then:

(a) *$\mathcal{D}$ consists only of null vectors, and thus $\dim \mathcal{D} \leq 1$.*

(b) *$\dim \mathcal{D} = 1$ if and only if $\mathrm{C} = (u \wedge v) \otimes u$, for some $u \in \mathcal{D} \smallsetminus \{0\}$ and unit vector $v \in \mathcal{D}^{\perp}$.*

(c) *In (b), u is unique up to sign and v is unique modulo $\mathcal{D}$.*

Proof.

Linear Algebra ◡̈ □

The main result

Theorem (T., 2023)

A compact three-dimensional pseudo-Riemannian manifold with parallel Cotton tensor must be conformally flat.

Proof.

Let (M^3, g) have $\nabla \mathrm{C} = 0$, but $\mathrm{C} \neq 0$.

Pull back all the geometry of (M, g) to its universal covering manifold $\widetilde{M}$, so that the covering projection $\pi \colon \widetilde{M} \to M$ becomes a local isometry between $(\widetilde{M}, \widetilde{\mathrm{g}})$ and (M, g).

Write $M = \widetilde{M}/\Gamma$, for a group $\Gamma \cong \pi_1 M$ acting freely and properly discontinuously on $(\widetilde{M}, \widetilde{\mathrm{g}})$ by deck isometries.

As $\widetilde{M}$ is simply-connected, there is a smooth vector field $\mathbf{u}$ and a rough vector field $\mathbf{v}$ such that $\mathrm{C} = (\mathbf{u} \wedge \mathbf{v}) \otimes \mathbf{u}$ on $\widetilde{M}$.

The main result

Proof. (cont'd)

As $\mathcal{D}$ is parallel, item (c) of our previous Lemma tells us that:

(i) $\mathbf{u}$ is a null parallel field spanning $\mathcal{D}$;

(ii) every $\gamma \in \Gamma$ pushes $\mathbf{u}$ forward onto either $\mathbf{u}$ or $-\mathbf{u}$.

Next, the fact that Ric is self-adjoint and $\mathcal{D}$-valued allows us to write $\mathrm{Ric} = -f\, \mathbf{u} \otimes \mathbf{u}$, for some smooth function $f \colon \widetilde{M} \to \mathbb{R}$.

On the other hand, it follows that $\mathrm{C} = (\mathbf{u} \wedge \nabla f) \otimes \mathbf{u}$.

Now, Γ-invariance of Ric and of $\mathbf{u} \otimes \mathbf{u}$ implies Γ-invariance of f.

Hence, f survives as a smooth function on the quotient $\widetilde{M}/\Gamma = M$.

If M were compact, f would have a critical point x: then $(\nabla f)_x = 0$ means that $\mathrm{C}_x = 0$, and so $\mathrm{C} = 0$ (because $\nabla \mathrm{C} = 0$).

□

References

[1] E. Calviño Louzao, E. García-Río, J. Seoane-Bascoy, R. Vásquez-Lorenzo, **Three-dimensional conformally symmetric manifolds**, Ann. Mat. Pura Appl. (4) **196** (2014), no. 6, pp. 1661–1670.

[2] I. Terek, **Conformal flatness of compact three-dimensional Cotton-parallel manifolds**, Proc. Amer. Math. Soc., vol. **152** (2024), no. 2, pp. 797–800.

[3] A. G. Walker, **Canonical forms II. Parallel partially null planes**, Quart. J. Math. Oxford Ser. (2), 1:147–152, 1950.

Thank you for your attention!

(scan here for more on my research)