Some conformal geometry formulas

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Fix a pseudo-Riemannian manifold (M, g) and a smooth function $\varphi\colon M \to \mathbb{R}$. With this, define a new pseudo-Riemannian metric $\widetilde{\mathrm{g}} = \mathrm{e}^{2\varphi}\mathrm{g}$. Let ∇ and $\widetilde{\nabla}$ be the Levi-Civita connections of (M, g) and $(M, \widetilde{\mathrm{g}})$, respectively, and let $U = \operatorname{grad}_{\mathrm{g}}\varphi$ (this field U is called the *conformal gradient* of (M, g)). It will be convenient to set $H = \operatorname{Hess}_{\mathrm{g}}\varphi$, so that $H(X, Y) = \mathrm{g}(\nabla_X U, Y)$, and also $(\mathrm{d}\varphi)^{\otimes 2} = \mathrm{d}\varphi \otimes \mathrm{d}\varphi$ in some proofs. Recall that H is symmetric because ∇ is torsion-free. We'll also write $\|\mathrm{d}\varphi\|^2_{\mathrm{g}} = \mathrm{g}(U, U)$, but this has to be interpreted as a formal symbol, as it might be zero or negative in case g is not Riemannian.

Theorem 1. *For all $X, Y \in \mathfrak{X}(M)$, the relation*

$$\widetilde{\nabla}_X Y = \nabla_X Y + \mathrm{g}(Y, U)X + \mathrm{g}(X, U)Y - \mathrm{g}(X, Y)U$$

holds.

Proof: Recall the Koszul formula

$$2\mathrm{g}(\nabla_X Y, Z) = (\mathscr{L}_Y \mathrm{g})(X, Z) + \mathrm{d}(Y_\flat)(X, Z),$$

where $\mathscr{L}$ is the Lie derivative and the exterior derivative of $Y_\flat = \mathrm{g}(Y, \cdot)$ is to be thought as the curl of Y. First, note that

$$\mathscr{L}_Y \widetilde{\mathrm{g}} = \mathscr{L}_Y(\mathrm{e}^{2\varphi}\mathrm{g}) = Y(\mathrm{e}^{2\varphi})\mathrm{g} + \mathrm{e}^{2\varphi}\mathscr{L}_Y \mathrm{g} = 2Y(\varphi)\mathrm{e}^{2\varphi}\mathrm{g} + \mathrm{e}^{2\varphi}\mathscr{L}_Y \mathrm{g}.$$

Secondly, that $Y_{\widetilde{\flat}} = \mathrm{e}^{2\varphi} Y_\flat$ implies that

$$\mathrm{d}(Y_{\widetilde{\flat}}) = 2\mathrm{e}^{2\varphi}\mathrm{d}\varphi \wedge Y_\flat + \mathrm{e}^{2\varphi}\mathrm{d}(Y_\flat),$$

so that

$$\begin{aligned} 2\widetilde{\mathrm{g}}(\widetilde{\nabla}_X Y, Z) &= (\mathscr{L}_Y \widetilde{\mathrm{g}})(X, Z) + \mathrm{d}(Y_{\widetilde{\flat}})(X, Z) \\ &= 2Y(\varphi)\mathrm{e}^{2\varphi}\mathrm{g}(X, Z) + \mathrm{e}^{2\varphi}(\mathscr{L}_Y \mathrm{g})(X, Z) + 2\mathrm{e}^{2\varphi}(\mathrm{d}\varphi \wedge Y_\flat)(X, Z) + \mathrm{e}^{2\varphi}\mathrm{d}(Y_\flat)(X, Z) \\ &= 2Y(\varphi)\mathrm{e}^{2\varphi}\mathrm{g}(X, Z) + 2\mathrm{e}^{2\varphi}\mathrm{g}(\nabla_X Y, Z) + 2\mathrm{e}^{2\varphi}(X(\varphi)\mathrm{g}(Y, Z) - Z(\varphi)\mathrm{g}(Y, X)). \end{aligned}$$

Divide everything through by $2\mathrm{e}^{2\varphi}$ and use that $\widetilde{\mathrm{g}} = \mathrm{e}^{2\varphi}\mathrm{g}$ to get

$$\mathrm{g}(\widetilde{\nabla}_X Y, Z) = Y(\varphi)\mathrm{g}(X, Z) + \mathrm{g}(\nabla_X Y, Z) + X(\varphi)\mathrm{g}(Y, Z) - Z(\varphi)\mathrm{g}(X, Y).$$

Rewriting the above using the conformal gradient U instead of $\mathrm{d}\varphi$ and reorganizing the right side in the form $\mathrm{g}(*, Z)$, we have that

$$\mathrm{g}(\widetilde{\nabla}_X Y, Z) = \mathrm{g}(\nabla_X Y + \mathrm{g}(Y,U)Z + \mathrm{g}(X,U)Y - \mathrm{g}(X,Y)U, Z).$$

Since Z is arbitrary and g is non-degenerate, we conclude that

$$\widetilde{\nabla}_X Y = \nabla_X Y + \mathrm{g}(Y,U)X + \mathrm{g}(X,U)Y - \mathrm{g}(X,Y)U,$$

as wanted. □

What about the curvature? First, we take $X, Y, Z \in \mathfrak{X}(M)$ and directly compute that:

$$\begin{aligned}
\widetilde{\nabla}_X\widetilde{\nabla}_Y Z &= \nabla_X \widetilde{\nabla}_Y Z + \mathrm{g}(\widetilde{\nabla}_Y Z, U)X + \mathrm{g}(X,U)\widetilde{\nabla}_Y Z - \mathrm{g}(X, \widetilde{\nabla}_Y Z)U \\
&= \nabla_X\nabla_Y Z + \mathrm{g}(\nabla_X Z, U)Y + \mathrm{g}(Z, \nabla_X U)Y + \mathrm{g}(Z,U)\nabla_X Y \\
&\quad + \mathrm{g}(\nabla_X Y, U)Z + \mathrm{g}(Y, \nabla_X U)Z + \mathrm{g}(Y,U)\nabla_X Z \\
&\quad - \mathrm{g}(\nabla_X Y, Z)U - \mathrm{g}(Y, \nabla_X Z)U - \mathrm{g}(Y,Z)\nabla_X U \\
&\quad + \mathrm{g}(\nabla_Y Z, U)X + \mathrm{g}(Z,U)\mathrm{g}(Y,U)X + \mathrm{g}(Y,U)\mathrm{g}(Z,U)X - \|\mathrm{d}\varphi\|_{\mathrm{g}}^2 \mathrm{g}(Y,Z)X \\
&\quad + \mathrm{g}(X,U)\nabla_Y Z + \mathrm{g}(X,U)\mathrm{g}(Z,U)Y + \mathrm{g}(X,U)\mathrm{g}(Y,U)Z - \mathrm{g}(X,U)\mathrm{g}(Y,Z)U \\
&\quad - \mathrm{g}(X, \nabla_Y Z)U - \mathrm{g}(Z,U)\mathrm{g}(X,Y)U - \mathrm{g}(Y,U)\mathrm{g}(X,Z)U + \mathrm{g}(Y,Z)\mathrm{g}(X,U)U \\
&= \nabla_X\nabla_Y Z + \mathrm{g}(\nabla_X Z, U)Y + H(X,Z)Y + \mathrm{g}(Z,U)\nabla_X Y \\
&\quad + \mathrm{g}(\nabla_X Y, U)Z + H(X,Y)Z + \mathrm{g}(Y,U)\nabla_X Z \\
&\quad - \mathrm{g}(\nabla_X Y, Z)U - \mathrm{g}(Y, \nabla_X Z)U - \mathrm{g}(Y,Z)\nabla_X U \\
&\quad + \mathrm{g}(\nabla_Y Z, U)X + \mathrm{g}(Z,U)\mathrm{g}(Y,U)X + \mathrm{g}(Y,U)\mathrm{g}(Z,U)X - \|\mathrm{d}\varphi\|_{\mathrm{g}}^2 \mathrm{g}(Y,Z)X \\
&\quad + \mathrm{g}(X,U)\nabla_Y Z + \mathrm{g}(X,U)\mathrm{g}(Z,U)Y + \mathrm{g}(X,U)\mathrm{g}(Y,U)Z \\
&\quad - \mathrm{g}(X, \nabla_Y Z)U - \mathrm{g}(Z,U)\mathrm{g}(X,Y)U - \mathrm{g}(Y,U)\mathrm{g}(X,Z)U.
\end{aligned}$$

With this in place, we may compute the curvature tensor.

Theorem 2. *For all $X, Y, Z, W \in \mathfrak{X}(M)$, we have that*

$$\widetilde{R} = \mathrm{e}^{2\varphi} R - 2\mathrm{e}^{2\varphi}\mathrm{g} \owedge \left(\mathrm{Hess}_{\mathrm{g}}\varphi - \mathrm{d}\varphi \otimes \mathrm{d}\varphi + \frac{\|\mathrm{d}\varphi\|_{\mathrm{g}}^2}{2}\mathrm{g} \right),$$

as $(0,4)$-tensors.

Proof: By definition, we have

$$\begin{aligned}
\widetilde{R}(X,Y)Z = {} & \nabla_X\nabla_Y Z + \mathrm{g}(\nabla_X Z, U)Y + H(X,Z)Y + \mathrm{g}(Z,U)\nabla_X Y \\
& + \mathrm{g}(\nabla_X Y, U)Z + H(X,Y)Z + \mathrm{g}(Y,U)\nabla_X Z \\
& - \mathrm{g}(\nabla_X Y, Z)U - \mathrm{g}(Y, \nabla_X Z)U - \mathrm{g}(Y,Z)\nabla_X U \\
& + \mathrm{g}(\nabla_Y Z, U)X + \mathrm{g}(Z,U)\mathrm{g}(Y,U)X + \mathrm{g}(Y,U)\mathrm{g}(Z,U)X - \|\mathrm{d}\varphi\|_{\mathrm{g}}^2 \mathrm{g}(Y,Z)X \\
& + \mathrm{g}(X,U)\nabla_Y Z + \mathrm{g}(X,U)\mathrm{g}(Z,U)Y + \mathrm{g}(X,U)\mathrm{g}(Y,U)Z
\end{aligned}$$

$$\begin{aligned}
&-\mathrm{g}(X,\nabla_Y Z)U-\mathrm{g}(Z,U)\mathrm{g}(X,Y)U-\mathrm{g}(Y,U)\mathrm{g}(X,Z)U\\
&-\nabla_Y\nabla_X Z-\mathrm{g}(\nabla_Y Z,U)X-H(Y,Z)X-\mathrm{g}(Z,U)\nabla_Y X\\
&-\mathrm{g}(\nabla_Y X,U)Z-H(Y,X)Z-\mathrm{g}(X,U)\nabla_Y Z\\
&+\mathrm{g}(\nabla_Y X,Z)U+\mathrm{g}(X,\nabla_Y Z)U+\mathrm{g}(X,Z)\nabla_Y U\\
&-\mathrm{g}(\nabla_X Z,U)Y-\mathrm{g}(Z,U)\mathrm{g}(X,U)Y-\mathrm{g}(X,U)\mathrm{g}(Z,U)Y+\|\mathrm{d}\varphi\|^2_{\mathrm{g}}\mathrm{g}(X,Z)Y\\
&-\mathrm{g}(Y,U)\nabla_X Z-\mathrm{g}(Y,U)\mathrm{g}(Z,U)X-\mathrm{g}(Y,U)\mathrm{g}(X,U)Z\\
&+\mathrm{g}(Y,\nabla_X Z)U+\mathrm{g}(Z,U)\mathrm{g}(Y,X)U+\mathrm{g}(X,U)\mathrm{g}(Y,Z)U\\
&-\nabla_{[X,Y]}Z-\mathrm{g}(Z,U)[X,Y]-\mathrm{g}([X,Y],U)Z+\mathrm{g}([X,Y],Z)U.
\end{aligned}$$

Since $\widetilde{R}$ is a tensor, it should depend linearly on X, Y and Z, no covariant derivatives of these fields should appear, we'll have terms with U, and terms with ∇U. The reader should take 5 or 10 minutes to parse the above carefully. With these cancellations, we obtain:

$$\begin{aligned}
\widetilde{R}(X,Y)Z = R(X,Y)Z &+ H(X,Z)Y - \mathrm{g}(Y,Z)\nabla_X U + \mathrm{g}(Y,U)\mathrm{g}(Z,U)X\\
&- \|\mathrm{d}\varphi\|^2_{\mathrm{g}}\mathrm{g}(Y,Z)X - \mathrm{g}(Y,U)\mathrm{g}(X,Z)U - H(Y,Z)X + \mathrm{g}(X,Z)\nabla_Y U\\
&- \mathrm{g}(X,U)\mathrm{g}(Z,U)Y + \|\mathrm{d}\varphi\|^2_{\mathrm{g}}\mathrm{g}(X,Z)Y + \mathrm{g}(X,U)\mathrm{g}(Y,Z)U.
\end{aligned}$$

So, we g-multiply by a fourth field W to get:

$$\begin{aligned}
\mathrm{g}(\widetilde{R}(X,Y)Z,W) = R(X,Y,Z,W) &+ H(X,Z)\mathrm{g}(Y,W) - \mathrm{g}(Y,Z)H(X,W)\\
&+\mathrm{g}(Y,U)\mathrm{g}(Z,U)\mathrm{g}(X,W) - \|\mathrm{d}\varphi\|^2_{\mathrm{g}}\mathrm{g}(Y,Z)\mathrm{g}(X,W)\\
&-\mathrm{g}(Y,U)\mathrm{g}(X,Z)\mathrm{g}(U,W) - H(Y,Z)\mathrm{g}(X,W)\\
&+\mathrm{g}(X,Z)H(Y,W) - \mathrm{g}(X,U)\mathrm{g}(Z,U)\mathrm{g}(Y,W)\\
&+\|\mathrm{d}\varphi\|^2_{\mathrm{g}}\mathrm{g}(X,Z)\mathrm{g}(Y,W) + \mathrm{g}(X,U)\mathrm{g}(Y,Z)\mathrm{g}(U,W)
\end{aligned}$$

By definition of the Kulkarni-Nomizu product $⃝\!\!\!\!\wedge$, we have that

$$\begin{aligned}
\mathrm{g}(\widetilde{R}(X,Y)Z,W) = R(X,Y,Z,W) &- 2(\mathrm{g} ⃝\!\!\!\!\wedge\, H)(X,Y,Z,W)\\
&- \|\mathrm{d}\varphi\|^2_{\mathrm{g}}(\mathrm{g} ⃝\!\!\!\!\wedge\, \mathrm{g})(X,Y,Z,W) + 2(\mathrm{g} ⃝\!\!\!\!\wedge\, (\mathrm{d}\varphi)^{\otimes 2})(X,Y,Z,W)
\end{aligned}$$

Multiply everything by $\mathrm{e}^{2\varphi}$ to conclude. □

Example. In the upper half-space $\mathbb{H}^{n+1} = \mathbb{R}^n \times \mathbb{R}_{>0}$, let $\mathrm{g} = |\mathrm{d}\boldsymbol{x}|^2 + \mathrm{d}y^2$ be the standard flat metric, and $\widetilde{\mathrm{g}} = (|\mathrm{d}\boldsymbol{x}|^2 + \mathrm{d}y^2)/y^2$ be the hyperbolic metric. Then $\mathrm{e}^{2\varphi} = y^{-2}$ means that the conformal factor is $\varphi(\boldsymbol{x},y) = -\log y$. We have that $\mathrm{d}\varphi = -\mathrm{d}y/y$ and also $\mathrm{Hess}_{\mathrm{g}}\varphi = (\mathrm{d}y \otimes \mathrm{d}y)/y^2$. In particular, $\|\mathrm{d}\varphi\|^2_{\mathrm{g}} = 1/y^2$. It follows that

$$\widetilde{R} = \frac{1}{y^2}\cdot 0 - \frac{2}{y^2}\mathrm{g} ⃝\!\!\!\!\wedge \left(\frac{\mathrm{d}y\otimes\mathrm{d}y}{y^2} - \frac{\mathrm{d}y\otimes\mathrm{d}y}{y^2} + \frac{1}{2y^2}\mathrm{g}\right) = -\widetilde{\mathrm{g}} ⃝\!\!\!\!\wedge\, \widetilde{\mathrm{g}}.$$

Hence $(\mathbb{H}^{n+1},\widetilde{\mathrm{g}})$ has constant sectional curvature equal to -1.

For all the other curvatures, we'll need metric traces. So:

Lemma 3. *Let T be a $(0,2)$-tensor field on M. Then* $\operatorname{tr}_{\widetilde{g}} T = e^{-2\varphi}\operatorname{tr}_g T$.

Proof: In coordinates. We have that $\widetilde{g}_{ij} = e^{2\varphi} g_{ij}$ implies $\widetilde{g}^{ij} = e^{-2\varphi} g^{ij}$, so that

$$\operatorname{tr}_{\widetilde{g}} T = \widetilde{g}^{ij} T_{ij} = e^{-2\varphi} g^{ij} T_{ij} = e^{-2\varphi}\operatorname{tr}_g T.$$

□

Theorem 4. *The relation between Ricci tensors is:*

$$\widetilde{\operatorname{Ric}} = \operatorname{Ric} - (n-2)(\operatorname{Hess}_g \varphi - d\varphi \otimes d\varphi) - (\triangle_g \varphi + (n-2)\|d\varphi\|_g^2) g,$$

where $\triangle_g \varphi$ is the g-Laplacian of φ.

Proof: It suffices to recall that if T is any symmetric $(0,2)$-tensor field on (M, g), then the relation $2\operatorname{Ric}(g \owedge T) = (n-2)T + \operatorname{tr}_g(T) g$ holds, where $\operatorname{Ric}(\cdot)$ is the abstract Ricci contraction of a curvaturelike tensor, on the first and last arguments. When applying the $\widetilde{g}$-trace, the previous lemma will cancel the $e^{2\varphi}$ factors on the right side, so recalling that $\operatorname{tr}_g((d\varphi)^{\otimes 2}) = \|d\varphi\|_g^2$, we obtain

$$\widetilde{\operatorname{Ric}} = \operatorname{Ric} - \left((n-2)\left(\operatorname{Hess}_g \varphi - (d\varphi)^{\otimes 2} + \frac{\|d\varphi\|_g^2}{2} g \right) + \left(\triangle_g \varphi - \|d\varphi\|_g^2 + \frac{n\|d\varphi\|_g^2}{2} \right) g \right).$$

Reorganize. □

Theorem 5. *The relation between scalar curvatures is:*

$$\widetilde{s} = e^{-2\varphi}\left(s - (2n-2)\triangle_g \varphi - (n-1)(n-2)\|d\varphi\|_g^2 \right).$$

Proof: Take $\widetilde{g}$-trace of both sides using the lemma to obtain

$$\begin{aligned} \widetilde{s} &= e^{-2\varphi} s - (n-2)e^{-2\varphi}(\triangle_g \varphi - \|d\varphi\|_g^2) - n e^{-2\varphi}(\triangle_g \varphi + (n-2)\|d\varphi\|_g^2) \\ &= e^{-2\varphi}\left(s - (2n-2)\triangle_g \varphi - (n-1)(n-2)\|d\varphi\|_g^2 \right). \end{aligned}$$

□

Remark (Korn-Lichtenstein theorem)**.** When $n = 2$, the above turns out to be the relation between Gaussian curvatures: $\widetilde{K} = e^{-2\varphi}(K - \triangle_g \varphi)$. If g is Riemannian, the PDE $\triangle_g \varphi = K$ is elliptic *and so has local solutions*. This shows the existence of isothermal coordinates in any surface or, in other words, any point in any surface has a neighborhood conformally equivalent to an open subset of $\mathbb{R}^2$ equipped with the standard flat metric. Of course, what makes this theorem extremely hard to prove is justifying the italicized part above.

Remark (The Yamabe Problem)**.** It is also convenient to recast the scalar curvature equation in a different form, by letting $e^{2\varphi} = u^{4/(n-2)}$. We have that

$$\varphi = \frac{2}{n-2}\log u \implies d\varphi = \frac{2}{n-2}\frac{du}{u} \implies \|d\varphi\|_g^2 = \frac{4}{(n-2)^2}\frac{\|du\|_g^2}{u^2}.$$

We also have that

$$\nabla(\mathrm{d}\varphi) = \frac{2}{n-2}\left(\frac{\nabla(\mathrm{d}u)}{u} - \frac{\mathrm{d}u\otimes\mathrm{d}u}{u^2}\right) \implies \triangle_{\mathrm{g}}\varphi = \frac{2}{n-2}\left(\frac{\triangle_{\mathrm{g}}u}{u} - \frac{\|\mathrm{d}u\|_{\mathrm{g}}^2}{u^2}\right).$$

With this, we obtain

$$\begin{aligned} u^{4/(n-2)}\widetilde{\mathrm{s}} &= \mathrm{s} - 2(n-1)\frac{2}{n-2}\left(\frac{\triangle_{\mathrm{g}}u}{u} - \frac{\|\mathrm{d}u\|_{\mathrm{g}}^2}{u^2}\right) - (n-1)(n-2)\frac{4}{(n-2)^2}\frac{\|\mathrm{d}u\|_{\mathrm{g}}^2}{u^2} \\ &= \mathrm{s} - \frac{4(n-1)}{n-2}\frac{\triangle_{\mathrm{g}}u}{u} + \frac{4(n-1)}{n-2}\frac{\|\mathrm{d}u\|_{\mathrm{g}}^2}{u^2} - \frac{4(n-1)}{n-2}\frac{\|\mathrm{d}u\|_{\mathrm{g}}^2}{u^2} \\ &= \mathrm{s} - \frac{4(n-1)}{n-2}\frac{\triangle_{\mathrm{g}}u}{u}. \end{aligned}$$

So we see that the advantage of the (a priori weird) power $4/(n-2)$ is to eliminate $\|\mathrm{d}u\|_{\mathrm{g}}^2$ from the equation. The classical Yamabe problem asks whether every Riemannian metric is conformally equivalent to a metric with constant scalar curvature. We see that it is equivalent to the existence of a constant c and a function u such that

$$\mathrm{s}u - \frac{4(n-1)}{n-2}\triangle_{\mathrm{g}}u + cu^{(n+2)/(n-2)} = 0.$$

There is always c for which this PDE (called a *Yamabe-type* equation) has a solution for the conformal factor u. One may also reorganize the equation as to make the coefficient of $\triangle_{\mathrm{g}}$ equal to 1, so

$$\triangle_{\mathrm{g}}u - \frac{(n-2)}{4(n-1)}\mathrm{s}u = cu^{(n+2)/(n-2)},$$

after renaming c. Thus, one defines a differential operator $\mathrm{Y}_{\mathrm{g}}\colon \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$ by

$$\mathrm{Y}_{\mathrm{g}} = \triangle_{\mathrm{g}} - \frac{(n-2)}{4(n-1)}\mathrm{s},$$

and properties of Y_{g} are used in the study of the Yamabe equation.

Recall that the Schouten tensor of (M, g) is defined by

$$\mathrm{Sch} = \mathrm{Ric} - \frac{\mathrm{s}}{2(n-1)}\mathrm{g},$$

and similarly for $(M, \widetilde{\mathrm{g}})$.

Theorem 6. *The relation between Schouten tensors is*

$$\widetilde{\mathrm{Sch}} = \mathrm{Sch} - (n-2)(\mathrm{Hess}_{\mathrm{g}}\varphi - \mathrm{d}\varphi\otimes\mathrm{d}\varphi) - \frac{(n-2)}{2}\|\mathrm{d}\varphi\|_{\mathrm{g}}^2\mathrm{g}.$$

Proof: It's a straightforward computation based on everything obtained so far:

$$\widetilde{\mathrm{Sch}} = \widetilde{\mathrm{Ric}} - \frac{\widetilde{\mathrm{s}}}{2(n-1)}\widetilde{\mathrm{g}}$$

$$
\begin{aligned}
&= \mathrm{Ric} - (n-2)(\mathrm{Hess}_{\mathrm{g}}\varphi - (\mathrm{d}\varphi)^{\otimes 2}) - (\triangle_{\mathrm{g}}\varphi + (n-2)\|\mathrm{d}\varphi\|_{\mathrm{g}}^2)\mathrm{g} \\
&\quad - \frac{\mathrm{e}^{-2\varphi}\left(\mathrm{s} - (2n-2)\triangle_{\mathrm{g}}\varphi - (n-1)(n-2)\|\mathrm{d}\varphi\|_{\mathrm{g}}^2\right)}{2(n-1)}\mathrm{e}^{2\varphi}\mathrm{g} \\
&= \mathrm{Ric} - (n-2)(\mathrm{Hess}_{\mathrm{g}}\varphi - (\mathrm{d}\varphi)^{\otimes 2}) - (\triangle_{\mathrm{g}}\varphi + (n-2)\|\mathrm{d}\varphi\|_{\mathrm{g}}^2)\mathrm{g} \\
&\quad - \frac{\mathrm{s}}{2(n-1)}\mathrm{g} + (\triangle_{\mathrm{g}}\varphi)\mathrm{g} + \frac{(n-2)}{2}\|\mathrm{d}\varphi\|_{\mathrm{g}}^2\mathrm{g} \\
&= \mathrm{Sch} - (n-2)(\mathrm{Hess}_{\mathrm{g}}\varphi - (\mathrm{d}\varphi)^{\otimes 2}) - \frac{(n-2)}{2}\|\mathrm{d}\varphi\|_{\mathrm{g}}^2\mathrm{g}.
\end{aligned}
$$

□

There's one last relevant tensor to discuss: the Weyl tensor of (M, g) is defined by

$$\mathrm{W} = R - \frac{2}{n-2}\mathrm{g} \owedge \mathrm{Sch}.$$

Similarly for $(M, \widetilde{\mathrm{g}})$.

Theorem 7. *The relation between the Weyl tensors is:*

(i) $\widetilde{\mathrm{W}} = \mathrm{e}^{2\varphi}\mathrm{W}$, *as* $(0,4)$*-tensors.*

(ii) $\widetilde{\mathrm{W}} = \mathrm{W}$, *as* $(1,3)$*-tensors.*

Explicitly: the $(1,3)$*-type Weyl tensor is a conformal invariant of* (M, g)*.*

Proof: Again, we compute:

$$
\begin{aligned}
\widetilde{\mathrm{W}} &= \widetilde{R} - \frac{2}{n-2}\widetilde{\mathrm{g}} \owedge \widetilde{\mathrm{Sch}} \\
&= \mathrm{e}^{2\varphi}R - 2\mathrm{e}^{2\varphi}\mathrm{g} \owedge \left(H - \mathrm{d}\varphi^{\otimes 2} + \frac{\|\mathrm{d}\varphi\|_{\mathrm{g}}^2}{2}\mathrm{g}\right) \\
&\quad - \frac{2}{n-2}\mathrm{e}^{2\varphi}\mathrm{g} \owedge \left(\mathrm{Sch} - (n-2)(H - (\mathrm{d}\varphi)^{\otimes 2}) - \frac{(n-2)}{2}\|\mathrm{d}\varphi\|_{\mathrm{g}}^2\mathrm{g}\right) \\
&= \mathrm{e}^{2\varphi}R - 2\mathrm{e}^{2\varphi}\mathrm{g} \owedge \left(H - \mathrm{d}\varphi^{\otimes 2} + \frac{\|\mathrm{d}\varphi\|_{\mathrm{g}}^2}{2}\mathrm{g}\right) \\
&\quad - \frac{2}{n-2}\mathrm{e}^{2\varphi}\mathrm{g} \owedge \mathrm{Sch} - 2\mathrm{e}^{2\varphi}\mathrm{g} \owedge \left(H - (\mathrm{d}\varphi)^{\otimes 2} + \frac{\|\mathrm{d}\varphi\|_{\mathrm{g}}^2}{2}\mathrm{g}\right) \\
&= \mathrm{e}^{2\varphi}\left(R - \frac{2}{n-2}\mathrm{g} \owedge \mathrm{Sch}\right) \\
&= \mathrm{e}^{2\varphi}\mathrm{W}.
\end{aligned}
$$

This means that $\widetilde{\mathrm{g}}(\widetilde{\mathrm{W}}(X,Y)Z, W) = \mathrm{e}^{2\varphi}\mathrm{g}(\mathrm{W}(X,Y)Z, W)$ for all $X, Y, Z, W \in \mathfrak{X}(M)$. Since $\widetilde{\mathrm{g}} = \mathrm{e}^{2\varphi}\mathrm{g}$, we may cancel the factor $\mathrm{e}^{2\varphi}$ everywhere to get $\widetilde{\mathrm{W}}(X,Y)Z = \mathrm{W}(X,Y)Z$ for all $X, Y, Z \in \mathfrak{X}(M)$, as wanted. □

Theorem 8. *For any smooth $f\colon M \to \mathbb{R}$, we have*

$$\triangle_{\widetilde{g}} f = e^{-2\varphi}(\triangle_g f + (n-2)g(d\varphi, df)).$$

Proof: First we compute the Hessian

$$\begin{aligned}(\widetilde{\nabla}_X(df))Y &= \widetilde{\nabla}_X(df(Y)) - df(\widetilde{\nabla}_X Y)\\ &= \nabla_X(df(Y)) - df(\nabla_X Y + g(Y,U)X + g(X,U)Y - g(X,Y)U)\\ &= \nabla_X(df(Y)) - df(\nabla_X Y) - g(Y,U)df(X) - g(X,U)df(Y) + g(X,Y)df(U)\\ &= (\nabla_X(df))Y - g(Y,U)df(X) - g(X,U)df(Y) + g(X,Y)df(U).\end{aligned}$$

Now apply the $\widetilde{g}$-trace on both sides to obtain

$$\begin{aligned}\triangle_{\widetilde{g}} f &= e^{-2\varphi}(\triangle_g f - df(U) - df(U) + n\, df(U))\\ &= e^{-2\varphi}(\triangle_g f + (n-2)g(d\varphi, df)),\end{aligned}$$

as required.

□