Compactifying rank-one Weyl-parallel manifolds

(joint work with Andrzej Derdzinski)

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May 26th, 2023
3:35 p.m. – 4:05 p.m. EDT

Graduate Student Conference in Algebra, Geometry, and Topology
May 26th to May 28th, 2023, Temple University

These slides can also be found at

https://www.asc.ohio-state.edu/terekcouto.1/texts/Temple_slides_may2023.pdf

The Weyl curvature tensor

We will start by recalling the definition of the Weyl curvature tensor W of a pseudo-Riemannian manifold (M, g).

The curvature tensor of $\mathbb{S}^n$ equipped with its round metric is given by

$$R(X,Y,Z,V) = \mathrm{g}(Y,Z)\mathrm{g}(X,V) - \mathrm{g}(X,Z)\mathrm{g}(Y,V)$$

$$R(X,Y,Z,V) = \underbrace{\mathrm{g}(Y,Z)\mathrm{g}(X,V) - \mathrm{g}(X,Z)\mathrm{g}(Y,V)}_{(\mathrm{g} \owedge \mathrm{g})(X,Y,Z,V)}$$

This is a quadratic expression in g. Polarize!

$$\begin{aligned} 2(T \owedge S)(X,Y,Z,V) \doteq{}& T(Y,Z)S(X,V) - T(X,Z)S(Y,V) \\ &+ S(Y,Z)T(X,V) - S(X,Z)T(Y,V) \end{aligned}$$

The $\owedge$-multiplication between symmetric type $(0,2)$ tensor fields is always a type $(0,4)$ tensor field with the "symmetries of a curvature".

In any pseudo-Riemannian manifold (M, g), we may $\owedge$-divide R by g:

$$R = \mathrm{g} \owedge P + W, \qquad W = \text{Weyl curvature tensor of } (M, \mathrm{g}).$$

Here are the main facts about W:

- W is the remainder of the $\owedge$-division of R by g.
- W is the "Ricci-traceless" part of R.
- W is the part of R not constrained by Einstein's field equations.
- R has $n^2(n^2 - 1)/12$ independent components, while Ric has $n(n+1)/2$: the remaining ones all come from W.
- $W = 0$ whenever $\dim M \leq 3$.
- If $\dim M \geq 4$, (M, g) is conformally flat if and only if $W = 0$.

The condition we are interested in is $\nabla W = 0$.

Definition (ECS manifold)

A pseudo-Riemannian manifold (M, g) is called *essentially conformally symmetric* if $\nabla W = 0$ but neither $W = 0$ nor $\nabla R = 0$.

What is known

ECS manifolds are objects of strictly indefinite nature:

Theorem (Roter, 1977)

For a Riemannian manifold (M, g)*:* $\nabla W = 0 \iff W = 0$ *or* $\nabla R = 0$.

Other important facts:

- The local structure of ECS manifolds has been completely described by Derdzinski and Roter in 2009.
- Every ECS manifold carries a distinguished null parallel distribution $\mathcal{D}$, whose rank equals 1 or 2. We call $\mathcal{D}$ the *Olszak distribution of* (M, g) and refer to the rank of $\mathcal{D}$ as the *rank of* (M, g).
- There are **compact** ECS manifolds of all dimensions of the form $3j + 2$, $j \geq 1$, realizing all indefinite metric signatures (Derdzinski-Roter, 2010).

A rank-one example

Example (Conformally symmetric pp-wave manifolds)

Let $(V, \langle\cdot,\cdot\rangle)$ be a pseudo-Euclidean vector space of dimension $n-2 \geq 2$, $A \in \mathfrak{sl}(V)$ be self-adjoint, $I \subseteq \mathbb{R}$ be an open interval and $f\colon I \to \mathbb{R}$ be a smooth function. Consider

$$(\widehat{M}, \widehat{\mathrm{g}}) = (I \times \mathbb{R} \times V, \kappa\,\mathrm{d}t^2 + \mathrm{d}t\,\mathrm{d}s + \langle\cdot,\cdot\rangle),$$

where $\kappa\colon \widehat{M} \to \mathbb{R}$ is given by $\kappa(t,s,v) = f(t)\langle v,v\rangle + \langle Av, v\rangle$.

Then $(\widehat{M}, \widehat{\mathrm{g}})$ has $\nabla W = 0$, with:

- $W = 0 \iff A = 0$;
- $\nabla R = 0 \iff f$ is constant.

In the ECS case, the Olszak distribution $\mathcal{D}$ is spanned by the null parallel coordinate vector field ∂_s, and $(V, \langle\cdot,\cdot\rangle)$ is isometrically identified with the vector space of parallel sections of $\mathcal{D}^\perp/\mathcal{D}$.

$(\widehat{M}, \widehat{\mathrm{g}})$ is complete if and only if $I = \mathbb{R}$ (which we'll assume from now on).

Intuition

We consider such examples because *any point in a rank-one ECS manifold* (M^n, g) *has a neighborhood isometric to an open subset of some* $(\widehat{M}, \widehat{\mathrm{g}})$.

The idea relies on two general facts about rank-one ECS manifolds:

- Ric is $\mathcal{D}$-valued.
- the connections induced on $\mathcal{D}$ and $\mathcal{D}^\perp/\mathcal{D}$ are flat.

Locally, consider: a null parallel vector field w spanning $\mathcal{D}$, and a function t such that $\mathrm{d}t = \mathrm{g}(w, \cdot)$. This way:

- $\mathrm{Ric} = (2-n)f(t)\,\mathrm{d}t \otimes \mathrm{d}t$ for some suitable function f.
- The Weyl tensor acts as a traceless self-adjoint endomorphism A of $V = \mathcal{D}^\perp/\mathcal{D}$ via $A(v + \mathcal{D}) = W(u, v)u + \mathcal{D}$ (where u is any vector field with $\mathrm{g}(u, w) = 1$).

Any null geodesic $t \mapsto x(t)$ with $\mathrm{g}(\dot{x}(t), w_{x(t)}) = 1$ gives rise to a mapping

$$F(t, s, v) = \exp_{x(t)}\left(v_{x(t)} + \frac{s w_{x(t)}}{2}\right), \quad \text{with} \quad F^*\mathrm{g} = \widehat{\mathrm{g}}.$$

The isometry group of $(\widehat{M}, \widehat{\mathrm{g}})$

Again: $(V, \langle\cdot,\cdot\rangle)$ with $\dim V = n-2 \geq 2$, $A \in \mathfrak{sl}(V) \smallsetminus \{0\}$ is self-adjoint, $f \in C^\infty(\mathbb{R})$ is nonconstant, and $\kappa(t,s,v) = f(t)\langle v,v\rangle + \langle Av, v\rangle$. Our "rank-one ECS model" is $(\widehat{M}, \widehat{\mathrm{g}}) = (\mathbb{R}^2 \times V, \kappa\,\mathrm{d}t^2 + \mathrm{d}t\,\mathrm{d}s + \langle\cdot,\cdot\rangle)$.

1. $\mathcal{S}$ is the group of the triples $\sigma = (q,p,C) \in \mathrm{Aff}(\mathbb{R}) \times \mathrm{O}(V)$ with $CAC^{-1} = q^2$ and $q^2 f(qt+p) = f(t)$.
2. $(\mathcal{E}, \Omega)$ is the symplectic vector space of solutions $u\colon \mathbb{R} \to V$ of $\ddot{u}(t) = f(t)u(t) + Au(t)$, with $\Omega(u,\hat{u}) = \langle \dot{u}, \hat{u}\rangle - \langle u, \dot{\hat{u}}\rangle$. Note: $\mathcal{S}$ acts on $\mathcal{E}$ and $\mathbb{R}$ via $(\sigma u)(t) = Cu(q^{-1}(t-p))$ and $\sigma t = qt + p$.
3. The Heisenberg group $\mathcal{H} = \mathbb{R} \times \mathcal{E}$ associated with $(\mathcal{E}, \Omega)$, with operation given by $(r,u)(\widehat{r},\widehat{u}) = (r + \widehat{r} - \Omega(u,\widehat{u}), u + \widehat{u})$.

Theorem

$\mathrm{Iso}(\widehat{M}, \widehat{\mathrm{g}})$ is isomorphic to a semidirect product $\mathcal{S} \ltimes \mathcal{H}$.

- $(\sigma, r, u)(\widehat{\sigma}, \widehat{r}, \widehat{u}) = (\sigma\widehat{\sigma}, r + q^{-1}\widehat{r} - \Omega(u, \sigma\widehat{u}), u + \sigma\widehat{u})$
- $(\sigma, r, u)(t,s,v) = (\sigma t, -\langle \dot{u}(\sigma t), 2\sigma v + u(\sigma t)\rangle + q^{-1}s + r, \sigma v + u(\sigma t))$

About compact examples

The 2010 compact ECS examples all have rank one, and were obtained by finding suitable subgroups $\Gamma \subseteq \mathrm{Iso}(\widehat{M}, \widehat{\mathrm{g}})$ acting freely and properly discontinuously on $(\widehat{M}, \widehat{\mathrm{g}})$ with compact quotient $M = \widehat{M}/\Gamma$.

The previously mentioned dimensions of the form $3j+2$ were a particularity of the construction performed then: a 5-dimensional example was obtained with $\dim V = 3$, but the construction was "compatible" with taking cartesian powers of $(V, \langle\cdot,\cdot\rangle)$, leading also to dimensions 8, 11, 14, etc.

Theorem (Derdzinski-T., 2022)

There exist compact rank-one ECS manifolds of all dimensions $\boldsymbol{n \geq 5}$ and all indefinite metric signatures, diffeomorphic to nontrivial torus bundles over the circle, geodesically complete, and not locally homogeneous. Moreover, in each fixed dimension and metric signature, there is an infinite-dimensional moduli space of local isometry types of such manifolds.

P.S.: we seem to just have found incomplete and locally homogeneous examples too!

Outline of proof (1/4): searching for Γ inside $\mathrm{Iso}(\widehat{M}, \widehat{\mathrm{g}})$

Fixing a period $p > 0$, we will look for subgroups $\Gamma \leq \mathrm{G}(p) \leq \mathrm{Iso}(\widehat{M}, \widehat{\mathrm{g}})$ producing a compact quotient $\widehat{M}/\Gamma$, for suitable choices of f and A.

Here, $\mathrm{G}(p) = \langle (1, p, \mathrm{Id}_V) \rangle \ltimes \mathcal{H} \cong \mathbb{Z} \ltimes \mathcal{H}$, and we consider the translation operator $T\colon \mathcal{E} \to \mathcal{E}$ given by $(Tu)(t) = u(t - p)$, associated with the "generator" $(1, p, \mathrm{Id}_V) \in \mathrm{Iso}(\widehat{M}, \widehat{\mathrm{g}})$.

Then $\mathrm{G}(p)$ acts isometrically on $(\widehat{M}, \widehat{\mathrm{g}})$ by

$$(k, r, u) \cdot (t, s, v) = \big(t + kp\,,\ s + r - \langle \dot{u}(t), 2v + u(t) \rangle\,,\ v + u(t)\big),$$

and has its group operation given by

$$(k, r, u) \cdot (\ell, \widehat{r}, \widehat{u}) = \big(k + \ell\,,\ r + \widehat{r} - \Omega(u, T^{\ell}\widehat{u})\,,\ T^{-\ell}u + \widehat{u}\big).$$

Outline of proof (2/4): first-order subspaces of $(\mathcal{E}, \Omega)$

Such a subgroup Γ would give rise to a "lattice" Λ inside a T-invariant *first-order* subspace $\mathcal{L}$ of $(\mathcal{E}, \Omega)$. But what is a first-order subspace? It is a subspace $\mathcal{L} \leq \mathcal{E}$ such that for every $t \in \mathbb{R}$, the evaluation map $\delta_t \colon \mathcal{L} \to V$ is an isomorphism.

$$\{\mathcal{L} \mid \mathcal{L} \text{ is a first-order subspace}\} \leftrightharpoons \{B \colon \mathbb{R} \to \mathrm{End}(V) \mid \dot{B} + B^2 = f + A\}$$
$$\mathcal{L} = \{u \in \mathcal{E} \mid \dot{u}(t) = B(t)u(t) \text{ for all } t \in \mathbb{R}\}$$

In this correspondence: $\mathcal{L}$ is Lagrangian $\iff$ each $B(t)$ is self-adjoint.

The goal here is to reverse-engineer Γ from the spectrum of $T|_{\mathcal{L}}$ while at the same time finding f and A.

The projection $\mathrm{G}(p) \to \mathbb{Z}$ restricts to a homomorphism $\Gamma \to \mathbb{Z}$ whose kernel Σ projects to a subset $\Lambda \subseteq \mathcal{E}$, which spans a first-order subspace $\mathcal{L}$.

Either $\mathcal{L}$ is Lagrangian and Σ is a lattice in $\mathbb{R} \times \mathcal{L}$ which projects isomorphically onto Λ, or Λ itself is a lattice in $\mathcal{L}$.

Outline of proof (3/4): reverse-engineering the spectrum

Next:

- Step 1: Choose mutually distinct positive reals $\lambda_1, \ldots, \lambda_{n-2}$, not equal to 1: then $\{\lambda_1, \ldots, \lambda_{n-2}\}$ is not of the form $\{\lambda\}$ or $\{\lambda, \lambda^{-1}\}$ for any $\lambda > 0$.
- Step 2: As $n \geq 5$, $\lambda_1, \ldots, \lambda_{n-2}$ are the roots of the characteristic polynomial P of a matrix in $\mathrm{GL}(n-2, \mathbb{Z})$.
- Step 3: Using the Implicit Function Theorem, we may obtain an infinite-dimensional space of p-periodic functions f for which there are a diagonal traceless nonzero matrix A and a curve $t \mapsto B(t)$ of diagonal matrices such that $\dot{B} + B^2 = f + A$ and

$$\mathrm{diag}(\log \lambda_1, \ldots, \log \lambda_{n-2}) = -\int_0^p B(t)\,\mathrm{d}t. \qquad (*)$$

Outline of proof (4/4): reconstructing Γ

- Step 4: For $\mathcal{L}$ corresponding to B obtained in Step 3, the spectrum of $T|_{\mathcal{L}}$ is – due to $(*)$ – precisely $\lambda_1, \ldots, \lambda_{n-2}$ and its characteristic polynomial is P, so that $T[\Lambda] = \Lambda$ for some lattice $\Lambda \subseteq \mathcal{L}$.
- Step 5: As $\mathcal{L}$ is Lagrangian, the action of Λ on $\mathcal{L}$ by vector space translations coincides with its action by left-translations with the group operation induced from $\mathcal{H} \hookrightarrow \mathrm{G}(p)$;
- Step 6: Fixing any $\theta > 0$, we let Γ be the group generated by $\{0\} \times \mathbb{Z}\theta \times \Lambda$ and the element $(1, 0, 0) \in \mathrm{G}(p)$. This works.

For instance, one possible compact fundamental domain for the action of Γ on $\widehat{M}$ is $K = \{(t, s, v) \in \widehat{M} \mid s \in [0, \theta] \text{ and } (s, v) \in K'\}$, where K' is the image under the diffeomorphism

$$\mathbb{R} \times \mathcal{L} \ni (t, w) \mapsto (t, w(t)) \in \mathbb{R} \times V$$

of $[0, p] \times K''$, K'' being a compact fundamental domain for $\Lambda \circlearrowright \mathcal{L}$.

Final considerations

Other features of $\widehat{M}/\Gamma$ stated in the Theorem follow from the construction:

- Γ not being virtually Abelian precludes coverings of M by tori;
- The map $\widehat{M} \ni (t, s, v) \mapsto t/p \in \mathbb{R}$ is Γ-equivariant and induces a fibration $M \to \mathbb{S}^1$.
- The fibers $(\{t\} \times \mathbb{R} \times V)/(\{0\} \times \mathbb{Z}\theta \times \Lambda)$ are tori, as they are diffeomorphic to $(\mathbb{R} \times \mathcal{L})/(\mathbb{Z}\theta \times \Lambda)$.

This bundle structure is not an accident:

Theorem (Derdzinski-T., 2022)

Every non-locally homogeneous compact rank-one ECS manifold is (up to a double isometric covering) diffeomorphic to a bundle over $\mathbb{S}^1$ in such a way that $\mathcal{D}^\perp$ becomes the vertical distribution.

References

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Thank you for your attention!