A classical exercise in do Carmo's *Differential Geometry of Curves and Surfaces* states that for any compact surface M in $\mathbb{R}^3$ there is $p \in M$ such that $K(p) > 0$, where $K\colon M \to \mathbb{R}$ denotes the Gaussian curvature function of M. Here we present the generalization of this result to higher dimensions. Unfortunately, I do not know where this result has first appeared.

Theorem

For any compact hypersurface $M^n \subseteq \mathbb{R}^{n+1}$ (without boundary) there is $p \in M$ such that $K(\Pi) > 0$, for every 2-plane $\Pi \subseteq T_pM$. Here, $K\colon \mathrm{Gr}(TM) \to \mathbb{R}$ denotes the sectional curvature function of M.

Proof: Consider the *energy density* function $f\colon M \to \mathbb{R}$ given by $f(x) = \|x\|^2/2$, and let $p \in M$ realize the maximum value of f (such p exists by compactness of M and continuity of f). As $\mathrm{d}f_p = 0$, we have that $T_pM = p^\perp$, so that writing $r = \|p\| > 0$, $N = p/r$ is a unit normal vector to M at p.

Now, let $v \in T_pM$ and choose a smooth curve $\gamma\colon (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$ and $\dot\gamma(0) = v$. As $t = 0$ realizes the maximum value of $(-\varepsilon, \varepsilon) \ni t \mapsto f(\gamma(t)) \in \mathbb{R}$, we know that

$$\begin{aligned} 0 \geq \left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} f(\gamma(t)) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \langle \dot\gamma(t), \gamma(t)\rangle \\ &= \langle \ddot\gamma(0), \gamma(0)\rangle + \langle \dot\gamma(0), \dot\gamma(0)\rangle \\ &= \langle \ddot\gamma(0), p\rangle + \langle v, v\rangle \\ &= \langle \ddot\gamma(0), N\rangle r + \|v\|^2, \end{aligned} \tag{0.1}$$

so that, by $\langle \ddot\gamma(0), N\rangle = \langle v, S(v)\rangle$, where $S\colon T_pM \to T_pM$ is the shape operator associated with N, it follows that

$$\langle v, S(v)\rangle r + \|v\|^2 \leq 0, \qquad \text{for all } v \in T_pM. \tag{0.2}$$

Recall that S is self-adjoint, and hence diagonalizable (with all real eigenvalues). Letting v be a unit eigenvector of S, with $S(v) = \lambda v$ and $\lambda \in \mathbb{R}$, relation (0.2) yields $\lambda \leq -1/r < 0$. Hence, $\langle\langle v, w\rangle\rangle = -\langle v, S(w)\rangle$ defines a positive-definite inner product on T_pM. Finally, let $\Pi \subseteq T_pM$ be an arbitrary 2-plane and $\{v, w\}$ be an orthonormal basis of Π. In view of the Gauss equation, we have that

$$\begin{aligned} K(\Pi) &= \langle v, S(v)\rangle\langle w, S(w)\rangle - \langle v, S(w)\rangle^2 \\ &= \langle\langle v, v\rangle\rangle\langle\langle w, w\rangle\rangle - \langle\langle v, w\rangle\rangle^2 \\ &> 0 \end{aligned} \tag{0.3}$$

by the Cauchy-Schwarz inequality applied for $\langle\langle \cdot, \cdot\rangle\rangle$, with linear independence of $\{v, w\}$ ensuring that the inequality is strict. □

Remark. The last step (0.3) only really uses the assumption that some point in M has a (positive or negative) definite shape operator, which follows from compactness. On the other hand, the conclusion becomes false when one considers instead compact submanifolds of higher codimension. For instance, the *Clifford torus* $\mathbb{S}^1 \times \mathbb{S}^1 \subseteq \mathbb{R}^4$, being flat, is an explicit counter-example in codimension two.