EQUIVALENT DEFINITIONS OF TANGENT SPACE

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Let M be an n-dimensional smooth manifold, and fix a point $p \in M$. In this note we present the three standard definitions of the tangent space T_pM, show that their vector-space operations are well-defined, and prove that they all have dimension n. Finally, we prove that the three definitions are equivalent, by exhibiting isomorphisms between the resulting tangent spaces.

1 The algebraic approach

On the set consisting of all the pairs (U, f) where $U \subseteq M$ is an open neighborhood of p and $f\colon U \to \mathbb{R}$ is smooth, we define a relation $\sim$ by declaring that $(U, f) \sim (V, g)$ is there is an open neighborhood $W \subseteq U \cap V$ of p such that $f|_W = g|_W$. Then, $\sim$ is an equivalence relation, an equivalence class $[(U, f)]$ is called a *smooth germ at p*, and the quotient set $\mathscr{G}_p^\infty(M)$ is called the *algebra of smooth germs at p*.

When U is a subset of $\mathbb{R}^n$, the germ $[(U, f)]$ clearly contains information about all partial derivatives of all orders of f at p. However, two functions having the same Taylor series at p do not necessarily have the same germ at p. Consider in the real line the function f given by $f(x) = \mathrm{e}^{-1/x^2}$ if $x > 0$, and $f(x) = 0$ if $x \leq 0$. All the derivatives of f at $p = 0$ vanish, but $[(\mathbb{R}, f)]$ is not the zero germ since f does not identically vanish on any interval around the origin.

Note that $\mathscr{G}_p^\infty(M)$ has a natural structure of an $\mathbb{R}$-algebra[1]:

$$\begin{aligned} &\text{(i)}\ [(U,f)] + [(V,g)] = [(U\cap V, f|_{U\cap V} + g|_{U\cap V})], \\ &\text{(ii)}\ [(U,f)]\cdot[(V,g)] = [(U\cap V, f|_{U\cap V}\, g|_{U\cap V})], \\ &\text{(iii)}\ \lambda\cdot[(U,f)] = [(U,\lambda f)], \end{aligned} \tag{1.1}$$

for all $[(U, f)], [(V, g)] \in \mathscr{G}_p^\infty(M)$ and $\lambda \in \mathbb{R}$. It has to be verified that these operations are well-defined. For example, if $[(U, f)] = [(U', f')]$, there is an open neighborhood $U'' \subseteq U \cap U''$ of p on which $f|_{U''} = f'|_{U''}$, so that $U'' \cap V \subseteq (U \cap V) \cap (U' \cap V)$ is also

[1] It is a real vector space X equipped with a product $X \times X \to X$ compatible with the vector space operations. Equivalently, it is a ring R equipped with a scalar multiplication $\mathbb{R} \times R \to R$ compatible with the ring operations.

an open neighborhood of p, and

$$\begin{aligned}(f|_{U\cap V}+g|_{U\cap V})|_{U''\cap V} &= (f|_{U\cap V})|_{U''\cap V}+(g|_{U\cap V})|_{U''\cap V}\\ &= f|_{U''\cap V}+g|_{U''\cap V}\\ &= f'|_{U''\cap V}+g|_{U''\cap V}\\ &= (f'|_{U'\cap V})|_{U''\cap V}+(g|_{U'\cap V})|_{U''\cap V}\\ &= (f|_{U'\cap V}+g|_{U'\cap V})|_{U''\cap V},\end{aligned} \tag{1.2}$$

showing that $[(U\cap V, f|_{U\cap V}+g|_{U\cap V})] = [(U'\cap V, f'|_{U'\cap V}+g|_{U'\cap V})]$. This means that the definition of $[(U,f)]+[(V,g)]$ does not depend on the choice of representative for $[(U,f)]$. One similarly shows that it does not depend on the choice of representative for $[(V,g)]$ either, and so addition in $\mathscr{G}_p^\infty(M)$ is well-defined. You should check now that the product $\cdot$ in $\mathscr{G}_p^\infty(M)$ is also well-defined.

The evaluation mapping $\delta_p\colon \mathscr{G}_p^\infty(M)\to\mathbb{R}$, naturally given by $\delta_p[(U,f)] = f(p)$, is an $\mathbb{R}$-algebra homomorphism (check your understanding: why is it well-defined?). This allows us to consider the algebra of derivations $\operatorname{Der}(\mathscr{G}_p^\infty(M),\delta_p)$, consisting of all $v\colon \mathscr{G}_p^\infty(M)\to\mathbb{R}$ such that

$$v([f]+[g]) = v[f]+v[g] \quad\text{and}\quad v([f][g]) = g(p)v[f]+f(p)v[g], \tag{1.3}$$

for all $[f],[g]\in\mathscr{G}_p^\infty(M)$. Note here the first instance of an abuse of notation: we denote a germ $[(U,f)]$ simply by $[f]$. It is justified since whenever $U'\subseteq U$ is an open neighborhood of p, we have the equality $[(U', f|_{U'})] = [(U,f)]$. The germ itself $[f]$ is not a function defined on any open neighborhood of p, but it can still be evaluated at the point p — this is what the homomorphism δ_p is really doing.

We call $(T_pM)_{\text{ALG.}} = \operatorname{Der}(\mathscr{G}_p^\infty(M),\delta_p)$ the *algebraic tangent space to M at p*. It is, for general algebraic reasons having nothing to do with topology or calculus, a real vector space. Note that any $v\in(T_pM)_{\text{ALG.}}$ also acts on any $C^\infty(U)$, where $U\subseteq M$ is an open neighborhood of p, by $v(f) = v[f]$.

It remains to show that $\dim(T_pM)_{\text{ALG.}} = n$. We will do this by exhibiting a basis containing n elements. Namely, consider a chart (U,φ) for M around p, and write its components as $\varphi = (x^1,\ldots,x^n)$. That is, denoting the Euclidean coordinate functions by $u^i\colon\mathbb{R}^n\to\mathbb{R}$, we have that $x^i = u^i\circ\varphi\colon U\to\mathbb{R}$. We then define

$$\left.\frac{\partial}{\partial x^i}\right|_p \in (T_pM)_{\text{ALG.}} \quad\text{by}\quad \left.\frac{\partial}{\partial x^i}\right|_p[f] = \frac{\partial(f\circ\varphi^{-1})}{\partial u^i}(\varphi(p)), \tag{1.4}$$

where f is a representative of the germ $[f]$, defined on some open neighborhood of p contained in U. The second relation in (1.3) for $\partial/\partial x^i|_p$ is nothing more than the product rule for the Euclidean partial derivatives $\partial/\partial u^i$. Note that

$$\left.\frac{\partial}{\partial x^i}\right|_p[x^j] = \frac{\partial u^j}{\partial u^i}(\varphi(p)) = \delta_i^j = \begin{cases}1, & \text{if } i=j\\ 0, & \text{if } i\neq j.\end{cases} \tag{1.5}$$

A consequence of (1.5) is that

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \ldots, \frac{\partial}{\partial x^n}\bigg|_p \right\} \text{ is linearly independent.} \tag{1.6}$$

If $a^1, \ldots, a^n \in \mathbb{R}$ are such that $\sum_{i=1}^n a^i \partial/\partial x^i|_p = 0$, evaluating both sides as $[x^j]$ leads to $\sum_{i=1}^n a^i \delta_i^j = 0$, that is, $a^j = 0$. And finally, we claim that

$$v = \sum_{i=1}^n v[x^i] \frac{\partial}{\partial x^i}\bigg|_p. \tag{1.7}$$

To establish (1.7), we must evaluate both sides at an arbitrary germ $[f]$ and check that they produce the same output. First note that $v(1) = 0$, by writing $1^1 = 1$ and applying the product rule. Hence $v(c) = 0$ for any $c \in \mathbb{R}$. Now we apply Hadamard's lemma to write

$$f = f(p) + \sum_{i=1}^n (x^i - x^i(p)) g_i, \tag{1.8}$$

where g_i are smooth functions defined on some open neighborhood of p satisfying $g_i(p) = (\partial(f \circ \varphi^{-1})/\partial u^i)(\varphi(p))$. Now:

$$\begin{aligned} v[f] &= v\left[f(p) + \sum_{i=1}^n (x^i - x^i(p)) g_i \right] = v[f(p)] + \sum_{i=1}^n v[(x^i - x^i(p)) g_i] \\ &= 0 + \sum_{i=1}^n \Big((x^i(p) - x^i(p)) g_i(p) + v[(x^i - x^i(p)] g_i(p) \Big) = \sum_{i=1}^n v[x^i] g_i(p) \\ &= \sum_{i=1}^n v[x^i] \frac{\partial (f \circ \varphi^{-1})}{\partial u^i}(\varphi(p)) = \sum_{i=1}^n v[x^i] \frac{\partial}{\partial x^i}\bigg|_p [f] \\ &= \left(\sum_{i=1}^n v[x^i] \frac{\partial}{\partial x^i}\bigg|_p \right) [f], \end{aligned} \tag{1.9}$$

as required.

2 The geometric approach

Fix a chart (U, φ) centered at p, that is, with $\varphi(p) = 0$. On the set of all smooth curves α in M defined on some interval around $0 \in \mathbb{R}$ and such that $\alpha(0) = p$, we define a relation $\approx$ by declaring that $\alpha \approx_\varphi \beta$ if $(\varphi \circ \alpha)'(0) = (\varphi \circ \beta)'(0)$. It is clear that $\approx_\varphi$ is an equivalence relation. It is less clear that $\approx_\varphi$ in fact does not depend on the choice of (U, φ): if (V, ψ) is a second chart centered at p and $\alpha \approx_\varphi \beta$, we claim that $\alpha \approx_\psi \beta$. Indeed, we may compute that

$$(\psi \circ \alpha)'(0) = (\psi \circ \varphi^{-1} \circ \varphi \circ \alpha)'(0) = D(\psi \circ \varphi^{-1})(\varphi(\alpha(0)))(\varphi \circ \alpha)'(0) \tag{2.1}$$

and, similarly, $(\psi\circ\beta)'(0) = D(\psi\circ\varphi^{-1})(\varphi(\beta(0)))(\varphi\circ\beta)'(0)$. Since $\varphi(\alpha(0)) = \varphi(\beta(0))$ and $(\varphi\circ\alpha)'(0) = (\varphi\circ\beta)'(0)$, we conclude that $(\psi\circ\alpha)'(0) = (\psi\circ\beta)'(0)$ as required. Note that, at this point, the chain rule may only be applied to functions between open subsets of Euclidean spaces, not on manifolds. Our use of the chain rule here is legal because $\psi\circ\varphi^{-1}$ is a smooth mapping between open subsets of $\mathbb{R}^n$, and its derivative at any point takes as inputs elements of $\mathbb{R}^n$, which $(\varphi\circ\alpha)'(0)$ and its friends are.

We therefore denote $\approx_\varphi$ simply by $\approx$. The *geometric tangent space to M at p* is the set $(T_pM)_{\text{GEOM.}}$ of all the equivalence classes according to $\approx$. This time, the real vector space structure of $(T_pM)_{\text{GEOM.}}$ is less clear. But here it is:

$$[\alpha]+[\beta] = [t\mapsto \varphi^{-1}(\varphi(\alpha(t))+\varphi(\beta(t)))] \quad\text{and}\quad \lambda\cdot[\alpha] = [t\mapsto\varphi^{-1}(\lambda\varphi(\alpha(t)))], \tag{2.2}$$

for all $[\alpha],[\beta]\in(T_pM)_{\text{GEOM.}}$ and $\lambda\in\mathbb{R}$. We again must check that such operations are well-defined. For instance, to see that $[\alpha]+[\beta]$ does not depend on the choice of representatives for $[\alpha]$ and $[\beta]$, assume that $[\alpha]=[\gamma]$ and $[\beta]=[\eta]$, and note that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\varphi\left(\varphi^{-1}(\varphi(\alpha(t))+\varphi(\beta(t)))\right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\varphi\left(\varphi^{-1}(\varphi(\gamma(t))+\varphi(\eta(t)))\right). \tag{2.3}$$

Namely, (2.3) reads as $(\varphi\circ\alpha)'(0)+(\varphi\circ\beta)'(0) = (\varphi\circ\gamma)'(0)+(\varphi\circ\eta)'(0)$, which is obviously true under the given assumptions. A similar calculation shows that scalar multiplication in $(T_pM)_{\text{GEOM}}$ is well-defined. The neutral element is $0=[c_p]$, where $c_p(t)=p$ for all t. The resulting vector space structure is also independent of the choice of chart (U,φ) in (2.2) because $\approx$ itself is independent of it (check your understanding: how exactly does this follow from what we did in the beginning of the section?).

Now, we may prove that $\dim(T_pM)_{\text{GEOM.}} = n$. For the second time, a chart (U,φ) (centered at p) will induce a basis of $(T_pM)_{\text{GEOM.}}$. Denoting by $(e_1,\ldots,e_n)$ the canonical basis of $\mathbb{R}^n$, we consider the curves $\gamma_j(t)=\varphi^{-1}(te_j)$, for $j=1,\ldots,n$. We claim that

$$\{[\gamma_1],\ldots,[\gamma_n]\}\text{ is a basis of }(T_pM)_{\text{GEOM.}} \tag{2.4}$$

For linear independence, consider $a^1,\ldots,a^n\in\mathbb{R}$, and assume that $\sum_{j=1}^n a^j[\gamma_j]=[c_p]$. Thus $[\varphi^{-1}(t\sum_{j=1}^n a^je_j)]=[c_p]$, and so $\sum_{j=1}^n a^je_j=0$ by definition of $\approx$. Hence, it follows that $a^1=\cdots=a^n=0$. Finally, for any $[\alpha]\in(T_pM)_{\text{GEOM.}}$, it holds that

$$[\alpha]=\sum_{j=1}^n (x^i\circ\alpha)'(0)[\gamma_i], \tag{2.5}$$

where x^i are the components of φ, as before. Indeed,

$$\varphi^{-1}(\alpha(t)) = (x^1(\alpha(t)),\ldots,x^n(\alpha(t)) \quad\text{and}\quad t\mapsto(t(x^1\circ\alpha)'(0),\ldots,t(x^n\circ\alpha)'(0)) \tag{2.6}$$

have the same value and derivative at $t=0$. This proves (2.4).

In practice, we denote $[\alpha]$ by $\alpha'(0)$, even though α takes values in M instead of $\mathbb{R}^n$. This is how we make sense of the velocity vector of a curve valued in a manifold. More generally, we have that $\alpha'(t)\in(T_{\alpha(t)}M)_{\text{GEOM.}}$ for all t in the domain of α.

3 The physics approach

On the set $\mathscr{A}_p \times \mathbb{R}^n$ consisting of the pairs $((U,\varphi),\boldsymbol{v})$, where (U,φ) is a chart for M around p and $\boldsymbol{v} \in \mathbb{R}^n$, we define a relation $\simeq$ by declaring that

$$((U,\varphi),\boldsymbol{v}) \simeq ((V,\psi),\boldsymbol{w}) \quad \text{if} \quad D(\psi\circ\varphi^{-1})(\varphi(p))\boldsymbol{v} = \boldsymbol{w}. \tag{3.1}$$

Transitivity of $\simeq$ amounts to the chain rule for mappings between open subsets of $\mathbb{R}^n$, while symmetry follows from $D(\psi\circ\varphi^{-1})(\varphi(p))^{-1} = D(\varphi\circ\psi^{-1})(\psi(p))$. The quotient set $(T_pM)_{\text{PHYS.}}$ is called the *physical tangent space to M at p*. The idea is that we look at a vector in $\mathbb{R}^n$ starting at the image point $\varphi(p)$, consider its images under all possible transition functions between charts around p, and identify all of them: this is the abstract tangent vector. The real vector space structure in $(T_pM)_{\text{PHYS.}}$ is defined by $\lambda\cdot[((U,\varphi),\boldsymbol{v})] = [((U,\varphi),\lambda\boldsymbol{v})]$ and

$$[((U,\varphi),\boldsymbol{v})] + [((V,\psi),\boldsymbol{w})] = [((U,\varphi),\boldsymbol{v} + D(\varphi\circ\psi^{-1})(\psi(p))\boldsymbol{w})], \tag{3.2}$$

for all $[((U,\varphi),\boldsymbol{v})], [((V,\psi),\boldsymbol{w})] \in (T_pM)_{\text{PHYS.}}$ and $\lambda\in\mathbb{R}$. The way to think about this definition of addition is that $\boldsymbol{w}$, initially seen from the perspective of the chart (V,ψ), must be transported to the perspective of (U,φ) before being added to $\boldsymbol{v}$. And it does not matter if $\boldsymbol{v}$ is transported to the perspective of (V,ψ) before being added to $\boldsymbol{w}$ instead, since

$$[((U,\varphi),\boldsymbol{v} + D(\varphi\circ\psi^{-1})(\psi(p))\boldsymbol{w})] = [((V,\psi),\boldsymbol{w} + D(\psi\circ\varphi^{-1})(\varphi(p))\boldsymbol{v})]. \tag{3.3}$$

The scalar multiplication in $(T_pM)_{\text{PHYS.}}$ is well-defined since derivatives are linear transformations. To see that the addition in (3.2) is well-defined, we note that it suffices to show that it does not depend on the choice of representative for $[((U,\varphi),\boldsymbol{v})]$, due to (3.3). Indeed, if $[((W,\zeta),\boldsymbol{z})] = [((U,\varphi),\boldsymbol{v})]$, we have

$$\begin{aligned} D(\zeta\circ\varphi^{-1})(\varphi(p))\Big(\boldsymbol{v} + D(\varphi\circ\psi^{-1})(\psi(p))\boldsymbol{w}\Big) &= \\ &= D(\zeta\circ\varphi^{-1})(\varphi(p))\boldsymbol{v} + D(\zeta\circ\varphi^{-1})(\varphi(p))D(\varphi\circ\psi^{-1})(\psi(p))\boldsymbol{w} \\ &= \boldsymbol{z} + D(\zeta\circ\varphi^{-1}\circ\varphi\circ\psi^{-1})(\psi(p))\boldsymbol{w} \\ &= \boldsymbol{z} + D(\zeta\circ\psi^{-1})(\psi(p))\boldsymbol{w}, \end{aligned} \tag{3.4}$$

showing that $[((U,\varphi),\boldsymbol{v} + D(\varphi\circ\psi^{-1})(\psi(p))\boldsymbol{w})] = [((W,\zeta),\boldsymbol{z} + D(\zeta\circ\psi^{-1})(\psi(p))\boldsymbol{w})]$. The neutral element of $(T_pM)_{\text{PHYS.}}$ is, of course, $[((U,\varphi),0)]$.

Lastly, we claim that $\dim(T_pM)_{\text{PHYS.}} = n$. If $(e_1,\ldots,e_n)$ is the canonical basis of $\mathbb{R}^n$ and a chart (U,φ) around p is fixed, then it is easy to see that

$$\{[((U,\varphi),e_1)],\ldots,[((U,\varphi),e_n)]\} \text{ is a basis of } (T_pM)_{\text{PHYS.}}. \tag{3.5}$$

Indeed, if $a^1,\ldots,a^n\in\mathbb{R}$ are such that $\sum_{i=1}^n a^i[((U,\varphi),e_i)] = [((U,\varphi),0)]$, then by definition of addition we have $[((U,\varphi),\sum_{i=1}^n a^ie_i)] = [((U,\varphi),0)]$, and so $\sum_{i=1}^n a^ie_i = 0$, leading to $a^1 = \cdots = a^n = 0$. On the other hand, if $[((U,\varphi),\boldsymbol{v})]\in(T_pM)_{\text{PHYS.}}$ is given, we may write $\boldsymbol{v} = \sum_{i=1}^n a^ie_i$ for suitable coefficients $a^1,\ldots,a^n\in\mathbb{R}$, and then $[((U,\varphi),\boldsymbol{v})] = \sum_{i=1}^n a^i[((U,\varphi),e_i)]$, as expected.

The way physicists actually think about this construction is that to each coordinate system $(x^1,\ldots,x^n)$ we assign a list of numbers $(a^1,\ldots,a^n)$, to be thought of the components (relative to the given coordinates) of the abstract tangent vector being defined, and these numbers must satisfy the *transformation law*

$$\widetilde{a}^j = \sum_{i=1}^n \frac{\partial \widetilde{x}^j}{\partial x^i}\bigg|_p a^i, \quad \text{for all } j = 1,\ldots,n, \tag{3.6}$$

whenever $(\widetilde{a}^1,\ldots,\widetilde{a}^n)$ are the components associated with the coordinates $(\widetilde{x}^1,\ldots,\widetilde{x}^n)$. These partial derivatives of course constitute the entries of the matrix representing $D(\varphi\circ\psi^{-1})(\psi(p))$ — it was a matter of style to consider total derivatives instead of matrix entries above. In any case, the condition (3.6) ensures that

$$\sum_{i=1}^n a^i \frac{\partial}{\partial x^i}\bigg|_p = \sum_{i=1}^n \widetilde{a}^i \frac{\partial}{\partial \widetilde{x}^i}\bigg|_p, \tag{3.7}$$

so that constructions involving coordinate descriptions of tangent vectors ultimately end up being coordinate-independent.

4 Equivalences

In this section, we will finally establish natural isomorphisms

$$(T_pM)_{\text{ALG.}} \cong (T_pM)_{\text{GEOM.}} \cong (T_pM)_{\text{PHYS.}}. \tag{4.1}$$

They will be essentially independent of the choice of any chart but, in the presence of one, they will also identify the bases

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \cdots, \frac{\partial}{\partial x^n}\bigg|_p \right\} \cong \{[\gamma_1],\ldots,[\gamma_n]\} \cong \{[((U,\varphi),e_1)],\ldots,[((U,\varphi),e_n)]\} \tag{4.2}$$

appearing in (1.6), (2.4), and (3.5). Consider the triangle:

$$\begin{array}{ccccc} & & (T_pM)_{\text{ALG.}} & & \\ & \overset{\Phi}{\nearrow} & & \overset{\Psi}{\searrow} & \\ (T_pM)_{\text{GEOM.}} & & \underset{\Theta}{\longleftarrow} & & (T_pM)_{\text{PHYS.}} \end{array} \tag{4.3}$$

Here, we have that

(i) $\Phi([\alpha])[f] = (f\circ\alpha)'(0)$,

(ii) $\Psi(v) = \left[\left((U,\varphi), \sum_{i=1}^n v[x^i]e_i\right)\right]$, (4.4)

(iii) $\Theta[((U,\varphi),\boldsymbol{v})] = [t\mapsto \varphi^{-1}(\varphi(p)+t\boldsymbol{v})]$.

Of course, one must check that such linear transformations are well-defined. For example, consider Θ. If $[((V,\psi),\boldsymbol{w})] = [((U,\varphi),\boldsymbol{v})]$, then both $t \mapsto \varphi^{-1}(\varphi(p)+t\boldsymbol{v})$ and $t \mapsto \psi^{-1}(\psi(p)+t\boldsymbol{w})$ are equal to p when $t = 0$ while, using (U,φ) as the reference chart, we compute the derivatives as

$$\begin{aligned} &\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi\big(\varphi^{-1}(\varphi(p)+t\boldsymbol{v})\big) = \boldsymbol{v} \quad \text{and} \\ &\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi\big(\psi^{-1}(\psi(p)+t\boldsymbol{w})\big) = D(\varphi\circ\psi^{-1})(\psi(p))\boldsymbol{w}, \end{aligned} \tag{4.5}$$

which agree by assumption. Check your understanding and verify that Φ and Ψ are also well-defined.

As all tangent spaces involved have the same finite dimension, we may simultaneously prove that Φ, Ψ, and Θ are isomorphisms without exhibiting their inverses (although we will soon do so, for completeness), just checking instead that $\Theta\circ\Psi\circ\Phi$ or any positive permutation of it equals the identity — that is, that the diagram (4.3) commutes. So, let $[\alpha] \in (T_pM)_{\text{GEOM.}}$, and consider $(\Theta\circ\Psi\circ\Phi)([\alpha])$. We simply unwind it as follows:

$$\begin{aligned} (\Theta\circ\Psi\circ\Phi)([\alpha]) &= \Theta\left[\left((U,\varphi), \sum_{i=1}^n \Phi([\alpha])[x^i]e_i\right)\right] \\ &= \Theta\left[\left((U,\varphi), \sum_{i=1}^n (x^i\circ\alpha)'(0)e_i\right)\right] \\ &= \Theta\left[\big((U,\varphi),(\varphi\circ\alpha)'(0)\big)\right] \\ &= \left[t\mapsto \varphi^{-1}(\varphi(p)+t(\varphi\circ\alpha)'(0))\right] \\ &= [\alpha], \end{aligned} \tag{4.6}$$

with the last equality due to the obvious relation

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi\big(\varphi^{-1}(\varphi(p)+t(\varphi\circ\alpha)'(0))\big) = (\varphi\circ\alpha)'(0). \tag{4.7}$$

Check your understanding and also verify that $(\Psi\circ\Phi\circ\Theta)[((U,\varphi),\boldsymbol{v})] = [((U,\varphi),\boldsymbol{v})]$ and $(\Phi\circ\Theta\circ\Psi)(v) = v$ for all $[((U,\varphi),\boldsymbol{v})] \in (T_pM)_{\text{PHYS.}}$ and $v \in (T_pM)_{\text{ALG.}}$.

As expected, also note that when a chart (U,φ) is fixed, we have that

$$\Phi([\gamma_i]) = \left.\frac{\partial}{\partial x^i}\right|_p, \quad \Psi\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = [((U,\varphi),e_i)], \quad \text{and} \quad \Theta[((U,\varphi),e_i)] = [\gamma_i], \tag{4.8}$$

establishing (4.2).

Finally, we register that the inverses of Φ, Ψ, and Θ, are given by

$$\begin{aligned}
&\text{(i)}\ \ \Phi^{-1}(v) = \left[t \mapsto \varphi^{-1}\left(\varphi(p) + t\textstyle\sum_{i=1}^n v[x^i]e_i\right)\right] \\
&\text{(ii)}\ \ \Psi^{-1}[((U,\varphi),\boldsymbol{v})][f] = \frac{\partial(f\circ\varphi^{-1})}{\partial \boldsymbol{v}}(\varphi(p)) \\
&\text{(iii)}\ \ \Theta^{-1}([\alpha]) = [((U,\varphi),(\varphi\circ\alpha)'(0))].
\end{aligned} \tag{4.9}$$

In the right side of (4.9-ii), $\partial/\partial \boldsymbol{v}$ is an Euclidean directional derivative. The right side of (4.9-i), however, seems to depend on the choice of chart (U,φ) used to build the curve representative of $\Phi^{-1}(v)$. This is not the case: if $(V,\psi = (y^1,\ldots,y^n))$ is a second chart and we use (U,φ) as the reference chart, we have that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\varphi\left(\varphi^{-1}\left(\varphi(p)+t\sum_{i=1}^n v[x^i]e_i\right)\right) = \sum_{i=1}^n v[x^i]e_i, \tag{4.10}$$

while

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\varphi\left(\psi^{-1}\left(\psi(p)+t\sum_{j=1}^n v[y^j]e_j\right)\right) &= D(\varphi\circ\psi^{-1})(\psi(p))\sum_{j=1}^n v[y^j]e_j \\
&= \sum_{i,j=1}^n v[y^j]\frac{\partial(\varphi\circ\psi^{-1})^i}{\partial y^j}(\psi(p))e_i.
\end{aligned} \tag{4.11}$$

But the relation

$$v[x^i] = \sum_{j=1}^n v[y^j]\frac{\partial(\varphi\circ\psi^{-1})^i}{\partial y^j}(\psi(p)) \tag{4.12}$$

must hold: we use Hadamard's lemma to write

$$x^i = x^i(p) + \sum_{j=1}^n (y^j - y^j(p))g^i_j \tag{4.13}$$

for some smooth functions g^i_j defined on an open neighborhood of p and such that $g^i_j(p) = (\partial(\varphi\circ\psi^{-1})^i/\partial y^j)(\psi(p))$ for all j, and then apply the derivation v to both sides of (4.13); so (4.12) follows. This means that

$$\left[t\mapsto\varphi^{-1}\left(\varphi(p)+t\sum_{i=1}^n v[x^i]e_i\right)\right] = \left[t\mapsto\psi^{-1}\left(\psi(p)+t\sum_{i=1}^n v[y^i]e_i\right)\right], \tag{4.14}$$

as wanted.