Some index computations with curvature tensors

Ivo Terek*

1 Definitions

Let (M^n, g), $n \geq 3$, be a (connected) pseudo-Riemannian manifold.

Definition 1 (Products).

(i) Given $\alpha \in \Omega^1(M)$ and $\beta \in \Omega^2(M)$, we regard β as an element of $\Omega^1(M; T^*M)$ and compute the actual wedge product $\alpha \wedge \beta \in \Omega^2(M; T^*M)$. It is given by $(\alpha \wedge \beta)(X, Y)Z = \alpha(X)\beta(Y, Z) - \alpha(Y)\beta(X, Z)$.

(ii) Given symmetric tensors $T, S \in \Gamma(T^*M^{\odot 2})$, we define their *Kulkarni-Nomizu product* as

$$2(T \owedge S)(X, Y, Z, W) = T(Y, Z)S(X, W) - T(X, Z)S(Y, W) + \mathrm{switch}(T \leftrightarrow S),$$

where by $\mathrm{switch}(T \leftrightarrow S)$ we mean the previous terms with T and S switched.

Remark. The relation $\nabla_X(T \owedge S) = (\nabla_X T) \owedge S + T \owedge (\nabla_X S)$ always holds. This is easily verified pointwise by using a geodesic frame centered at the arbitrary chosen point.

Definition 2 (Curvatures).

(i) The Riemann curvature tensor of (M, g) is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

(ii) The Ricci tensor is $\mathrm{Ric}(Y, Z) = \mathrm{tr}\,(X \mapsto R(X, Y)Z)$.

(iii) The scalar curvature is $\mathrm{s} = \mathrm{tr}_g \mathrm{Ric}$.

With the operation $\owedge$, we have that (M, g) has constant curvature c if $R = c(g \owedge g)$. In general, we may decompose R as

$$R = \frac{\mathrm{s}}{n(n-1)} g \owedge g + \frac{2}{n-2} g \owedge \left(\mathrm{Ric} - \frac{\mathrm{s}}{n} g\right) + \mathrm{W},$$

*terekcouto.1@osu.edu

where W is called the *Weyl tensor* of (M, g). Essentially, this decomposition consists in writing R as a sum of: a multiple of $g \owedge g$, a term $g \owedge E$ where $\operatorname{tr}_g E = 0$, and a term W whose abstract Ricci contraction vanishes. The tensor W controls conformal flatness, and while we can just solve for W in the above formula, it is more convenient to write it as

$$\mathrm{W} = R - \frac{2}{n-2} g \owedge \mathrm{Sch},$$

where

$$\mathrm{Sch} = \mathrm{Ric} - \frac{\mathrm{s}}{2(n-1)} g$$

is called the *Schouten tensor* of (M, g).

Definition 3 (Divergence)**.**

(i) If $T \in \Gamma(T^*M^{\otimes k})$ is a tensor field, we define the g-divergence $\delta T \in \Gamma(T^*M^{\otimes(k-1)})$ as

$$(\delta T)(X_1, \ldots, X_{k-1}) = \operatorname{tr}_g((X, Y) \mapsto (\nabla_X T)(X_1, \ldots, X_{k-1}, Y)).$$

(ii) If T is a type $(1, k)$-tensor, we define the g-divergence δT as the $(0, k)$ tensor given by

$$\delta T(X_1, \ldots, X_k) = \operatorname{tr}(X \mapsto (\nabla_X T)(X_1, \ldots, X_k))$$

Definition 4 (Exterior derivative)**.** Let $T \in \Gamma(T^*M^{\otimes 2})$ be a tensor field. We define the exterior derivative $\mathrm{d}^\nabla T \in \Gamma(T^*M^{\otimes 3})$ by

$$(\mathrm{d}^\nabla T)(X, Y)Z = (\nabla_X T)(Y, Z) - (\nabla_Y T)(X, Z).$$

Remark. Since ∇ is torsion-free, the above is the same as regarding T as an element of $\Omega^1(M; T^*M)$ and then taking the exterior derivative $\mathrm{d}^\nabla T \in \Omega^2(M; T^*M)$ with the aid of ∇.

Using coordinates, and denoting covariant derivatives in direction of coordinate fields with a semi-colon, we have that:

- $(\alpha \wedge \beta)_{ijk} = \alpha_i \beta_{jk} - \alpha_j \beta_{ik}$
- $2(T \owedge S)_{ijk\ell} = T_{jk} S_{i\ell} - T_{ik} S_{j\ell} + S_{jk} T_{i\ell} - S_{ik} T_{j\ell}$
- $R(\partial_i, \partial_j)\partial_k = R_{ijk}{}^{\ell} \partial_\ell$
- $\mathrm{Ric}(\partial_i, \partial_j) = R_{jk}$ with $R_{jk} = R_{ijk}{}^{i}$
- $\mathrm{s} = g^{ij} R_{ij}$
- $(\delta T)_{i_1 \cdots i_{k-1}} = g^{ij} T_{i_1 \cdots i_{k-1} i; j}$ if T is of type $(0, k)$
- $(\delta T)_{i_1 \cdots i_k} = T_{i_1 \cdots i_k}{}^{j}{}_{;j}$ if T is of type $(1, k)$.
- $(\mathrm{d}^\nabla T)_{ijk} = T_{jk;i} - T_{ik;j}$

- $R_{ijk\ell} + R_{jki\ell} + R_{kij\ell} = 0$ (first Bianchi identity $\mathrm{d}^\nabla \tau = 0$)
- $R_{ijk\ell;r} + R_{jrk\ell;i} + R_{rik\ell;j} = 0$ (second Bianchi identity $\mathrm{d}^\nabla R = 0$).

Lastly, recall that if $E \to M$ is a vector bundle equipped with a connection (also to be denoted by ∇), and $\psi \in \Gamma(E)$, one may define the second covariant derivative of ψ by $(\nabla_X(\nabla\psi))Y = \nabla_X\nabla_Y\psi - \nabla_{\nabla_X Y}\psi$. Using coordinates (x^j) in M and a local trivialization (e_a), we may write

$$\psi = \psi^a e_a, \quad \nabla_{\partial_j}\psi = \psi^a{}_{;j} e_a \quad \text{and} \quad (\nabla_{\partial_k}(\nabla\psi))\partial_j = \psi^a{}_{;jk} e_a.$$

In the last expression, note that the index k is the second index in $\psi^a{}_{;jk}$ because ∇_{∂_k} is the last derivative to be applied. In particular, we have that $\psi^a{}_{;kj} - \psi^a{}_{;jk} = R_{jkb}{}^a\psi^b$, by definition of curvature.

This should be enough to get us going. Unless said otherwise, f stands for an arbitrary $f \in \mathscr{C}^\infty(M)$.

2 Formulas

Proposition 5. *$\delta(\nabla X) - \mathrm{d}(\delta X) = \mathrm{Ric}(\cdot, X)$.*

Corollary 6. *For $X = \nabla f$, we get $\delta(\operatorname{Hess} f) = \mathrm{Ric}(\nabla f, \cdot) + \mathrm{d}(\triangle f)$.*

Proof: Make $i = j$ in $X^i{}_{;kj} - X^i{}_{;jk} = R_{jk\ell}{}^i X^\ell$. □

Proposition 7. *$(\mathrm{d}^\nabla(\operatorname{Hess} f))(X,Y)Z = R(X,Y,\nabla f, Z)$.*

Proof:

$$\begin{aligned}(\mathrm{d}^\nabla(\operatorname{Hess} f))_{ijk} &= (\operatorname{Hess} f)_{jk;i} - (\operatorname{Hess} f)_{ik;j} \\ &= f_{;jki} - f_{;ikj} \\ &= f_{;kji} - f_{;kij} \\ &= R_{ij\ell k} f^{;\ell}\end{aligned}$$

□

Proposition 8. *$\mathrm{d}^\nabla(fT) = \mathrm{d}f \wedge T + f\,\mathrm{d}^\nabla T$.*

Corollary 9. *For $T = g$, we get $\mathrm{d}^\nabla(fg) = \mathrm{d}f \wedge g$.*

Proof:

$$\begin{aligned}\mathrm{d}^\nabla(fT) &= (fT)_{jk;i} - (fT)_{ik;j} \\ &= (fT_{jk})_{;i} - (fT_{ik})_{;j} \\ &= f_{;i}T_{jk} + fT_{jk;i} - f_{;j}T_{ik} - fT_{ik;j} \\ &= f_{;i}T_{jk} - f_{;j}T_{ik} + f(T_{jk;i} - T_{ik;j}) \\ &= (\mathrm{d}f \wedge T)_{ijk} + (f\mathrm{d}^\nabla T)_{ijk}\end{aligned}$$

□

Proposition 10. $\delta(fT) = T(\cdot, \nabla f) + f\delta T$

Corollary 11. *For* $T = g$, $\delta(fg) = \mathrm{d}f$.

Proof:

$$\begin{aligned}(\delta(fT))_i &= (fT)_{ij;}{}^{j} \\ &= (fT_{ij})_{;}{}^{j} \\ &= f^{;j}T_{ij} + fT_{ij;}{}^{j} \\ &= f^{;j}T_{ij} + f(\delta T)_i\end{aligned}$$

□

Proposition 12. $\delta T = \mathrm{tr}_{1,3}(\mathrm{d}^\nabla T) + \mathrm{d}(\mathrm{tr}\, T)$, *if* T *has rank* 2.

Proof:

$$(\delta T)_i = g^{jk}T_{ij;k} = g^{jk}((\mathrm{d}^\nabla T)_{kij} + T_{kj;i}) = (\mathrm{tr}_{1,3}(\mathrm{d}^\nabla T))_i + (\mathrm{tr}\, T)_{;i}$$

□

Proposition 13. $\delta R = \mathrm{d}^\nabla \mathrm{Ric}$.

Proof:

$$\begin{aligned}(\delta R)_{ijk} &= R_{ijk}{}^{\ell}{}_{;\ell} \\ &= g^{\ell r}R_{ijkr;\ell} \\ &= -g^{\ell r}R_{j\ell kr;i} - g^{\ell r}R_{\ell ikr;j} \\ &= g^{\ell r}R_{\ell jkr;i} - g^{\ell r}R_{\ell ikr;j} \\ &= R_{jk;i} - R_{ik;j} \\ &= (\mathrm{d}^\nabla \mathrm{Ric})_{ijk}\end{aligned}$$

□

Proposition 14. $\delta\mathrm{Ric} = \mathrm{d}s/2$.

Remark. This also follows from Proposition 12, as $\mathrm{tr}_{1,3}(\mathrm{d}^\nabla \mathrm{Ric}) = -\delta \mathrm{Ric}$.

Proof:

$$\begin{aligned}(\delta \mathrm{Ric})_i &= R_{ij;}{}^{j} \\ &= g^{k\ell}R_{kij\ell;}{}^{j} \\ &= g^{rj}g^{k\ell}R_{kij\ell;r} \\ &= -g^{rj}g^{k\ell}R_{irj\ell;k} - g^{rj}g^{k\ell}R_{rkj\ell;i} \\ &= -g^{k\ell}R_{i\ell;k} + g^{rj}R_{rj;i} \\ &= -(\delta\mathrm{Ric})_i + \mathrm{s}_{;i}\end{aligned}$$

□

Proposition 15. $2\delta(g \owedge S) = \delta S \wedge g + \mathrm{d}^\nabla S$.

Proof:

$$\begin{aligned}
(2\delta(g \owedge S))_{ijk} &= (2g \owedge S)_{ijk\ell;}{}^{\ell} \\
&= (g_{jk}S_{i\ell} - g_{ik}S_{j\ell} + S_{jk}g_{i\ell} - S_{ik}g_{j\ell})_{;}{}^{\ell} \\
&= g_{jk}S_{i\ell;}{}^{\ell} - g_{ik}S_{j\ell;}{}^{\ell} + S_{jk;}{}^{\ell}g_{i\ell} - S_{ik;}{}^{\ell}g_{j\ell} \\
&= g_{jk}(\delta S)_i - g_{ik}(\delta S)_j + S_{jk;i} - S_{ik;j} \\
&= (\delta S \wedge g)_{ijk} + (\mathrm{d}^\nabla S)_{ijk} \\
&= (\delta S \wedge g + \mathrm{d}^\nabla S)_{ijk}
\end{aligned}$$

□

Proposition 16. $\delta \mathrm{Sch} = \dfrac{n-2}{2(n-1)}\mathrm{ds}$.

Proof:

$$\begin{aligned}
\delta \mathrm{Sch} &= \delta\left(\mathrm{Ric} - \frac{\mathrm{s}}{2(n-1)}g\right) \\
&= \delta \mathrm{Ric} - \frac{1}{2(n-1)}\delta(\mathrm{s}g) \\
&= \frac{\mathrm{ds}}{2} - \frac{\mathrm{ds}}{2(n-1)} \\
&= \frac{n-2}{2(n-1)}\mathrm{ds}
\end{aligned}$$

□

Proposition 17. $\delta \mathrm{W} = \dfrac{n-3}{n-2}\mathrm{d}^\nabla \mathrm{Sch}$.

Proof:

$$\begin{aligned}
\delta \mathrm{W} &= \delta\left(R - \frac{2}{n-2}g \owedge \mathrm{Sch}\right) \\
&= \delta R - \frac{2\delta(g \owedge \mathrm{Sch})}{n-2} \\
&= \mathrm{d}^\nabla \mathrm{Ric} - \frac{\delta \mathrm{Sch} \wedge g}{n-2} - \frac{1}{n-2}\mathrm{d}^\nabla \mathrm{Sch} \\
&= \mathrm{d}^\nabla \mathrm{Ric} - \frac{\mathrm{ds} \wedge g}{2(n-1)} - \frac{1}{n-2}\mathrm{d}^\nabla \mathrm{Sch} \\
&= \mathrm{d}^\nabla \mathrm{Sch} - \frac{1}{n-2}\mathrm{d}^\nabla \mathrm{Sch} \\
&= \frac{n-3}{n-2}\mathrm{d}^\nabla \mathrm{Sch}
\end{aligned}$$

□

Proposition 18. $\delta R = \dfrac{\mathrm{ds}\wedge g}{2(n-1)} + \dfrac{n-2}{n-3}\delta \mathrm{W}$.

Proof: This is equivalent to the previous proposition. □

Proposition 19. $\delta \mathrm{W} = 0 \implies \mathrm{d}^\nabla \mathrm{W} = 0$.

Remark. This justifies the name "harmonic" Weyl curvature, as W will be both closed and co-closed.

Proof: Computing $(\mathrm{d}^\nabla \mathrm{W})_{ijk\ell m}$ by definition gives a sum of six terms of the form $g_{jk}(\mathrm{d}^\nabla \mathrm{Sch})_{mi\ell}$. □

3 Immediate consequences

Corollary 20. *If (M,g) is locally symmetric (that is, $\nabla R = 0$), then we also have $\nabla T = 0$ (hence $\delta T = 0$), for $T \in \{\mathrm{Ric}, \mathrm{s}, \mathrm{Sch}, \mathrm{W}\}$.*

Corollary 21 (Schur). *If $n \geq 3$ and $\mathrm{Ric} = fg$ for some $f \in \mathscr{C}^\infty(M)$, then f is automatically constant so that (M,g) is Einstein (and has constant scalar curvature).*

Proof: Applying tr to $\mathrm{Ric} = fg$ gives $f = \mathrm{s}/n$, while applying δ gives $\mathrm{ds}/2 = \mathrm{ds}/n$, so $\mathrm{ds} = \mathrm{d}f = 0$. □

Corollary 22. *(M,g) has harmonic curvature $\iff$ Ric is closed. In this case, s is constant.*

Proof:

$$\mathrm{s}_{;k} = g^{ij}R_{ij;k} = g^{ij}R_{kj;i} = R_{kj;}{}^{j} = (\delta \mathrm{Ric})_k = \frac{\mathrm{s}_{;k}}{2} \implies \mathrm{s}_{;k} = 0.$$

□

Corollary 23. *If $n \geq 3$, $\delta \mathrm{Sch} = 0 \iff$ s is constant.*

Corollary 24. *If $n \geq 4$, (M,g) is has harmonic Weyl curvature $\iff$ Sch is closed.*

Corollary 25. *If $n \geq 4$, then (M,g) has harmonic curvature $\iff$ it has harmonic Weyl curvature and s is constant.*

Proof: If (M,g) has harmonic curvature, note that s is constant by Proposition 22, so $\mathrm{d}^\nabla \mathrm{Ric} = 0$ implies $\mathrm{d}^\nabla \mathrm{Sch} = 0$ by definition of Sch. The converse is clear from Proposition 18. □

Remark. For instance, every Einstein manifold and every locally symmetric manifold satisfy the assumptions above.