A magnetic version of E. Hopf's theorem

(joint work with Valerio Assenza and James Marshall Reber)

Ivo Terek

Williams College

January 8th, 2025
1:00 p.m. – 1:20 p.m. PST

AMS Special Session on Metric Geometry and Topology, II
2025 Joint Mathematics Meeting
January 8th to January 11th, 2025, Seattle, WA

These slides can also be found at

https://www.web.williams.edu/it3/texts/JMM_slides_january2025.pdf

- V. Assenza, J. Marshall Reber, I. Terek; *Magnetic flatness and E. Hopf's Theorem for magnetic systems*, arXiv 2404.17726. To appear in **Communications in Mathematical Physics**.

Review: Geodesic flow

Let (M, g) be a **compact and connected** Riemannian manifold.

The geodesic equation

$$\frac{\mathrm{D}\dot{\gamma}}{\mathrm{d}t}(t) = 0$$

induces a flow $\Phi^{\mathrm{g}} \colon \mathbb{R} \times TM \to TM$ on the tangent bundle

$$TM = \{(x, v) : x \in M \text{ and } v \in T_x M\}.$$

It is given by $\Phi^{\mathrm{g}}(t, (x, v)) = (\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t))$, where $\gamma_{(x,v)} \colon \mathbb{R} \to M$ is the unique solution of the IVP

$$\begin{cases} \dfrac{\mathrm{D}\dot{\gamma}}{\mathrm{d}t}(t) = 0 \\ \gamma(0) = x, \quad \dot{\gamma}(0) = v. \end{cases}$$

Review: Geodesic flow

Each sphere bundle

$$\Sigma_s = \{(x, v) \in TM : \|v\| = s\}, \qquad s > 0,$$

is Φ^{g}-invariant, and so Φ^{g} restricts to a flow $\Phi^{\mathrm{g}}_s \colon \mathbb{R} \times \Sigma_s \to \Sigma_s$.

The dynamics of Φ^{g}_s is closely related to the geometry of (M, g), and has interesting properties when (M, g) is negatively curved (Anosov, Sinai, Arnold, Avez — 1967):

- Closed geodesics have vanishing Morse index.
- $\mathrm{Iso}(M, \mathrm{g})$ is finite.
- (M, g) has no conjugate points.
- Φ^{g}_s is of Anosov type (there is a $\mathrm{d}\Phi^{\mathrm{g}}_s$-invariant center-stable-unstable splitting $T\Sigma_s = \mathbb{R}\mathbf{X}^{\mathrm{g}} \oplus E^s \oplus E^u$).
- Φ^{g}_s is ergodic (for the so-called Liouville measure on Σ_s)
- Φ^{g}_s has dense periodic orbits.

Hopf's theorem

Theorem (E. Hopf, 1948)

If (M, g) is a closed Riemannian surface without conjugate points, then

$$\int_M K^{\mathrm{g}}\, \mathrm{d}\nu_{\mathrm{g}} \leq 0,$$

and equality holds if and only if (M, g) is a flat torus.

Above, K^{g} and ν_{g} are the Gaussian curvature and area form of (M, g).

The Gauss-Bonnet theorem then trivially implies that

every Riemannian metric without conjugate points on $\mathbb{T}^2$ is flat.

What about higher dimensions?

Green's Theorem

Theorem (Green, 1958)

If (M, g) is a closed Riemannian manifold without conjugate points, then

$$\int_M \mathrm{scal}^{\mathrm{g}}\,\mathrm{d}\nu_{\mathrm{g}} \leq 0,$$

and equality holds if and only if (M, g) is flat.

Above, $\mathrm{scal}^{\mathrm{g}}$ and ν_{g} are the scalar curvature and volume form of (M, g).

What about the topological conclusion in the equality case?

The best we know is:

Theorem (Burago-Ivanov, 1994)

Every Riemannian metric without conjugate points on $\mathbb{T}^n$ is flat.

Magnetic systems

We want to model, using differential geometry, trajectories of particles on Riemannian manifolds subject to the action of a magnetic field.

Definition (Anosov & Sinai, 1967?)

A magnetic system on a smooth manifold M is a pair (g, σ), where g is a Riemannian metric and σ is a closed 2-form on M. The Lorentz force operator of (g, σ) is the endomorphism $\mathbf{Y}\colon TM \to TM$ characterized by

$$\sigma_x(v, w) = \mathrm{g}_x(\mathbf{Y}_x(v), w),$$

for all $x \in M$ and $v, w \in T_xM$. The 2-form σ is called the magnetic form and, in this context, it is called uniform if $\nabla\sigma = 0$.

Variational characterizations

The geodesic equation gets replaced with the Landau-Hall equation:

$$\frac{\mathrm{D}\dot{\gamma}}{\mathrm{d}t}(t) = \mathbf{Y}_{\gamma(t)}(\dot{\gamma}(t)).$$

When $\dim M = 3$ and M is orientable, every skew-adjoint operator is given as a cross product, and we have the Lorentz force law:

$$\frac{\mathrm{D}\dot{\gamma}}{\mathrm{d}t}(t) = q\,\mathbf{B}_{\gamma(t)} \times \dot{\gamma}(t).$$

For any magnetic system (g, σ) on M, magnetic geodesics have constant speed and, when the magnetic form $\sigma = \mathrm{d}A$ is exact, they can be characterized as critical points of the Landau-Hall functional:

$$\mathcal{LH}(\gamma) = \int_a^b \left(\frac{1}{2}\|\dot{\gamma}(t)\|^2 + A_{\gamma(t)}(\dot{\gamma}(t))\right)\mathrm{d}t.$$

The magnetic flow

The Landau-Hall equation

$$\frac{\mathrm{D}\dot{\gamma}}{\mathrm{d}t}(t) = \mathbf{Y}_{\gamma(t)}(\dot{\gamma}(t))$$

induces a flow $\Phi^{\mathrm{g},\sigma}\colon \mathbb{R} \times TM \to TM$ on the tangent bundle

$$TM = \{(x, v) : x \in M \text{ and } v \in T_xM\}.$$

It is given by $\Phi^{\mathrm{g},\sigma}(t, (x, v)) = (\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t))$, where $\gamma_{(x,v)}\colon \mathbb{R} \to M$ is the unique solution of the IVP

$$\begin{cases} \dfrac{\mathrm{D}\dot{\gamma}}{\mathrm{d}t}(t) = \mathbf{Y}_{\gamma(t)}(\dot{\gamma}(t)) \\ \gamma(0) = x, \quad \dot{\gamma}(0) = v. \end{cases}$$

The magnetic flow

The sphere bundles

$$\Sigma_s = \{(x, v) \in TM : \|v\| = s\}, \qquad s > 0,$$

are invariant under the magnetic flow, which can then be restricted to a flow $\Phi_s^{\mathrm{g},\sigma} \colon \mathbb{R} \times \Sigma_s \to \Sigma_s$.

But this time, since the Landau-Hall equation is not homogeneous, the dynamical properties of $\Phi_s^{\mathrm{g},\sigma}$ depend heavily on the value of $s > 0$.

The value marking the change in dynamical behavior is called the Mañé critical value of (g, σ) — it is generally difficult to compute.

Magnetic curvature in dimension 2

There is also a natural notion of conjugate points for magnetic systems, and this time it depends on the energy level $s > 0$.

But what about curvature?

When $\dim M = 2$ we may write $\sigma = b\,\nu_{\mathrm{g}}$ for some $b \in \mathrm{C}^\infty(M)$. The magnetic form is uniform if $b \in \mathbb{R}$ is constant.

Definition (M. & P. Paternain, 1996)

The magnetic Gaussian curvature $K_s^{\mathrm{g},b}\colon SM \to \mathbb{R}$ is defined by

$$K_s^{\mathrm{g},b}(x,v) = s^2 K^{\mathrm{g}}(x) - s\,\mathrm{d}b_x(\mathrm{i}v) + b(x)^2,$$

for all $(x,v) \in SM$.

When $K_s^{\mathrm{g},b} < 0$, the flow $\Phi_s^{\mathrm{g},b}$ has no conjugate points.

Going towards a magnetic Hopf

Theorem (Gouda, 1996)

For a closed surface M equipped with a uniform magnetic system (g, b) without conjugate points for energy $s = 1$,

$$\int_M (K^{\mathrm{g}} + b^2)\, \mathrm{d}\nu_{\mathrm{g}} \leq 0,$$

with equality if and only if (M, g) is a flat torus and $b = 0$.

Even with the restrictive assumption that the magnetic system is uniform, this already generalizes Hopf's 1948 theorem!

Gouda was not aware of the definition of $K_s^{\mathrm{g},b}$ at the time, but we can rewrite his result as

$$\int_{SM} K_1^{\mathrm{g},b}\, \mathrm{d}\mu_{\mathrm{g}} \leq 0,$$

where $\mathrm{d}\mu_{\mathrm{g}}$ is the Liouville measure in SM.

Going towards a magnetic Green

What about higher dimensions?

Theorem (Gouda, 1996)

For a closed n-manifold M equipped with a uniform magnetic system (g, σ) without conjugate points for energy $s = 1$,

$$\frac{1}{\mathrm{vol}(M, \mathrm{g})} \int_M \mathrm{scal}^{\mathrm{g}} \, \mathrm{d}\nu_{\mathrm{g}} \leq -\frac{n}{4} \mathrm{tr}(\mathbf{Y}^\dagger \mathbf{Y}),$$

with equality if and only if (M, g) is flat and $\sigma = 0$.

Again, this generalizes Green's 1958 theorem — but now it is not obvious how to rewrite this as the integral of a magnetic curvature.

And what even is the magnetic curvature for higher dimensional systems?

Higher-dimensional magnetic curvature

We consider the vector bundle $E \to SM$ whose fibers are given by the orthogonal complements $E_{(x,v)} = v^\perp \subseteq T_xM$, and $E^1_{(x,v)} = E_{(x,v)} \cap S_xM$.

Definition (Assenza, 2023)

Let (g, σ) be a magnetic system on a smooth n-manifold M, and $s > 0$. The magnetic curvature operator is $M_s^{\mathrm{g},\sigma}\colon E \to E$ given by $M_s^{\mathrm{g},\sigma} = R_s^{\mathrm{g},\sigma} + A^{\mathrm{g},\sigma}$, where

$$(R_s^{\mathrm{g},\sigma})_{(x,v)}(w) = s^2 R_x(w,v)v + \cdots \quad \text{and} \quad A^{\mathrm{g},\sigma}_{(x,v)}(w) = \cdots$$

Then $\sec^{\mathrm{g},\sigma}\colon E^1 \to \mathbb{R}$, $\mathrm{Ric}_s^{\mathrm{g},\sigma}\colon SM \to \mathbb{R}$, and $\mathrm{scal}_s^{\mathrm{g},\sigma}\colon M \to \mathbb{R}$, are defined as

$$(\sec_s^{\mathrm{g},\sigma})_x(v,w) = \langle (M_s^{\mathrm{g},\sigma})_{(x,v)}(w), w\rangle, \qquad \mathrm{Ric}_s^{\mathrm{g},\sigma}(x,v) = \mathrm{tr}(M_s^{\mathrm{g},\sigma})_{(x,v)},$$

$$\text{and} \quad \mathrm{scal}_s^{\mathrm{g},\sigma}(x) = \frac{n}{\mathrm{vol}(\mathbb{S}^{n-1})}\int_{S_xM} \mathrm{Ric}_s^{\mathrm{g},\sigma}(x,v)\,\mathrm{d}\mu_x(v).$$

Our main result

We have finally generalized the previous results to possibly non-uniform magnetic systems of arbitrary signature:

Theorem (Assenza, Marshall-Reber, T., 2024)

Let (g, σ) be any magnetic system on a closed n-manifold M, without conjugate points for energy s. Then

$$\int_M \mathrm{scal}_s^{\mathrm{g},\sigma}\, \mathrm{d}\nu_{\mathrm{g}} \leq 0,$$

with equality if and only if $M_s^{\mathrm{g},\sigma} = 0$.

It remains to understand the true meaning of magnetic flatness.

Magnetic flatness

Theorem (Assenza, Marshall-Reber, T., 2024)

Let (g, σ) *be any nontrivial magnetic system (i.e., with* $\sigma \neq 0$*) on a smooth manifold* M*, and assume that there is* $c \in \mathbb{R}$ *such that* $\sec_s^{\mathrm{g},\sigma} \equiv c$*.*

Then $\nabla\sigma = 0$ *and* σ *is nowhere-vanishing, and one of the following options must hold:*

1. (M, g) *is an oriented surface with constant Gaussian curvature* $K^{\mathrm{g}} = (c - \|\mathbf{Y}\|^2)/s^2$*, and* $\sigma = \|\mathbf{Y}\|^{-1}\nu_{\mathrm{g}}$*.*
2. $\dim M \geq 4$*,* $c = 0$*, and* $J = \|\mathbf{Y}\|^{-1}\mathbf{Y}$ *is a complex structure turning* (M, g) *into a Kähler manifold with constant negative holomorphic sectional curvature* $-\|\mathbf{Y}\|^2/s^2$*.*

If $c = 0$ *in the first case, then* s *equals the Mañé critical value of* (g, σ) *in both cases.*

Proof idea if there's still time left

- Introduce a "magnetic connector" $\mathcal{K}^{\mathrm{g},\sigma}\colon TTM \to TM$ and use it to get a horizontal-vertical decomposition $T\Sigma_s = \widehat{H}^{\mathrm{g},\sigma} \oplus V$.
- Obtain a symplectic vector bundle $Q = T\Sigma_s/\mathbf{X}^{\mathrm{g},\sigma} \to \Sigma_s$ via Marsden-Weinstein reduction.
- Move the projected vertical distribution V with the quotient Hamiltonian flow induced by the derivative of $\Phi_s^{\mathrm{g},\sigma}$ to obtain a curve $E(t)$ of Lagrangian subbundles of Q.
- Express the limit $\lim_{t\to+\infty} E(t)$ as the graph of a self-adjoint bundle morphism and use it to build a solution of the Riccati equation

$$\mathrm{tr}\,\dot{U}_v(t) + \mathrm{tr}(U_v(t)^2) + \mathrm{Ric}_s^{\mathrm{g},\sigma}(\Phi_s^{\mathrm{g},\sigma}(t,v)/s) = 0.$$

- Integrate.

Thank you for your attention!

(scan here for more on my research)