The topology of compact Lorentzian manifolds with parallel Weyl curvature

(joint work with Andrzej Derdzinski)

Ivo Terek

The Ohio State University

January 31st, 2024
12:15 p.m. – 12:40 p.m. CST

XI International Meeting on Lorentzian Geometry
January 29th to February 2nd, 2024, Universidad Autónoma de Yucatán

These slides can also be found at

https://www.asc.ohio-state.edu/terekcouto.1/texts/Merida_slides_january2024.pdf

Setup

The curvature tensor of a pseudo-Riemannian manifold (M, g), $n = \dim M$, admits the orthogonal decomposition

$$R = \frac{\mathrm{s}}{n(n-1)}\mathrm{g} \owedge \mathrm{g} + \frac{2}{n-2}\mathrm{g} \owedge \left(\mathrm{Ric} - \frac{\mathrm{s}}{n}\mathrm{g}\right) + W,$$

where Ric, s and W are the Ricci, scalar, and Weyl curvatures of (M, g).

We will assume throughout this talk that $n \geq 4$, in which case (M, g) is conformally flat if and only if $W = 0$.

The condition we are interested in is $\nabla W = 0$.

Definition (ECS manifold)

A pseudo-Riemannian manifold (M, g) is called essentially conformally symmetric if $\nabla W = 0$ but neither $W = 0$ nor $\nabla R = 0$.

The metric signature

ECS manifolds are objects of strictly indefinite nature:

Theorem (Roter, 1977)

For a Riemannian manifold (M, g)*:* $\nabla W = 0 \iff W = 0$ *or* $\nabla R = 0$*.*

Roter has also shown that ECS manifolds exist in all dimensions starting from 4, and realizing all possible indefinite metric signatures.

Every ECS manifold carries a distinguished null parallel distribution, which helps control its geometry:

Definition

The Olszak distribution of an ECS manifold (M, g) is $\mathcal{D} \hookrightarrow TM$ given by

$$\mathcal{D}_x = \{v \in T_xM \mid \mathrm{g}_x(v, \cdot) \wedge W_x(v', v'', \cdot, \cdot) = 0, \text{ for all } v', v'' \in T_xM\},$$

for every $x \in M$.

More on the Olszak distribution

The Olszak distribution was originally introduced for the more general study of conformally recurrent manifolds, and in this setting it is already true that $\mathcal{D}$ is indeed smooth, parallel and null.

In the ECS case, the rank of $\mathcal{D}$ is always equal to 1 or 2. For this reason, we speak of rank-one/rank-two ECS manifolds.

Theorem (Derdzinski-Roter, 2009)

Let (M, g) be an ECS manifold, and $\mathcal{D}$ be its Olszak distribution. Then:

1. *The Ricci endomorphism of (M, g) is $\mathcal{D}$-valued.*
2. *The connection induced in the quotient bundle $\mathcal{D}^\perp/\mathcal{D}$ over M is flat.*
3. *The connection induced in $\mathcal{D}$ itself is flat when (M, g) is of rank one.*

Lorentzian ECS manifolds are all of rank-one and, up to a double isometric covering, pp-wave spacetimes.

A rank-one example

Example (Conformally symmetric pp-wave manifolds)

Let $(V, \langle \cdot, \cdot \rangle)$ be a pseudo-Euclidean vector space of dimension $n-2 \geq 2$, $A \in \mathfrak{sl}(V)$ be self-adjoint, $I \subseteq \mathbb{R}$ be an open interval and $f \colon I \to \mathbb{R}$ be a smooth function. Consider

$$(\widehat{M}, \widehat{\mathrm{g}}) = (I \times \mathbb{R} \times V, \kappa\, \mathrm{d}t^2 + \mathrm{d}t\, \mathrm{d}s + \langle \cdot, \cdot \rangle),$$

where $\kappa \colon \widehat{M} \to \mathbb{R}$ is given by $\kappa(t, s, v) = f(t)\langle v, v \rangle + \langle Av, v \rangle$.

Then $(\widehat{M}, \widehat{\mathrm{g}})$ has $\nabla W = 0$, with:

- $W = 0 \iff A = 0$;
- $\nabla R = 0 \iff f$ is constant.

In the ECS case, the Olszak distribution $\mathcal{D}$ is spanned by the null parallel coordinate vector field ∂_s, and $(V, \langle \cdot, \cdot \rangle)$ is isometrically identified with the vector space of parallel sections of $\mathcal{D}^\perp/\mathcal{D}$.

Intuition

We consider such examples because *any point in a rank-one ECS manifold* (M^n, g) *has a neighborhood isometric to an open subset of some* $(\widehat{M}, \widehat{\mathrm{g}})$.

The idea relies on two general facts about rank-one ECS manifolds:

- Ric is $\mathcal{D}$-valued.
- the connections induced on $\mathcal{D}$ and $\mathcal{D}^\perp/\mathcal{D}$ are flat.

Locally, consider: a null parallel vector field w spanning $\mathcal{D}$, and a function t such that $\mathrm{d}t = \mathrm{g}(w, \cdot)$. This way:

- $\mathrm{Ric} = (2-n)f(t)\,\mathrm{d}t \otimes \mathrm{d}t$ for some suitable function f.
- The Weyl tensor acts as a traceless self-adjoint endomorphism A of $V = \mathcal{D}^\perp/\mathcal{D}$ via $A(v+\mathcal{D}) = W(u,v)u + \mathcal{D}$ (where u is any vector field with $\mathrm{g}(u,w) = 1$).

Any null geodesic $t \mapsto x(t)$ with $\mathrm{g}(\dot{x}(t), w_{x(t)}) = 1$ gives rise to a mapping

$$F(t,s,v) = \exp_{x(t)}\left(v_{x(t)} + \frac{s w_{x(t)}}{2}\right), \quad \text{with} \quad F^*\mathrm{g} = \widehat{\mathrm{g}}.$$

About compact ECS manifolds

With the local structure of ECS manifolds being fully understood, the next step is to address global aspects. The first question is whether compact ECS manifolds exist.

Theorem (Derdzinski-Roter, 2010)

In every dimension $n = 3j + 2$, $j = 1, 2, 3, \ldots$, there exists a compact rank-one ECS manifold (M, g) of any prescribed indefinite metric signature, which is diffeomorphic to a torus bundle over $\mathbb{S}^1$, but not homeomorphic to (or even covered by) a torus.

These examples are all of the form $M = \widehat{M}/\Gamma$, where Γ is some cocompact subgroup of $\mathrm{Iso}(\widehat{M}, \widehat{\mathrm{g}})$ acting freely and properly discontinuously on $\widehat{M}$.

The strange dimensions $n = 3j + 2$ were a particularity of their construction, which obtained a 5-dimensional example with $\dim V = 3$, but turned out to be "compatible" with taking cartesian powers of $(V, \langle\cdot,\cdot\rangle)$, leading also to dimensions 8, 11, 14, etc..

More compact examples

Generalizing the construction:

Theorem (Derdzinski-T., 2022)

There exist compact rank-one ECS manifolds of all dimensions $n \geq 5$ and all indefinite metric signatures, forming the total space of a nontrivial torus bundle over $\mathbb{S}^1$ with its fibers being the leaves of $\mathcal{D}^\perp$, all geodesically complete, and none locally homogeneous. In each fixed dimension and metric signature, there is an infinite-dimensional moduli space of local-isometry types.

Compact pp-wave spacetimes are complete (Leistner-Schliebner, 2016).

More generally, a compact Lorentzian manifold carrying a parallel null vector field is complete (Mehidi-Zeghib, 2022).

There are geodesically incomplete and locally homogeneous odd-dimensional examples too (Derdzinski-T., 2023), but none of them is Lorentzian.

The topological structure

Q: What do all known compact rank-one ECS manifolds presented so far have in common?

A: They are all bundles over $\mathbb{S}^1$, and $\mathcal{D}^\perp$ appears as the vertical distribution.

We will see next that this is **not an accident**.

The main result

Theorem (Derdzinski-T., 2022)

Every non-locally-homogeneous compact rank-one ECS manifold (M, g) for which the orthogonal distribution $\mathcal{D}^\perp$ is transversely orientable is the total space of a locally trivial fibration over $\mathbb{S}^1$ whose fibers are the leaves of $\mathcal{D}^\perp$. In addition, every leaf L of $\widetilde{\mathcal{D}}^\perp$ in $\widetilde{M}$ is simply connected and $\widetilde{M}$ is diffeomorphic to $\mathbb{R} \times L$.

The transverse orientability of $\mathcal{D}^\perp$ can be achieved by replacing (M, g) with a suitable isometric double covering, if necessary.

This is a generalization to arbitrary indefinite signature of:

Theorem (Derdzinski-Roter, 2008)

Let (M, g) be a compact Lorentzian ECS manifold. Then some two-fold covering of M is the total space of a C^∞ bundle over $\mathbb{S}^1$, the fiber of which admits a flat torsionfree connection with a nonzero parallel vector field.

How does the generalization happen?

To understand how our main result generalizes the 2008 one, it remains to argue that Lorentzian ECS manifolds cannot be locally homogeneous.

Here, we write $M = \widetilde{M}/\Gamma$ and again consider the space $(V, \langle\cdot,\cdot\rangle)$ of parallel sections of $\mathcal{D}^\perp/\mathcal{D}$, with the self-adjoint $A \in \mathfrak{sl}(V)$ resulting from W.

For every $\gamma \in \Gamma$ there are $(q, p) \in \mathrm{Aff}(\mathbb{R})$ and $C \in \mathrm{O}(V, \langle\cdot,\cdot\rangle)$ such that

$$t \circ \gamma = qt + p, \quad q^2 f(qt + p) = f(t), \quad \text{and} \quad CAC^{-1} = q^2 A.$$

If the image K of the obvious homomorphism $\Gamma \to \mathrm{Aff}(\mathbb{R})$ has an element (q, p) with $|q| \neq 1$ and (M, g) is Lorentzian, then $A = 0$ and hence $W = 0$.

But if (M, g) is Lorentzian and locally homogeneous, then $K \subseteq \mathbb{R}$ is dense in $\mathbb{R}$, making f constant and thus (M, g) becomes locally symmetric.

The strategy

The central concept used in the proof is what we call the dichotomy property for a codimension-one foliation $\mathcal{V}$ in a smooth manifold M, which has two alternatives (NC) and (AC) imposed on its compact leaves.

The reason why we care about this property is that it turns out that if M is compact, $\mathcal{V}$ is transversely orientable, and some compact leaf of $\mathcal{V}$ satisfies (AC), then there is a locally trivial bundle projection $M \to \mathbb{S}^1$ whose fibers are the leaves of $\mathcal{V}$.

There are two big steps to carry out:

1. Establishing the dichotomy property for $\mathcal{D}^\perp$ (when transversely orientable) in a rank-one ECS manifold (M, g).
2. Showing that some compact leaf of $\mathcal{D}^\perp$ satisfies (AC) when M is compact.

Step (i) does not use compactness of M, and local homogeneity is an obstacle for (ii).

The dichotomy property

Definition

A codimension-one foliation $\mathcal{V}$ in a smooth manifold M has the *dichotomy property* if every compact leaf L of $\mathcal{V}$ has a neighborhood U in M such that the leaves of $\mathcal{V}$ intersecting $U \smallsetminus L$ are either:

NC: all *noncompact*, or

AC: *all compact*, and some neighborhood of L in M saturated by compact leaves of $\mathcal{V}$ may be diffeomorphically identified with the product $\mathbb{R} \times L$ in such a way that $\mathcal{V}$ corresponds to the foliation $\{\{s\} \times L\}_{s\in\mathbb{R}}$.

Example

If both M and $\mathcal{V}$ are real-analytic and $\mathcal{V}$ is transversely orientable, then $\mathcal{V}$ has the dichotomy property. If a compact leaf L of $\mathcal{V}$ does not satisfy (NC), there are compact leaves of $\mathcal{V}$ arbitrarily close to L. Now analyticity implies that L satisfies (AC).

More examples of the dichotomy property

Example

If $\mathcal{V}$ is transversely orientable and has a finite number of compact leaves, then $\mathcal{V}$ clearly has the dichotomy property. Examples of this situation include the Reeb foliation on $\mathbb{S}^3$, and foliations on products $\mathbb{T}^2 \times K$ coming from foliations on $\mathbb{T}^2$ having themselves a finite number of leaves.

Example

Let M be an orientable line bundle over a compact and connected manifold L, equipped with a flat connection ∇, and let $\mathcal{V}$ be the horizontal distribution on M associated with ∇. The compact leaf L (and hence all others) satisfies (NC) or (AC) according to whether the holonomy group $\mathrm{Hol}(\nabla)$ is infinite or trivial.

Establishing the dichotomy property for $\mathcal{D}^\perp$

The last example illuminates the way to proceed:

Theorem

Let (M, g) be a compact rank-one ECS manifold with transversely orientable $\mathcal{D}^\perp$, and let L be a compact leaf of $\mathcal{D}^\perp$. Then, there is some neighborhood U of L in M which can be identified with a neighborhood U' of the zero section $L \hookrightarrow \mathcal{D}^*_L$ as to make the distribution $\mathcal{D}^\perp$ in U correspond in U' to the horizontal distribution of the flat connection in $\mathcal{D}^*_L$.

Sketch of proof: Let $t\colon \widetilde{M} \to \mathbb{R}$ is a function whose parallel gradient $\mathbf{w}$ spans $\widetilde{\mathcal{D}}$, and ϕ be a flow on M which is transverse to $\mathcal{D}^\perp$. Define $U = \phi[(-\varepsilon, \varepsilon) \times L]$ and $\Psi\colon U \to U' = \Psi[U]$ by

$$\Psi(\phi(\tau, x)) = [t(\widetilde{\phi}(\tau, y)) - t(y)]\xi_y \circ (\mathrm{d}\pi_y)^{-1},$$

where $\widetilde{\phi}$ is a lift of ϕ to $\widetilde{M}$, ξ is the parallel section of $\widetilde{\mathcal{D}}^*$ with $\xi(\mathbf{w}) = 1$, and $y \in \pi^{-1}(x)$ is chosen at will. This works.

Establishing the dichotomy property for $\mathcal{D}^\perp$

The last example illuminates the way to proceed:

Theorem

Let (M, g) be a compact rank-one ECS manifold with transversely orientable $\mathcal{D}^\perp$, and let L be a compact leaf of $\mathcal{D}^\perp$. Then, there is some neighborhood U of L in M which can be identified with a neighborhood U' of the zero section $L \hookrightarrow \mathcal{D}^*_L$ as to make the distribution $\mathcal{D}^\perp$ in U correspond in U' to the horizontal distribution of the flat connection in $\mathcal{D}^*_L$.

So:

Theorem

If (M, g) is a rank-one ECS manifold with transversely orientable $\mathcal{D}^\perp$, then $\mathcal{D}^\perp$ satisfies the dichotomy property. Namely, for a compact leaf L of $\mathcal{D}^\perp$, alternatives (AC) and (NC) correspond to whether the holonomy group of the natural flat connection in the line bundle $\mathcal{D}^*_L$ is finite or infinite.

Towards a compact leaf with (AC): cohomology, $\mathcal{F}$ & P

Our next goal is to show that some compact leaf of $\mathcal{D}^\perp$ in M satisfies alternative (AC) of the dichotomy property.

Closedness of a continuous 1-form ζ means its locally being the differential of a C^1 function. Thus it makes sense to consider a cohomology class $[\zeta] \in H^1_{\mathrm{dR}}(M) \cong \mathrm{Hom}(\pi_1(M), \mathbb{R})$.

We fix again the universal covering $(\widetilde{M}, \widetilde{\mathrm{g}})$ of (M, g), a function $t\colon \widetilde{M} \to \mathbb{R}$ whose parallel gradient spans $\widetilde{\mathcal{D}}$, and express $M = \widetilde{M}/\Gamma$ with $\Gamma \cong \pi_1(M)$.

Considering the space $\mathcal{F}$ of all continuous functions $\chi\colon \widetilde{M} \to \mathbb{R}$ such that $\chi\,\mathrm{d}t$ is closed and Γ-invariant, we may consider the operator

$$P\colon \mathcal{F} \to H^1_{\mathrm{dR}}(M), \qquad \text{given by} \qquad P\chi = [\chi\,\mathrm{d}t].$$

Special functions

Considering the space $\mathcal{F}$ of all continuous functions $\chi\colon \widetilde{M} \to \mathbb{R}$ such that $\chi\,\mathrm{d}t$ is closed and Γ-invariant, we may consider the operator

$$P\colon \mathcal{F} \to H^1_{\mathrm{dR}}(M), \qquad \text{given by} \qquad P\chi = [\chi\,\mathrm{d}t].$$

Theorem

Let (M, g) be a compact rank-one ECS manifold such that $\mathcal{D}^\perp$ is transversely orientable. If (M, g) is not locally homogeneous, then there exists a nonconstant function $\mu \in \mathrm{C}^1(M)$ which is constant along $\mathcal{D}^\perp$.

Sketch of proof: It mainly consists in showing that either

1. $\dim \mathcal{F} < \infty$ and (M, g) is locally homogeneous, or
2. $\dim \mathcal{F} = \infty$ and such μ exists.

In case (i), set-theoretical reasons imply that $f(t) = \varepsilon(t-b)^{-2}$, where $\mathrm{Ric} = (2-n)f(t)\,\mathrm{d}t \otimes \mathrm{d}t$. In case (ii), let $\chi \in \ker P \smallsetminus \{0\}$ and take μ such that $\mathrm{d}\mu$ equals the projected $\chi\,\mathrm{d}t$.

From special functions to compact leaves satisfying (AC)

Let $\mu \in \mathrm{C}^1(M)$ be nonconstant, but constant along $\mathcal{D}^\perp$.

By Sard's theorem, the image of μ in $\mathbb{R}$ contains an open interval of regular values of μ. Any connected component of a level set $\mu^{-1}(c)$, with c in a such open interval, is a compact leaf of $\mathcal{D}^\perp$ with (AC).

This completes the proof of our main result.

Note: Sard's theorem usually applies for a C^k function from an n-manifold into an m-manifold, where $k \geq \max\{n - m + 1, 1\}$. Here, $k = m = 1$ and $n \geq 4$, but compactness of M together with μ being locally a function of t allows us to apply Sard with $n = 1$ instead of $n \geq 4$.

Appendix: why $\dim \mathcal{F} < \infty$ gives (M, g) LH

The "set-theoretical reasons" mentioned before ultimately boil down to:

Lemma

Let X be a set and $\mathcal{F} \subseteq \mathbb{R}^{\mathrm{X}}$ be a finite-dimensional subspace which is closed under the absolute value function $|\cdot|$ and has the property that

$$|\psi_1 \dots \psi_k|^{1/k} \in \mathcal{F}, \text{ whenever } k \geq 1 \text{ and } \psi_1, \dots, \psi_k \in \mathcal{F}.$$

Then, writing $m = \dim \mathcal{F}$, there is a basis $\{\chi_1, \dots, \chi_m\}$ of $\mathcal{F}$ consisting of nonnegative functions with pairwise disjoint supports.

In other words, there is a disjoint-union decomposition

$$\mathrm{X} = \mathrm{X}_0 \cup \mathrm{X}_1 \cup \dots \cup \mathrm{X}_m,$$

with X_j non-empty for each $j = 1, \dots, m$, such that $\chi_j > 0$ on X_j and $\chi_j = 0$ on $\mathrm{X} \setminus \mathrm{X}_j$.

Appendix: why $\dim \mathcal{F} < \infty$ gives (M, g) LH

Applying this lemma to our space $\mathcal{F}$ (all $\chi \in \mathrm{C}^0(\widetilde{M})$ with $\chi\, \mathrm{d}t$ closed and Γ-invariant), we see that $|\dot{f}|^{1/3}/|f|^{1/2}$ is locally constant where $f \neq 0$.

Then it follows that $f \neq 0$ everywhere: otherwise, at a boundary point of the zero set of f, the linear function $|f|^{-1/2}$ would be unbounded on a bounded t-interval.

It now follows that

(i) $f = \varepsilon(t-b)^{-2}$,

and we can arrange for $b = 0$ and $I = (0, \infty)$ by making an affine substitution on t.

Whitney's theorem from algebraic geometry now implies that

(ii) for every $q \in (0, \infty)$ there is $C \in \mathrm{O}(V)$ such that $CAC^{-1} = q^2 A$.

Using (i) and (ii), one establishes local homogeneity of (M, g) through homogeneity of the corresponding model $(\widehat{M}, \widehat{\mathrm{g}})$.

References

[1] A. Derdzinski and W. Roter, **On compact manifolds admitting indefinite metrics with parallel Weyl tensor**, J. Geom. Phys., vol. **58** (2008), pp. 1137-1147.

[2] A. Derdzinski and W. Roter, **The local structure of conformally symmetric manifolds**, Bull. Belgian Math. Soc., vol. **16** (2009), pp. 117–128.

[3] A. Derdzinski and I. Terek, **New examples of compact Weyl-parallel manifolds**. Monatsh. Math. (published online: Sep. 27, 2023).

[4] A. Derdzinski and I. Terek, **The topology of compact rank-one ECS manifolds**. Proc. Edinb. Math. Soc., vol **66** (2023), no. 3, pp. 789–809.

[5] A. Derdzinski and I. Terek, **Compact locally homogeneous manifolds with parallel Weyl tensor**. To appear in Adv. Geom. Preprint available at arXiv:2306.01600.

[6] T. Leistner and D. Schliebner, **Completeness of compact Lorentz manifolds with abelian holonomy**, Math. Ann. (3-4) **364** (2016), 1469–1503.

[7] L. Mehidi and A. Zeghib, **On completeness and dynamics of compact Brinkmann spacetimes**. Preprint available at arXiv:2205.07243.

Thank you for your attention!

(scan here for more
on my research)