Notes on Killing fields

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Fix throughout this text a (connected) pseudo-Riemannian manifold $(M^n, \langle\cdot,\cdot\rangle)$, and its Levi-Civita connection ∇. The Riemann curvature tensor of $(M, \langle\cdot,\cdot\rangle)$ is the $(1,3)$-tensor field defined by

$$R(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z} \doteq \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\boldsymbol{Z} - \nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\boldsymbol{Z} - \nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{Z},$$

where $[\boldsymbol{X},\boldsymbol{Y}]$ denotes the Lie Bracket of $\boldsymbol{X}$ and $\boldsymbol{Y}$. The Ricci curvature is the $(0,2)$-tensor defined by

$$\mathrm{Ric}(\boldsymbol{Y},\boldsymbol{Z}) = \mathrm{tr}\big(\boldsymbol{X} \mapsto R(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z}\big),$$

and the scalar curvature is the smooth function $\mathrm{s} = \mathrm{tr}_{\langle\cdot,\cdot\rangle}(\mathrm{Ric})$. Proceeding: we can compute the Lie derivative of tensor fields with respect to vector fields. For example, for $\langle\cdot,\cdot\rangle$ we have

$$(\mathscr{L}_{\boldsymbol{X}}\langle\cdot,\cdot\rangle)(\boldsymbol{Y},\boldsymbol{Z}) \doteq \mathscr{L}_{\boldsymbol{X}}\langle\boldsymbol{Y},\boldsymbol{Z}\rangle - \langle\mathscr{L}_{\boldsymbol{X}}\boldsymbol{Y},\boldsymbol{Z}\rangle - \langle\boldsymbol{Y},\mathscr{L}_{\boldsymbol{X}}\boldsymbol{Z}\rangle,$$

which can also be rewritten as

$$(\mathscr{L}_{\boldsymbol{X}}\langle\cdot,\cdot\rangle)(\boldsymbol{Y},\boldsymbol{Z}) = \boldsymbol{X}\langle\boldsymbol{Y},\boldsymbol{Z}\rangle - \langle[\boldsymbol{X},\boldsymbol{Y}],\boldsymbol{Z}\rangle - \langle\boldsymbol{Y},[\boldsymbol{X},\boldsymbol{Z}]\rangle.$$

In terms of the Levi-Civita connection ∇, we can also write

$$(\mathscr{L}_{\boldsymbol{X}}\langle\cdot,\cdot\rangle)(\boldsymbol{Y},\boldsymbol{Z}) = \langle\nabla_{\boldsymbol{Y}}\boldsymbol{X},\boldsymbol{Z}\rangle + \langle\boldsymbol{Y},\nabla_{\boldsymbol{Z}}\boldsymbol{X}\rangle,$$

in view of the metric compatibility $\nabla\langle\cdot,\cdot\rangle = 0$. In other words, the $(0,2)$-tensor $\mathscr{L}_{\boldsymbol{X}}\langle\cdot,\cdot\rangle$ transformed into an endomorphism with the aid of $\langle\cdot,\cdot\rangle$ itself is just $\nabla\boldsymbol{X} + (\nabla\boldsymbol{X})^*$.

Definition. A vector field $\boldsymbol{\xi} \in \mathfrak{X}(M)$ is a *Killing field* if $\mathscr{L}_{\boldsymbol{\xi}}\langle\cdot,\cdot\rangle = 0$.

This means that $\langle\cdot,\cdot\rangle$ is preserved by the flow of $\boldsymbol{\xi}$, which then consists of isometries, where defined. Moreover, we see that the covariant differential $\nabla\boldsymbol{\xi} \in \mathrm{End}(\mathfrak{X}(M))$ is skew-adjoint. To give us a first and quick way to identify Killing fields in practice, we ask this first question: when are coordinate vector fields Killing?

Proposition. *Let (x^i) be a local coordinate system for M. Then the coordinate vector field ∂_k is a Killing field if and only if $\partial_k g_{ij} = 0$ for all i and j.*

Proof: By definition, the conclusion holds if and only if for all i and j the equality $0 = \partial_k g_{ij} - \langle[\partial_k,\partial_i],\partial_j\rangle - \langle\partial_i,[\partial_k,\partial_j]\rangle$ holds, but $[\partial_k,\partial_i] = [\partial_k,\partial_j] = \mathbf{0}$. $\square$

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More generally, we have the "full" coordinate versions of Killing's equations:

Proposition. *Let (x^i) be a local coordinate system for M. Assume that the field $\boldsymbol{\xi}$ is written as $\boldsymbol{\xi} = \xi^k \partial_k$ in the domain of the given coordinate system. The following are equivalent:*

(i) $\boldsymbol{\xi}$ *is Killing;*

(ii) $\xi^k \partial_k g_{ij} + (\partial_i \xi^k) g_{kj} + (\partial_j \xi^k) g_{ik} = 0$ *for all i, j and k;*

(iii) $\xi_{i;j} + \xi_{j;i} = 0$ *for all i and j.*

Proof: Just compute $\mathscr{L}_{\boldsymbol{\xi}}\langle\cdot,\cdot\rangle$ using the expressions we have seen just now. On one hand we have $(\mathscr{L}_{\boldsymbol{\xi}}\langle\cdot,\cdot\rangle)(\partial_i,\partial_j) = \boldsymbol{\xi}(g_{ij}) - \langle[\boldsymbol{\xi},\partial_i],\partial_j\rangle - \langle\partial_i,[\boldsymbol{\xi},\partial_j]\rangle$, which can be rewritten as $(\mathscr{L}_{\boldsymbol{\xi}}\langle\cdot,\cdot\rangle)(\partial_i,\partial_j) = \xi^k\partial_k g_{ij} + (\partial_i\xi^k)g_{kj} + (\partial_j\xi^k)g_{ik}$ by using that standard identities $[\boldsymbol{\xi},\partial_i] = [\xi^k\partial_k,\partial_i] = -(\partial_i\xi^k)\partial_k$. On the other hand, if we write $\nabla_{\partial_i}\boldsymbol{\xi} = \xi^k{}_{;i}\partial_k$, then we have $(\mathscr{L}_{\boldsymbol{\xi}}\langle\cdot,\cdot\rangle)(\partial_i,\partial_j) = \langle\xi^k{}_{;i}\partial_k,\partial_j\rangle + \langle\partial_i,\xi^k{}_{;k}\partial_k\rangle = \xi^k{}_{;i}g_{kj} + \xi^k{}_{;j}g_{ki} = \xi_{i;j} + \xi_{j;i}$. $\square$

Example.

(1) Let $A = (a_{ij})_{i,j=1}^n \in \mathrm{GL}(n,\mathbb{R})$ be a symmetric matrix and consider in $\mathbb{R}^n$ the pseudo-Riemannian metric $\langle\cdot,\cdot\rangle_A = a_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$. Then all the natural coordinate fields are Killing fields, as all of the a_{ij} are constants. This means that translations are isometries for $(\mathbb{R}^n,\langle\cdot,\cdot\rangle_A)$. Another class of fields that we can consider are linear vector fields, $\boldsymbol{X} \in \mathfrak{X}(\mathbb{R}^n)$ for which $p \mapsto \boldsymbol{X}_p$ is a linear map. The Levi-Civita connection of $\langle\cdot,\cdot\rangle_A$ is the standard flat connection in $\mathbb{R}^n$, so that $\nabla\boldsymbol{X}$ is equal to the total derivative of $p \to \boldsymbol{X}_p$, i.e., $\boldsymbol{X}$ itself. But computing $(\nabla\boldsymbol{X})^*$, in turn, requires considering the matrix A. For instance, we can do this by relating $\langle\cdot,\cdot\rangle_A$ with the standard inner product $\langle\cdot,\cdot\rangle_{\mathrm{Id}_n}$ via $\langle\cdot,\cdot\rangle_A = \langle\cdot,A\cdot\rangle_{\mathrm{Id}_n}$. Thus, given $\boldsymbol{v},\boldsymbol{w} \in \mathbb{R}^n$, regarded as column vectors, we have that:

$$\boldsymbol{v}^\top A\boldsymbol{X}\boldsymbol{w} = (\boldsymbol{v}^\top A\boldsymbol{X}\boldsymbol{w})^\top = \boldsymbol{w}^\top\boldsymbol{X}^\top A\boldsymbol{v} = \boldsymbol{w}^\top AA^{-1}\boldsymbol{X}^\top A\boldsymbol{v},$$

which says that $(\nabla\boldsymbol{X})^* = A^{-1}\boldsymbol{X}^\top A$. Thus $\boldsymbol{X}$ is Killing if and only if

$$\boldsymbol{X} + A^{-1}\boldsymbol{X}^\top A = 0,$$

which is equivalent to $A\boldsymbol{X} + \boldsymbol{X}^\top A = 0$. In particular, we note that a linear field in $\mathbb{R}^n$ is Killing if and only if it is actually skew-symmetric. Of course, this can also be recovered by noting that

$$\mathrm{O}(\mathbb{R}^n,\langle\cdot,\cdot\rangle_A) = \{T \in \mathrm{GL}(n,\mathbb{R}) \mid T^\top AT = A\}$$

and thus

$$\mathfrak{so}(\mathbb{R}^n,\langle\cdot,\cdot\rangle_A) = \{X \in \mathfrak{gl}(n,\mathbb{R}) \mid X^\top A + AX = 0\},$$

by taking derivatives.

(2) Let $\mathbb{R}^2$ be equipped with the standard flat metric $\mathrm{d}x^2 + \mathrm{d}y^2$. Restricting it to the suitable domain $\mathbb{R}^2 \setminus \{(x,0) \in \mathbb{R}^2 \mid x < 0\}$, we may rewrite it using polar coordinates (r,θ), and $\mathrm{d}x^2 + \mathrm{d}y^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$. So ∂_θ is a Killing field, and the isometries it generates are the rotations around the origin, given by the matrices

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

as θ ranges over $[0, 2\pi[$.

(3) Let $I \subseteq \mathbb{R}$ be an interval and $f\colon I \to \mathbb{R}_{>0}$ be a smooth function. Any surface of revolution in $\mathbb{R}^3$ (generated by a rotation around the z axis, say) is modeled, with a suitable f, by a warped product $I \times_f \mathbb{S}^1$, with metric $\langle \cdot, \cdot \rangle = \mathrm{d}s^2 + f(s)^2\,\mathrm{d}\theta^2$, where s is the natural coordinate in I and $\mathrm{d}\theta$ is the (non-exact) angle-form in $\mathbb{S}^1$. We see that the field ∂_θ is a Killing field. The isometries generated by ∂_θ are further revolutions around the original axis of rotation. The field ∂_s, in turn, is not Killing.

(4) Let $\mathrm{M} > 0$ and $\hbar\colon \mathbb{R}_{>0} \to \mathbb{R}$ be given by $\hbar(r) = 1 - 2\mathrm{M}/r$. Consider the Schwarzschild half-plane $P_I = \{(t,r) \in \mathbb{R}^2 \mid r > 2\mathrm{M}\}$, equipped with the Lorentzian metric $-\hbar(r)\,\mathrm{d}t^2 + \hbar(r)^{-1}\,\mathrm{d}r^2$, and let the warped product $P_I \times_r \mathbb{S}^2$ be the full Schwarzschild space, with metric

$$\langle \cdot, \cdot \rangle = -h(r)^2\,\mathrm{d}t^2 + \hbar(r)^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2 \sin^2\theta\,\mathrm{d}\varphi^2,$$

where (θ, φ) are spherical coordinates in the $\mathbb{S}^2$ factor. Then ∂_t is a Killing field, and the isometries generated by it are translations in the time coordinate. The field ∂_φ is also Killing, and the flow generated by it consists on rotations in the $\mathbb{S}^2$ factor. The fields ∂_r and ∂_θ are not Killing.

(5) Focusing on P_I only, we can make a more detailed analysis. One may compute that the Gaussian curvature is just $K(t,r) = 2\mathrm{M}/r^3$. Since isometries preserve K, any isometry $F\colon P_I \to P_I$ necessarily has the form $F(t,r) = (f(t,r), r)$. So in particular we have that

$$\mathrm{d}F(\partial_t) = \frac{\partial f}{\partial t}\partial_t \quad \text{and} \quad \mathrm{d}F(\partial_r) = \frac{\partial f}{\partial r}\partial_t + \partial_r,$$

whence $\mathrm{d}F(\partial_t)$ being unit timelike forces $\partial f/\partial t = \pm 1$, and thus $\mathrm{d}F(\partial_t)$ and $\mathrm{d}F(\partial_r)$ being orthogonal gives that $\partial f/\partial r = 0$. This means that $f(t,r) = \pm t + c$ for some constant $c \in \mathbb{R}$, and so $F(t,r) = (\pm t + c, r)$. That being understood, we know that every Killing field arises as the vector field associated to a 1-parameter family of isometries. Since our only choices are $F_s(t,r) = (\pm t + c(s), r)$, for any functions c of the flow parameter s, we compute

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} F_s(t,r) = c'(0)\partial_t.$$

And $c'(0)$ can be any real number. The conclusion is that all Killing fields in P_I are multiples of ∂_t.

(6) Let $(G, \langle\cdot,\cdot\rangle)$ be a Lie group equipped with a left-invariant pseudo-Riemannian metric. So right-invariant vector fields are Killing fields, because their flows consist of left translations: if $\boldsymbol{X} \in \mathfrak{X}^R(G)$, then $\Phi_{\boldsymbol{X}}(t,g) = \Phi_{\boldsymbol{X}}(t,e)g$. Similarly, if the metric is right-invariant, then left-invariant vector fields are Killing fields.

Proposition. *Let $\boldsymbol{\xi}'$ and $\boldsymbol{\xi}''$ be two Killing fields. Then $[\boldsymbol{\xi}', \boldsymbol{\xi}'']$ is also a Killing field. Thus the space of Killing fields is a Lie algebra, denoted by $\mathfrak{iso}(M, \langle\cdot,\cdot\rangle)$. In particular, we see that the dimension of $\mathfrak{iso}(M, \langle\cdot,\cdot\rangle)$ is at most*[1] *$n(n+1)/2$.*

Proof: Just directly compute

$$\mathscr{L}_{[\boldsymbol{\xi}',\boldsymbol{\xi}'']}\langle\cdot,\cdot\rangle = \mathscr{L}_{\boldsymbol{\xi}'}(\mathscr{L}_{\boldsymbol{\xi}''}\langle\cdot,\cdot\rangle) - \mathscr{L}_{\boldsymbol{\xi}''}(\mathscr{L}_{\boldsymbol{\xi}'}\langle\cdot,\cdot\rangle) = \mathscr{L}_{\boldsymbol{\xi}'}(0) - \mathscr{L}_{\boldsymbol{\xi}''}(0) = 0.$$

□

Example.

(1) Consider the usual Euclidean space $(\mathbb{R}^3, \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$. This turns out to have the maximum $3(3+1)/2 = 6$ of linearly independent Killing fields. Let's exhibit them all, divided in two types:

- $T_s(x,y,z) = (x+s,y,z)$ is a 1-parameter family of isometries, since we have that $DT_s(x,y,z) = \mathrm{Id}_{\mathbb{R}^3}$. Thus

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} T_s(x,y,z) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}(x+s,y,z) = \partial_x$$

  is a Killing field (we already knew that). Similarly we recover that ∂_y and ∂_z are Killing fields.

- We have three 1-parameter families of rotations, around each axis, and the associated Killing fields are given by:

$$\left.\frac{\mathrm{d}}{\mathrm{d}\theta}\right|_{\theta=0}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = -z\partial_y + y\partial_z,$$

  and similarly:

$$\left.\frac{\mathrm{d}}{\mathrm{d}\theta}\right|_{\theta=0}\begin{pmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = -z\partial_x + x\partial_z$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}\theta}\right|_{\theta=0}\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = -y\partial_x + x\partial_y.$$

[1] Locally, the degrees of freedom we would have are the n components of $\boldsymbol{\xi}$ and more $n(n-1)/2$ derivatives, in view of the equation $\xi_{i;j} + \xi_{j;i} = 0$. And $n + n(n-1)/2 = n(n+1)/2$.

(2) Consider now Lorentz-Minkowski space $(\mathbb{R}^3, \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2)$. Like before, this will have the maximum number allowed of linearly independent Killing fields. We already have three: ∂_x, ∂_y and ∂_z. Now, the Euclidean rotations around the (timelike) z-axis are also Lorentz transformations, so $-y\partial_x + x\partial_y$ enters the list. For the other two coordinate directions we have hyperbolic rotations instead:

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varphi}\right|_{\varphi=0}\begin{pmatrix}1 & 0 & 0\\ 0 & \cosh\varphi & \sinh\varphi\\ 0 & \sinh\varphi & \cosh\varphi\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = z\partial_y + y\partial_z$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varphi}\right|_{\varphi=0}\begin{pmatrix}\cosh\varphi & 0 & \sinh\varphi\\ 0 & 1 & 0\\ \sinh\varphi & 0 & \cosh\varphi\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = z\partial_x + x\partial_z.$$

One more perhaps interesting Killing field (which may expressed as a combination of the previous 6, for dimensional reasons) is generated by a family of Lorentz boosts along a lightlike line:

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\begin{pmatrix}1 & -s & -s\\ s & 1-s^2/2 & s^2/2\\ s & -s^2/2 & 1+s^2/2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} &= \begin{pmatrix}0&-1&-1\\1&0&0\\1&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}\\ &= -(y+z)\partial_x + x(\partial_y + \partial_z).\end{aligned}$$

Let's continue to register some general facts:

Proposition. *Let $\boldsymbol{\xi}$ be a Killing field of constant length. Then the integral curves of $\boldsymbol{\xi}$ are geodesics of $(M, \langle\cdot,\cdot\rangle)$.*

Proof: The hypothesis says that $\boldsymbol{X}\langle\boldsymbol{\xi},\boldsymbol{\xi}\rangle = 0$ for all $\boldsymbol{X} \in \mathfrak{X}(M)$. But this equals $2\langle\nabla_{\boldsymbol{X}}\boldsymbol{\xi},\boldsymbol{\xi}\rangle = 0$, and it follows that $\nabla_{\boldsymbol{X}}\boldsymbol{\xi}$ is always orthogonal to $\boldsymbol{\xi}$. Now, Killing's equation reads $0 = (\mathscr{L}_{\boldsymbol{\xi}}\langle\cdot,\cdot\rangle)(\boldsymbol{\xi},\boldsymbol{X}) = \langle\nabla_{\boldsymbol{\xi}}\boldsymbol{\xi},\boldsymbol{X}\rangle + \langle\boldsymbol{\xi},\nabla_{\boldsymbol{X}}\boldsymbol{\xi}\rangle = \langle\nabla_{\boldsymbol{\xi}}\boldsymbol{\xi},\boldsymbol{X}\rangle$. Non-degeneracy of the metric finally says that $\nabla_{\boldsymbol{\xi}}\boldsymbol{\xi} = \mathbf{0}$, as wanted. $\square$

Example.

(1) For any symmetric $A \in \mathrm{GL}(n,\mathbb{R})$, straight lines are geodesics of $(\mathbb{R}^n, \langle\cdot,\cdot\rangle_A)$ (of course one can easily see that the Christoffel symbols for the natural coordinates in $\mathbb{R}^n$ vanish, but this is a more elegant argument).

(2) Meridians of surfaces of revolution in $\mathbb{R}^3$ are pre-geodesics (i.e., can be reparametrized as *bona fide* geodesics).

(3) If $(G, \langle\cdot,\cdot\rangle)$ is a Lie group equipped with a bi-invariant[2] pseudo-Riemannian metric, then both left-invariant and right-invariant vector fields are Killing fields with constant length. This implies that the geodesics are $(G, \langle\cdot,\cdot\rangle)$ are precisely translations of 1-parameter subgroups of G.

Proposition. *For any Killing field $\boldsymbol{\xi}$, we have* $\operatorname{div}\boldsymbol{\xi} = 0$.

[2]By the way, it is a result due to Milnor that a Lie group admits a bi-invariant Riemannian metric if and only if it is isomorphic to a direct product $K \times H$ with K compact and H abelian.

Proof: The right side of coordinate expression $\xi^j{}_{;j} = g^{ij}\xi_{i;j}$ for the divergence of $\boldsymbol{\xi}$ is both symmetric and skew-symmetric in i and j, hence vanishes. $\square$

Remark. Another proof is noting that $\operatorname{div}\boldsymbol{\xi}$ is the trace of the skew-symmetric endomorphism $\nabla\boldsymbol{\xi}$.

With this, we can try to understand one more question: when are Killing fields gradient fields?

Proposition. *Let $\boldsymbol{\xi}$ be a Killing field and assume that $\boldsymbol{\xi} = \operatorname{grad} f$ for some smooth function $f\colon M \to \mathbb{R}$. Then $\nabla\boldsymbol{\xi} = 0$ and $\triangle f = 0$.*

Proof: In the conditions of the statement, we have that $\langle \nabla_{\boldsymbol{X}}\boldsymbol{\xi}, \boldsymbol{Y}\rangle = \operatorname{Hess}(f)(\boldsymbol{X}, \boldsymbol{Y})$ for any vector fields $\boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{X}(M)$. This is skew-symmetric in $\boldsymbol{X}$ and $\boldsymbol{Y}$ since $\nabla\boldsymbol{\xi}$ is skew-adjoint. On the other hand, it is also symmetric, since torsion-free connections produce symmetric Hessian tensors. So $\langle \nabla_{\boldsymbol{X}}\boldsymbol{\xi}, \boldsymbol{Y}\rangle = 0$ for all $\boldsymbol{Y}$ implies that $\nabla_{\boldsymbol{X}}\boldsymbol{\xi} = \mathbf{0}$ for all $\boldsymbol{X}$, and so $\nabla\boldsymbol{\xi} = 0$. On the other hand, since $\operatorname{Hess}(f) = 0$, taking the trace we obtain $\triangle f = 0$ as well. $\square$

Remark.

- Even in the above setup, $\boldsymbol{\xi}$ might have no zeros whatsoever. Consider in the space $\mathbb{R}^n$ (equipped with the usual Riemannian metric) any natural coordinate field $\operatorname{grad}(x^i)$.
- The condition $\triangle f = 0$ is a quick test for deciding when a given gradient field is a Killing field.
- The above proof actually shows the converse: if $\operatorname{Hess}(f) = 0$, then $\operatorname{grad}(f)$ is Killing.

Proposition. *Assume that M is compact, Riemannian and oriented. If $\boldsymbol{\xi}$ is a Killing field and there is a smooth function $f\colon M \to \mathbb{R}$ such that $\boldsymbol{\xi} = \operatorname{grad} f$, then f is constant and $\boldsymbol{\xi} = \mathbf{0}$.*

Proof: We have that $\triangle f = \operatorname{div}\operatorname{grad} f = \operatorname{div}\boldsymbol{\xi} = 0$, so that[3]

$$\triangle(f^2) = 2f\triangle f + 2\|\operatorname{grad} f\|^2 = 2\|\boldsymbol{\xi}\|^2,$$

and thus

$$0 \overset{(*)}{=} \int_M \triangle(f^2)\,\mathrm{d}M = \int_M 2\|\boldsymbol{\xi}\|^2\,\mathrm{d}M \implies \|\boldsymbol{\xi}\| = 0 \implies \boldsymbol{\xi} = \mathbf{0},$$

as wanted, where $(*)$ follows from Stokes' Theorem and the general divergence expression[4] in terms of the volume form $\mathrm{d}M$: $\mathrm{d}(\iota_{\boldsymbol{X}}\mathrm{d}M) = (\operatorname{div}\boldsymbol{X})\,\mathrm{d}M$ for any vector field $\boldsymbol{X} \in \mathfrak{X}(M)$. $\square$

[3] The general formula $\triangle(fg) = f\triangle g + g\triangle f + 2\langle \operatorname{grad} f, \operatorname{grad} g\rangle$ holds for pseudo-Riemannian manifolds.

[4] If M is any smooth manifold, ∇ is a torsion-free connection in TM and $\boldsymbol{X}$ is any vector field, for any $\beta \in \Omega^k(M)$ we have the relation

$$\mathscr{L}_{\boldsymbol{X}}\beta = \nabla_{\boldsymbol{X}}\beta + \sum_{i=1}^{k} \beta(\cdot, \ldots, (\nabla\boldsymbol{X})_{(i)}, \ldots, \cdot),$$

where the subscript (i) indicates that $\nabla\boldsymbol{X}$ acts on the ith slot. In particular, if β has top degree, we have that $\mathscr{L}_{\boldsymbol{X}}\beta = \nabla_{\boldsymbol{X}}\beta + (\operatorname{div}\boldsymbol{X})\beta$. So the relation $\mathrm{d}(\iota_{\boldsymbol{X}}\omega) = (\operatorname{div}\boldsymbol{X})\omega$ holds for any top degree parallel form, due to Cartan's magic formula.

At this point it is convenient to recall the following general fact about vector bundles:

Lemma. *Let $E \to M$ be any vector bundle equipped with a Koszul connection ∇. If two sections $\phi, \psi \in \Gamma(E)$ are such that $\phi(p) = \psi(p)$ for some $p \in M$, and $v \in T_pM$ is given, then $\mathrm{d}\phi_p(v) = \mathrm{d}\psi_p(v)$ if and only if $\nabla_v\phi = \nabla_v\psi$.*

Proof: We may assume without loss of generality that $\phi(p) = \psi(p) = 0$, and show that $\mathrm{d}\phi_p(v) = 0$ if and only if $\nabla_v\phi = 0$. Let (x^j) be local coordinates for M near p and (e_a) be local trivializing sections for E near p. We have the coordinate formula $\nabla_v\phi = (v(\phi^b) + \Gamma^b_{ja}v^j\phi^a(p))e_b$, but $\phi^a(p) = 0$ and $v(\phi^b)e_b$ is identified with $\mathrm{d}\phi_p(v)$ under the canonical isomorphism $E_p \cong T_{\phi_p}(E_p)$. That is to say, $\nabla_v\phi \cong \mathrm{d}\psi_p(v)$. $\square$

Corollary. *Consider any vector bundle over M equipped with a connection. If a section is parallel along some curve in M and it vanishes at some point in the curve, then it vanishes identically along the curve.*

Proof: Let $E \to M$ be the bundle and ∇ its connection. Let $x\colon I \to M$ be a smooth curve and $\phi \in \Gamma(E)$ parallel along x, such that $\phi(x(t_0)) = 0$ for some $t_0 \in I$. Now

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(x(t)) = \mathrm{d}\phi_{x(t)}(\dot{x}(t)) \cong \nabla_{\dot{x}(t)}\phi = 0,$$

by the Lemma. $\square$

Proposition (Kostant formula). *If $\boldsymbol{\xi}$ is Killing, then $\nabla_{\boldsymbol{X}}(\nabla\boldsymbol{\xi}) = R(\boldsymbol{X},\boldsymbol{\xi})$, for all fields $\boldsymbol{X} \in \mathfrak{X}(M)$.*

Proof: Start with the Killing equation $\langle\nabla_{\boldsymbol{X}}\boldsymbol{\xi},\boldsymbol{Y}\rangle + \langle\boldsymbol{X},\nabla_{\boldsymbol{Y}}\boldsymbol{\xi}\rangle = 0$. Differentiate with respect to a third field $\boldsymbol{Z}$ to get, after considering cyclic permutations, the three relations:

$$\begin{aligned}
\langle\nabla_{\boldsymbol{Z}}\nabla_{\boldsymbol{X}}\boldsymbol{\xi},\boldsymbol{Y}\rangle + \langle\nabla_{\boldsymbol{X}}\boldsymbol{\xi},\nabla_{\boldsymbol{Z}}\boldsymbol{Y}\rangle + \langle\nabla_{\boldsymbol{Z}}\boldsymbol{X},\nabla_{\boldsymbol{Y}}\boldsymbol{\xi}\rangle + \langle\boldsymbol{X},\nabla_{\boldsymbol{Z}}\nabla_{\boldsymbol{Y}}\boldsymbol{\xi}\rangle &= 0,\\
\langle\nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\boldsymbol{\xi},\boldsymbol{Z}\rangle + \langle\nabla_{\boldsymbol{Y}}\boldsymbol{\xi},\nabla_{\boldsymbol{X}}\boldsymbol{Z}\rangle + \langle\nabla_{\boldsymbol{X}}\boldsymbol{Y},\nabla_{\boldsymbol{Z}}\boldsymbol{\xi}\rangle + \langle\boldsymbol{Y},\nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Z}}\boldsymbol{\xi}\rangle &= 0,\\
\langle\nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{Z}}\boldsymbol{\xi},\boldsymbol{X}\rangle + \langle\nabla_{\boldsymbol{Z}}\boldsymbol{\xi},\nabla_{\boldsymbol{Y}}\boldsymbol{X}\rangle + \langle\nabla_{\boldsymbol{Y}}\boldsymbol{Z},\nabla_{\boldsymbol{X}}\boldsymbol{\xi}\rangle + \langle\boldsymbol{Z},\nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\boldsymbol{\xi}\rangle &= 0.
\end{aligned}$$

Using that ∇ is torsion-free and that $\nabla\boldsymbol{\xi}$ is skew, we may add the second and third equations, and subtract the first one to get:

$$R(\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{\xi},\boldsymbol{X}) + R(\boldsymbol{X},\boldsymbol{Z},\boldsymbol{\xi},\boldsymbol{Y}) + \langle\nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\boldsymbol{\xi} + \nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\boldsymbol{\xi},\boldsymbol{Z}\rangle + \langle\nabla_{\boldsymbol{X}}\boldsymbol{Y} + \nabla_{\boldsymbol{Y}}\boldsymbol{X},\nabla_{\boldsymbol{Z}}\boldsymbol{\xi}\rangle = 0.$$

Using the definition of $\nabla(\nabla\boldsymbol{\xi})$, symmetries of R and that $\nabla\boldsymbol{\xi}$ is skew, we may reorganize this into:

$$\langle(\nabla_{\boldsymbol{X}}(\nabla\boldsymbol{\xi}))\boldsymbol{Y} + (\nabla_{\boldsymbol{Y}}(\nabla\boldsymbol{\xi}))\boldsymbol{X},\boldsymbol{Z}\rangle = R(\boldsymbol{X},\boldsymbol{\xi},\boldsymbol{Y},\boldsymbol{Z}) + R(\boldsymbol{Y},\boldsymbol{\xi},\boldsymbol{X},\boldsymbol{Z}).$$

Thus, non-degenerability of $\langle\cdot,\cdot\rangle$ yields that

$$(\nabla_{\boldsymbol{X}}(\nabla\boldsymbol{\xi}))\boldsymbol{Y} + (\nabla_{\boldsymbol{Y}}(\nabla\boldsymbol{\xi}))\boldsymbol{X} = R(\boldsymbol{X},\boldsymbol{\xi})\boldsymbol{Y} + R(\boldsymbol{Y},\boldsymbol{\xi})\boldsymbol{X}.$$

We are almost done. Now, recall the *Ricci identity*:

$$\begin{aligned}
(\nabla_{\boldsymbol{Y}}(\nabla\boldsymbol{\xi}))\boldsymbol{X} &= \nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\boldsymbol{\xi} - \nabla_{\nabla_{\boldsymbol{Y}}\boldsymbol{X}}\boldsymbol{\xi} \\
&= \nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\boldsymbol{\xi} - \nabla_{\nabla_{\boldsymbol{X}}\boldsymbol{Y}}\boldsymbol{\xi} + \nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{\xi} \\
&= -\nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\boldsymbol{\xi} + \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\boldsymbol{\xi} + \nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\boldsymbol{\xi} - \nabla_{\nabla_{\boldsymbol{X}}\boldsymbol{Y}}\boldsymbol{\xi} + \nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{\xi} \\
&= (\nabla_{\boldsymbol{X}}(\nabla\boldsymbol{\xi}))\boldsymbol{Y} - R(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{\xi}.
\end{aligned}$$

This leads to

$$2(\nabla_{\boldsymbol{X}}(\nabla\boldsymbol{\xi}))\boldsymbol{Y} = R(\boldsymbol{X},\boldsymbol{\xi})\boldsymbol{Y} + R(\boldsymbol{Y},\boldsymbol{\xi})\boldsymbol{X} + R(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{\xi} = 2R(\boldsymbol{X},\boldsymbol{\xi})\boldsymbol{Y},$$

by the Bianchi identity applied to the last two terms. So $\nabla_{\boldsymbol{X}}(\nabla\boldsymbol{\xi}) = R(\boldsymbol{X},\boldsymbol{\xi})$, as wanted. □

Remark.

- Another proof goes like this: compute the Lie derivative $\mathscr{L}_{\boldsymbol{\xi}}\nabla$ (which is a tensor, since the space of connections is affine with translation space of tensors) as follows:

$$\begin{aligned}
(\mathscr{L}_{\boldsymbol{\xi}}\nabla)_{\boldsymbol{X}}\boldsymbol{Y} &= \mathscr{L}_{\boldsymbol{\xi}}\nabla_{\boldsymbol{X}}\boldsymbol{Y} - \nabla_{\mathscr{L}_{\boldsymbol{\xi}}\boldsymbol{X}}\boldsymbol{Y} - \nabla_{\boldsymbol{X}}\mathscr{L}_{\boldsymbol{\xi}}\boldsymbol{Y} \\
&= [\boldsymbol{\xi},\nabla_{\boldsymbol{X}}\boldsymbol{Y}] - \nabla_{[\boldsymbol{\xi},\boldsymbol{X}]}\boldsymbol{Y} - \nabla_{\boldsymbol{X}}[\boldsymbol{\xi},\boldsymbol{Y}] \\
&= \nabla_{\boldsymbol{\xi}}\nabla_{\boldsymbol{X}}\boldsymbol{Y} - \nabla_{\nabla_{\boldsymbol{X}}\boldsymbol{Y}}\boldsymbol{\xi} - \nabla_{[\boldsymbol{\xi},\boldsymbol{X}]}\boldsymbol{Y} - \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{\xi}}\boldsymbol{Y} + \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\boldsymbol{\xi} \\
&= R(\boldsymbol{\xi},\boldsymbol{X})\boldsymbol{Y} + (\nabla_{\boldsymbol{X}}(\nabla\boldsymbol{\xi}))\boldsymbol{Y}.
\end{aligned}$$

  If $\boldsymbol{\xi}$ is Killing, its flow consists of isometries, which preserve the Levi-Civita connection ∇. This means that $\mathscr{L}_{\boldsymbol{\xi}}\nabla = 0$, and the conclusion follows.

- As a consequence, we get a second proof that the Lie bracket of Killing fields is again a Killing field. Assume $\boldsymbol{\xi}'$ and $\boldsymbol{\xi}''$ are Killing. Then $\nabla\boldsymbol{\xi}'$ and $\nabla\boldsymbol{\xi}''$ are skew, and our goal is to prove that $\nabla[\boldsymbol{\xi}',\boldsymbol{\xi}'']$ is also skew. We have that

$$\begin{aligned}
[\nabla\boldsymbol{\xi}',\nabla\boldsymbol{\xi}''](X) &= \nabla\boldsymbol{\xi}'(\nabla_{\boldsymbol{X}}\boldsymbol{\xi}'') - \nabla\boldsymbol{\xi}''(\nabla_{\boldsymbol{X}}\boldsymbol{\xi}') \\
&= \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{\xi}''}\boldsymbol{\xi}' - (\nabla_{\boldsymbol{X}}(\nabla\boldsymbol{\xi}'))\boldsymbol{\xi}'' - \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{\xi}'}\boldsymbol{\xi}'' + (\nabla_{\boldsymbol{X}}(\nabla\boldsymbol{\xi}''))\boldsymbol{\xi}' \\
&= \nabla_{\boldsymbol{X}}[\boldsymbol{\xi}'',\boldsymbol{\xi}'] - R(\boldsymbol{X},\boldsymbol{\xi}')\boldsymbol{\xi}'' + R(\boldsymbol{X},\boldsymbol{\xi}'')\boldsymbol{\xi}' \\
&= -(\nabla[\boldsymbol{\xi}',\boldsymbol{\xi}''])\boldsymbol{X} + R(\boldsymbol{\xi}',\boldsymbol{\xi}'')\boldsymbol{X},
\end{aligned}$$

  so $\nabla[\boldsymbol{\xi}',\boldsymbol{\xi}''] = R(\boldsymbol{\xi}',\boldsymbol{\xi}'') - [\nabla\boldsymbol{\xi}',\nabla\boldsymbol{\xi}'']$ is the difference of skew-adjoint operators, hence skew-adjoint as well.

For one consequence of this formula, recall that given a geodesic $\gamma\colon [a,b] \to M$, a *Jacobi field* along γ is a vector field $\boldsymbol{J}$ along γ satisfying the differential equation

$$\frac{D^2\boldsymbol{J}}{\mathrm{d}t^2} = R(\gamma',\boldsymbol{J})\gamma'.$$

Corollary. *If $\boldsymbol{\xi}$ is a Killing field and $\gamma\colon [a,b] \to M$ is a geodesic, then $\boldsymbol{\xi}$ restricts to a Jacobi field along γ.*

Proof: Compute:

$$R(\gamma',\boldsymbol{\xi})\gamma' = (\nabla_{\gamma'}(\nabla\boldsymbol{\xi}))\gamma' = \nabla_{\gamma'}\nabla_{\gamma'}\boldsymbol{\xi} - \nabla_{\nabla_{\gamma'}\gamma'}\boldsymbol{\xi} = \frac{D^2\boldsymbol{\xi}}{\mathrm{d}t^2},$$

using that $\nabla_{\gamma'}\gamma' = \mathbf{0}$.

□

Another interesting consequence:

Proposition. *Assume M is even dimensional, compact, Riemannian and has positive sectional curvature. Then if $\boldsymbol{\xi}$ is a Killing field, $\boldsymbol{\xi}$ has a zero in M.*

Proof: Consider the smooth function $f\colon M \to \mathbb{R}$ given by $f(x) = \|\boldsymbol{\xi}_x\|_x^2/2$. We start computing the differential and the Hessian of f. Clearly $\mathrm{d}f(\boldsymbol{X}) = \langle \nabla_{\boldsymbol{X}}\boldsymbol{\xi},\boldsymbol{\xi}\rangle$, while

$$\begin{aligned}\mathrm{Hess}(f)(\boldsymbol{X},\boldsymbol{Y}) &= \boldsymbol{X}\boldsymbol{Y}\langle\boldsymbol{\xi},\boldsymbol{\xi}\rangle/2 - \nabla_{\boldsymbol{X}}\boldsymbol{Y}\langle\boldsymbol{\xi},\boldsymbol{\xi}\rangle/2 \\ &= \boldsymbol{X}\langle\nabla_{\boldsymbol{Y}}\boldsymbol{\xi},\boldsymbol{\xi}\rangle - \langle\nabla_{\nabla_{\boldsymbol{X}}\boldsymbol{Y}}\boldsymbol{\xi},\boldsymbol{\xi}\rangle \\ &= -\boldsymbol{X}\langle\boldsymbol{Y},\nabla_{\boldsymbol{\xi}}\boldsymbol{\xi}\rangle + \langle\nabla_{\boldsymbol{X}}\boldsymbol{Y},\nabla_{\boldsymbol{\xi}}\boldsymbol{\xi}\rangle \\ &= -\langle\boldsymbol{Y},\nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{\xi}}\boldsymbol{\xi}\rangle,\end{aligned}$$

for all $\boldsymbol{X},\boldsymbol{Y} \in \mathfrak{X}(M)$. Now, by compactness, f has a global minimum $p \in M$. We have to show that $\boldsymbol{\xi}_p = 0$. Assume not. Then from the formula for $\mathrm{d}f_p$ we see that $(\nabla\boldsymbol{\xi})_p$ restricts to a map $(\nabla\boldsymbol{\xi})_p\colon \boldsymbol{\xi}_p^\perp \to \boldsymbol{\xi}_p^\perp$. But $\dim\boldsymbol{\xi}_p^\perp$ is odd and $(\nabla\boldsymbol{\xi})_p$ is skew-adjoint, so it is singular and we may take $\boldsymbol{v} \in \boldsymbol{\xi}_p^\perp \cap \ker(\nabla\boldsymbol{\xi})_p$. In particular, we have that $\mathrm{Hess}(f)_p(\boldsymbol{v},\boldsymbol{v}) \geq 0$, since p is a global minimum of f. With this set in place, we compute

$$\begin{aligned}0 = \langle\nabla_{\boldsymbol{v}}\boldsymbol{\xi},\nabla_{\boldsymbol{v}}\boldsymbol{\xi}\rangle &= -\langle\boldsymbol{v},\nabla_{\nabla_{\boldsymbol{v}}\boldsymbol{\xi}}\boldsymbol{\xi}\rangle \\ &= \langle\boldsymbol{v},(\nabla_{\boldsymbol{v}}(\nabla\boldsymbol{\xi}))\boldsymbol{\xi} - \nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{\xi}}\boldsymbol{\xi}\rangle \\ &= \langle(\nabla_{\boldsymbol{v}}(\nabla\boldsymbol{\xi}))\boldsymbol{\xi},\boldsymbol{v}\rangle - \langle\boldsymbol{v},\nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{\xi}}\boldsymbol{\xi}\rangle \\ &= R(\boldsymbol{v},\boldsymbol{\xi},\boldsymbol{\xi},\boldsymbol{v}) + \mathrm{Hess}(f)_p(\boldsymbol{v},\boldsymbol{v}) \\ &> 0.\end{aligned}$$

This contradiction concludes the proof. □

Note the important role that the *energy density* f of $\boldsymbol{\xi}$ played in the above proof. It is also the key for the so-called *Bochner's vanishing theorem*:

Theorem (Bochner, 1946)**.** *Assume again that M is Riemannian. Let $\boldsymbol{\xi}$ be Killing and consider again the energy density $f\colon M \to \mathbb{R}$ given by $f(x) = \|\boldsymbol{\xi}_x\|^2/2$. Then the relation*

$$\triangle f = -\mathrm{Ric}(\boldsymbol{\xi},\boldsymbol{\xi}) + \|\nabla\boldsymbol{\xi}\|^2$$

holds. Thus:

(a) If M is compact with $\mathrm{Ric} \leq 0$, *then* $\nabla\boldsymbol{\xi} = \mathbf{0}$.

(b) If M is compact with $\mathrm{Ric} < 0$*, then* $\boldsymbol{\xi} = \mathbf{0}$.

Proof: We prove the formula for the Laplacian of the energy density with a local computation, using a local orthonormal frame $(\boldsymbol{E}_i)_{i=1}^n$. On one hand, we have that the (Hilbert-Schmidt) norm of $\nabla\boldsymbol{\xi}$ is given by

$$\|\nabla\boldsymbol{\xi}\|^2 = \sum_{i=1}^n \|\nabla_{\boldsymbol{E}_i}\boldsymbol{\xi}\|^2,$$

in view of orthonormal expansion. Moreover, note that $\langle \nabla_{\nabla_{\boldsymbol{E}_i}\boldsymbol{E}_i}\boldsymbol{\xi}, \boldsymbol{\xi}\rangle = 0$ for every $i = 1, \ldots, n$, due to skew-symmetry of $\nabla\boldsymbol{\xi}$. With this, Kostant's formula kicks in to give

$$\begin{aligned}
\triangle f &= \sum_{i=1}^n \nabla_{\boldsymbol{E}_i}\nabla_{\boldsymbol{E}_i} f = \sum_{i=1}^n \nabla_{\boldsymbol{E}_i}\langle \nabla_{\boldsymbol{E}_i}\boldsymbol{\xi}, \boldsymbol{\xi}\rangle \\
&= \sum_{i=1}^n \left(\langle \nabla_{\boldsymbol{E}_i}\nabla_{\boldsymbol{E}_i}\boldsymbol{\xi}, \boldsymbol{\xi}\rangle + \|\nabla_{\boldsymbol{E}_i}\boldsymbol{\xi}\|^2\right) = \sum_{i=1}^n \langle \nabla_{\boldsymbol{E}_i}(\nabla\boldsymbol{\xi})\boldsymbol{E}_i, \boldsymbol{\xi}\rangle + \|\nabla\boldsymbol{\xi}\|^2 \\
&= \sum_{i=1}^n R(\boldsymbol{E}_i, \boldsymbol{\xi}, \boldsymbol{E}_i, \boldsymbol{\xi}) + \|\nabla\boldsymbol{\xi}\|^2 = -\mathrm{Ric}(\boldsymbol{\xi}, \boldsymbol{\xi}) + \|\nabla\boldsymbol{\xi}\|^2.
\end{aligned}$$

Now, assuming that M is compact and $\mathrm{Ric} \leq 0$, integrate this relation to obtain

$$0 \geq \int_M \mathrm{Ric}(\boldsymbol{\xi}, \boldsymbol{\xi})\,\mathrm{d}M = \int_M \|\nabla\boldsymbol{\xi}\|^2\,\mathrm{d}M \geq 0,$$

which leads to $\|\nabla\boldsymbol{\xi}\|^2 = 0$ (and so $\nabla\boldsymbol{\xi} = 0$) and also $\mathrm{Ric}(\boldsymbol{\xi}, \boldsymbol{\xi}) = 0$. If instead of $\mathrm{Ric} \leq 0$ we have $\mathrm{Ric} < 0$, this last conclusion implies that $\boldsymbol{\xi} = \mathbf{0}$. □

Remark. The frame computation in the above proof may be avoided by reusing the Hessian relation

$$\mathrm{Hess}(f)(\boldsymbol{X}, \boldsymbol{Y}) = -R(\boldsymbol{X}, \boldsymbol{\xi}, \boldsymbol{\xi}, \boldsymbol{Y}) + \langle \nabla_{\boldsymbol{X}}\boldsymbol{\xi}, \nabla_{\boldsymbol{Y}}\boldsymbol{\xi}\rangle,$$

seen previously. Taking the metric trace in the variables $\boldsymbol{X}$ and $\boldsymbol{Y}$ immediately gives $\triangle f = -\mathrm{Ric}(\boldsymbol{\xi}, \boldsymbol{\xi}) + \|\nabla\boldsymbol{\xi}\|^2$ all the same.

Another stronge result that also follows from Kostant's formula is the following characterization:

Theorem. *A Killing field is determined by its* 1*-jet at any point. That is, given Killing fields $\boldsymbol{\xi}'$ and $\boldsymbol{\xi}''$ such that $\boldsymbol{\xi}'_p = \boldsymbol{\xi}''_p$ and $(\nabla\boldsymbol{\xi}')_p = (\nabla\boldsymbol{\xi}'')_p$ for some $p \in M$, then $\boldsymbol{\xi}' = \boldsymbol{\xi}''$.*

Proof: Define in the vector bundle $TM \oplus \mathrm{End}(TM)$ the connection

$$\overline{\nabla}_{\boldsymbol{X}}(\boldsymbol{Y}, F) \doteq (\nabla_{\boldsymbol{X}}\boldsymbol{Y} - F(\boldsymbol{X}), \nabla_{\boldsymbol{X}}F - R(\boldsymbol{X}, \boldsymbol{Y})).$$

This is indeed a connection, as it is equal to a connection plus a tensor. Now, assume that $\boldsymbol{\xi}$ is a Killing field. Then

$$\overline{\nabla}_{\boldsymbol{X}}(\boldsymbol{\xi}, \nabla\boldsymbol{\xi}) = (\nabla_{\boldsymbol{X}}\boldsymbol{\xi} - \nabla\boldsymbol{\xi}(\boldsymbol{X}), \nabla_{\boldsymbol{X}}(\nabla\boldsymbol{\xi}) - R(\boldsymbol{X}, \boldsymbol{\xi})) = (\mathbf{0}, 0).$$

Our goal is to show that if a Killing field vanishes at one point, it vanishes identically. Now consider the "zero-set" of $\boldsymbol{\xi}$, $\mathrm{Z}(\boldsymbol{\xi}) = \{p \in M \mid \boldsymbol{\xi}_p = \mathbf{0} \text{ and } (\nabla\boldsymbol{\xi})_p = 0\}$. By assumption, $\mathrm{Z}(\boldsymbol{\xi})$ is non-empty. It is also closed, by continuity of $\boldsymbol{\xi}$. Since M is assumed connected, we only have to show that $\mathrm{Z}(\boldsymbol{\xi})$ is also open. Here's how: let $p \in \mathrm{Z}(\boldsymbol{\xi})$ and choose a path-connected neighborhood U of p in M. Given $q \in U$, there is a curve joining p and q. The section $(\boldsymbol{\xi}, \nabla\boldsymbol{\xi})$ is parallel along such a curve and vanishes at p, so it also vanishes at q. This shows that $U \subseteq \mathrm{Z}(\boldsymbol{\xi})$, so $\mathrm{Z}(\boldsymbol{\xi})$ is open. So $\mathrm{Z}(\boldsymbol{\xi}) = M$ and $\boldsymbol{\xi} = \mathbf{0}$, as wanted. □

Corollary. *An isometry is determined by its* 1*-jet at any point. That is, if* $\phi_1, \phi_2 \colon M \to N$ *are two isometries onto some second pseudo-Riemannian manifold* M *with* $\phi_1(x) = \phi_2(x)$ *and* $\mathrm{d}(\phi_1)_x = \mathrm{d}(\phi_2)_x$ *for some* $x \in M$*, then* $\phi_1 = \phi_2$*.*

If we restrict the connection defined in the above proof to $\mathfrak{so}(TM)$, the bundle of skew-adjoint endomorphisms of TM, we have that Killing fields are in one-to-one correspondence with $\overline{\nabla}$-parallel sections of $TM \oplus \mathfrak{so}(TM)$. Without this restriction, parallel sections of $TM \oplus \mathrm{End}(TM)$ are in correspondence with affine vector fields – those whose flows preserve the Levi-Civita connection ∇. Killing fields are particular examples, as we have seen.

Now, recall that a connection in a vector bundle $E \to M$ is flat if and only if given any point p in M and any element in the fiber E_p, there is a parallel section defined in a neighborhood of the point which realizes said fiber element – the issue here being that the local section is parallel along the whole neighborhood and not only at the chosen point. With this in mind, let's compute the curvature of the connection $\overline{\nabla}$ defined in $TM \oplus \mathfrak{so}(TM)$, to try to understand further geometric obstructions for the local existance of Killing fields. To begin with, we have

$$\begin{aligned}\overline{\nabla}_{\boldsymbol{X}}\overline{\nabla}_{\boldsymbol{Y}}(\boldsymbol{Z}, F) &= \overline{\nabla}_{\boldsymbol{X}}(\nabla_{\boldsymbol{Y}}\boldsymbol{Z} - F(\boldsymbol{Y}), \nabla_{\boldsymbol{Y}}F - R(\boldsymbol{Y}, \boldsymbol{Z})) \\ &= (\nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\boldsymbol{Z} - \nabla_{\boldsymbol{X}}F(\boldsymbol{Y}) - (\nabla_{\boldsymbol{Y}}F)(\boldsymbol{X}) + R(\boldsymbol{Y}, \boldsymbol{Z})\boldsymbol{X}, \\ &\qquad \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}F - \nabla_{\boldsymbol{X}}R(\boldsymbol{Y}, \boldsymbol{Z}) - R(\boldsymbol{X}, \nabla_{\boldsymbol{Y}}\boldsymbol{Z}) + R(\boldsymbol{X}, F(\boldsymbol{Y}))\,.\end{aligned}$$

Let's compute $\overline{R}(\boldsymbol{X}, \boldsymbol{Y})(\boldsymbol{Z}, F)$.

- The vector component[5] is:

$$\begin{aligned}&= R(\boldsymbol{X}, \boldsymbol{Y})\boldsymbol{Z} - \nabla_{\boldsymbol{X}}F(\boldsymbol{Y}) + \nabla_{\boldsymbol{Y}}F(\boldsymbol{X}) + F([\boldsymbol{X}, \boldsymbol{Y}]) \\ &\qquad - (\nabla_{\boldsymbol{Y}}F)(\boldsymbol{X}) + (\nabla_{\boldsymbol{X}}F)(\boldsymbol{Y}) + R(\boldsymbol{Y}, \boldsymbol{Z})\boldsymbol{X} - R(\boldsymbol{X}, \boldsymbol{Z})\boldsymbol{Y} \\ &= -\nabla_{\boldsymbol{X}}F(\boldsymbol{Y}) + \nabla_{\boldsymbol{Y}}F(\boldsymbol{X}) + F(\nabla_{\boldsymbol{X}}\boldsymbol{Y}) - F(\nabla_{\boldsymbol{Y}}\boldsymbol{X}) - (\nabla_{\boldsymbol{Y}}F)(\boldsymbol{X}) + (\nabla_{\boldsymbol{X}}F)(\boldsymbol{Y}) \\ &= \mathbf{0},\end{aligned}$$

  where we use, in order, the Bianchi identity and the definition of ∇F (to cancel all the $\nabla_{\boldsymbol{X}}$ terms, and all $\nabla_{\boldsymbol{Y}}$ terms, separately).

[5] Actually this computation can be done for arbitrary connections in TM, and the vector component part of $\overline{R}$ is then $(\mathrm{d}^{\nabla}\tau^{\nabla})(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}) - F(\tau^{\nabla}(\boldsymbol{X}, \boldsymbol{Y}))$, where d^{∇} is the covariant exterior differential.

- The endomorphism component is:

$$\begin{aligned}
&= R(\boldsymbol{X},\boldsymbol{Y})F - \nabla_{\boldsymbol{X}} R(\boldsymbol{Y},\boldsymbol{Z}) + \nabla_{\boldsymbol{Y}} R(\boldsymbol{X},\boldsymbol{Z}) - R(\boldsymbol{X},\nabla_{\boldsymbol{Y}}\boldsymbol{Z}) \\
&\qquad + R(\boldsymbol{Y},\nabla_{\boldsymbol{X}}\boldsymbol{Z}) + R(\boldsymbol{X},F(\boldsymbol{Y})) - R(\boldsymbol{Y},F(\boldsymbol{X})) + R([\boldsymbol{X},\boldsymbol{Y}],\boldsymbol{Z}) \\
&= [R(\boldsymbol{X},\boldsymbol{Y}),F] - (\nabla_{\boldsymbol{X}} R)(\boldsymbol{Y},\boldsymbol{Z}) - R(\nabla_{\boldsymbol{X}}\boldsymbol{Y},\boldsymbol{Z}) - R(\boldsymbol{Y},\nabla_{\boldsymbol{X}}\boldsymbol{Z}) + (\nabla_{\boldsymbol{Y}} R)(\boldsymbol{X},\boldsymbol{Z}) \\
&\qquad + R(\nabla_{\boldsymbol{Y}}\boldsymbol{X},\boldsymbol{Z}) + R(\boldsymbol{X},\nabla_{\boldsymbol{Y}}\boldsymbol{Z}) - R(\boldsymbol{X},\nabla_{\boldsymbol{Y}}\boldsymbol{Z}) + R(\boldsymbol{Y},\nabla_{\boldsymbol{X}}\boldsymbol{Z}) \\
&\qquad + R(\boldsymbol{X},F(\boldsymbol{Y})) + R(F(\boldsymbol{X}),\boldsymbol{Y}) + R([\boldsymbol{X},\boldsymbol{Y}],\boldsymbol{Z}) \\
&= [R(\boldsymbol{X},\boldsymbol{Y}),F] - (\nabla_{\boldsymbol{X}} R)(\boldsymbol{Y},\boldsymbol{Z}) + (\nabla_{\boldsymbol{Y}} R)(\boldsymbol{X},\boldsymbol{Z}) + R(F(\boldsymbol{X}),\boldsymbol{Y}) + R(\boldsymbol{X},F(\boldsymbol{Y})) \\
&= [R(\boldsymbol{X},\boldsymbol{Y}),F] + (\nabla_{\boldsymbol{Z}} R)(\boldsymbol{X},\boldsymbol{Y}) + R(F(\boldsymbol{X}),\boldsymbol{Y}) + R(\boldsymbol{X},F(\boldsymbol{Y})),
\end{aligned}$$

  where in the last step we use the second Bianchi identity $\mathrm{d}^{\nabla} R = 0$.

Proposition.

(i) If $\overline{\nabla}$ is flat in $TM \oplus \mathrm{End}(TM)$, then ∇ is flat.

(ii) If $\overline{\nabla}$ is flat in $TM \oplus \mathfrak{so}(TM)$ and $\dim M \geq 3$, then (M,g) has constant sectional curvature.

Proof:

(i) In this case, we may take $\boldsymbol{Z} = \boldsymbol{0}$ and $F = \mathrm{Id}$, so that $2R(\boldsymbol{X},\boldsymbol{Y}) = 0$. So $R = 0$ and ∇ is flat.

(ii) In this case, we will do a coordinate computation, as we cannot simply choose $F = \mathrm{Id}$ anymore. Write $R(\partial_i,\partial_j) = R_{ijk}{}^{\ell}\,\mathrm{d}x^k \otimes \partial_\ell$ and $F = F^i{}_j \mathrm{d}x^j \otimes \partial_i$. First, we observe that

$$[\mathrm{d}x^k \otimes \partial_\ell, \mathrm{d}x^s \otimes \partial_r] = \delta^k_r\,\mathrm{d}x^s \otimes \partial_\ell - \delta^s_\ell\,\mathrm{d}x^k \otimes \partial_r.$$

With this, we set $\boldsymbol{X} = \partial_i$, $\boldsymbol{Y} = \partial_j$ and $\boldsymbol{Z} = \boldsymbol{0}$ in the formula for the endomorphism component of $\overline{R}(\boldsymbol{X},\boldsymbol{Y})(\boldsymbol{Z},F)$ to get

$$\begin{aligned}
0 &= [R_{ijk}{}^{\ell}\,\mathrm{d}x^k \otimes \partial_\ell, F^r{}_s\,\mathrm{d}x^s \otimes \partial_r] + R(F^r{}_i\partial_r,\partial_j) + R(\partial_i, F^r{}_j\partial_r) \\
&= R_{ijk}{}^{\ell} F^r{}_s(\delta^k_r\,\mathrm{d}x^s \otimes \partial_\ell - \delta^s_\ell\,\mathrm{d}x^k \otimes \partial_r) + R_{rjk}{}^{\ell} F^r{}_i\,\mathrm{d}x^k \otimes \partial_\ell + R_{irk}{}^{\ell} F^r{}_j\,\mathrm{d}x^k \otimes \partial_\ell \\
&= R_{ijk}{}^{\ell} F^k{}_s\,\mathrm{d}x^s \otimes \partial_\ell - R_{ijk}{}^{\ell} F^r{}_\ell\,\mathrm{d}x^k \otimes \partial_r + R_{rjk}{}^{\ell} F^r{}_i\,\mathrm{d}x^k \otimes \partial_\ell + R_{irk}{}^{\ell} F^r{}_j\,\mathrm{d}x^k \otimes \partial_\ell \\
&= \big(R_{ijk}{}^{\ell} F^k{}_s\,\mathrm{d}x^s - R_{ijk}{}^{r} F^\ell{}_r\,\mathrm{d}x^k + R_{rjk}{}^{\ell} F^r{}_i\,\mathrm{d}x^k + R_{irk}{}^{\ell} F^r{}_j\,\mathrm{d}x^k\big) \otimes \partial_\ell \\
&= \big(R_{ijr}{}^{\ell} F^r{}_k - R_{ijk}{}^{r} F^\ell{}_r + R_{rjk}{}^{\ell} F^r{}_i + R_{irk}{}^{\ell} F^r{}_j\big)\,\mathrm{d}x^k \otimes \partial_\ell,
\end{aligned}$$

by conveniently renaming dummy indices. This means that

$$R_{ijr}{}^{\ell} F^r{}_k - R_{ijk}{}^{r} F^\ell{}_r + R_{rjk}{}^{\ell} F^r{}_i + R_{irk}{}^{\ell} F^r{}_j = 0$$

for all choices of i, j, k and ℓ. Now, one source of skew-adjoint operators consists of bivectors regarded as operators. Namely, if $\boldsymbol{X},\boldsymbol{Y} \in \mathfrak{X}(M)$, we look at the map

$X \wedge Y \colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by $(X \wedge Y)Z = \langle Z, Y\rangle X - \langle Z, X\rangle Y$. Our choice for the coefficients F can be

$$(\partial_s \wedge \partial_t)^\ell{}_k = \mathrm{d}x^\ell\big((\partial_s \wedge \partial_t)\partial_k\big) = \mathrm{d}x^\ell\big(g_{tk}\partial_s - g_{sk}\partial_t\big) = g_{tk}\delta^\ell_s - g_{sk}\delta^\ell_t,$$

for any given indices s and t. So we obtain

$$R_{ijr}{}^\ell(g_{tk}\delta^r_s - g_{sk}\delta^r_t) - R_{ijk}{}^r(g_{tr}\delta^\ell_s - g_{sr}\delta^\ell_t) + R_{rjk}{}^\ell(g_{ti}\delta^r_s - g_{si}\delta^r_t) + R_{irk}{}^\ell(g_{tj}\delta^r_s - g_{sj}\delta^r_t) = 0,$$

which boils down to

$$R_{ijs}{}^\ell g_{tk} - R_{ijt}{}^\ell g_{sk} - R_{ijkt}\delta^\ell_s + R_{ijks}\delta^\ell_t + R_{sjk}{}^\ell g_{ti} - R_{tjk}{}^\ell g_{si} + R_{isk}{}^\ell g_{tj} - R_{itk}{}^\ell g_{sj} = 0,$$

for all choices of indices i, j, k, ℓ, s and t. Let $\mathrm{Ric} = R_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j$ be the local expression for the Ricci tensor of (M, g). Then make $\ell = i$ and contract to get

$$R_{js}g_{tk} - R_{jt}g_{sk} - R_{sjkt} + R_{tjks} + R_{sjkt} - R_{tjks} + R_{sk}g_{tj} - R_{tk}g_{sj} = 0.$$

All the Riemann components cancel. To proceed and produce the scalar curvature s, attack the above with g^{js} and sum:

$$\mathrm{s}g_{tk} - R_{kt} + R_{tk} - nR_{tk} = 0 \implies \mathrm{s}g_{tk} = nR_{tk}.$$

Thus $\mathrm{Ric} = \mathrm{s}g/n$. Now return to the original uncontracted expression, make $\ell = s$ and contract again to get

$$\begin{aligned} 0 &= 0 - R_{ijtk} - nR_{ijkt} + R_{ijkt} + R_{jk}g_{ti} - R_{tjki} - R_{ik}g_{tj} - R_{itk\ell} \\ &= (2-n)R_{ijkt} + \frac{\mathrm{s}}{n}(g_{jk}g_{ti} - g_{ik}g_{tj}) + R_{tjik} + R_{itjk} \\ &= (2-n)R_{ijkt} + \frac{\mathrm{s}}{n}(g_{jk}g_{ti} - g_{ik}g_{tj}) - R_{jitk} \\ &= (1-n)R_{ijkt} + \frac{\mathrm{s}}{n}(g_{jk}g_{ti} - g_{ik}g_{tj}). \end{aligned}$$

So rename $t \to \ell$ and raise ℓ to get

$$R_{ijk}{}^\ell = \frac{\mathrm{s}}{n(n-1)}(g_{jk}\delta^\ell_i - g_{ik}\delta^\ell_j) = \frac{\mathrm{s}}{n(n-1)}(\partial_i \wedge \partial_j)^\ell{}_k,$$

which implies that $R(\partial_i, \partial_j) = \dfrac{\mathrm{s}}{n(n-1)}\partial_i \wedge \partial_j$ for all i and j. By linearity it follows that

$$R(X, Y) = \frac{\mathrm{s}}{n(n-1)} X \wedge Y$$

for all $X, Y \in \mathfrak{X}(M)$. So the sectional curvature of the plane spanned by X and Y is $\frac{\mathrm{s}}{n(n-1)}$, which depends only on the base point in the manifold where we evaluate X and Y. So (M, g) is isotropic, and the condition $\dim M \geq 3$ allows us to conclude that the sectional curvature of (M, g) is constant, by Schur's lemma.

$\square$

A slight generalization of the concept of Killing field is given in the:

Definition. A vector field $\xi \in \mathfrak{X}(M)$ is a *conformal vector field* is $\mathscr{L}_\xi\langle\cdot,\cdot\rangle = \lambda\langle\cdot,\cdot\rangle$ for some smooth positive function $\lambda\colon M \to \mathbb{R}$. If λ is constant, we say that ξ is a *homothetic vector field*.

Like before, the Lie bracket of two conformal vector fields is again a conformal vector field. But more importantly:

Proposition. *Let $\xi \in \mathfrak{X}(M)$ be a λ-conformal vector field. Then $\lambda = (2/n)\operatorname{div}\xi$. So if M is compact and Riemannian, necessarily ξ is a Killing field.*

Proof: Since $(\nabla\xi) + (\nabla\xi)^* = \lambda\,\mathrm{Id}$, tracing gives $2\operatorname{div}\xi = n\lambda$. Integrating gives that

$$\int_M \lambda\,\mathrm{d}M = 0,$$

and so $\lambda = 0$ and ξ is Killing. $\square$