Comparing principal and vector bundles

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This is another set of notes I ended up writing to organize a few things in my mind. They're not meant to be comprehensive. I will try to point references to everything I don't feel like proving here.

Section 1 is a very brief review on vector bundles, (Koszul) connections and curvature tensors – we give examples, relate the condition $\nabla g = 0$ for a pseudo-Euclidean fiber metric in $E \to M$ with compatibility of ∇ with the musical isomorphisms $E \cong E^*$; describe the horizontal distribution associated to ∇ and establish the equivalence between its integrabiity and the flatness of ∇; and conclude the section describing the covariant exterior differentiation operator d^∇. Here, things are not always spelled out in full detail.

In Section 2 we register the definition of a principal G-bundle and work through a decent amount of examples, frequently making observations and general comments (either on the body of the text or on footnotes). Particular attention is given to the construction of frame bundles. We'll also briefly discuss sections (gauges) of a principal bundle, and we'll conclude the Section by discussing associated vector bundles.

In Section 3, we discuss right-invariant horizontal distributions in principal bundles and its relation with connection 1-forms. Plenty of examples of Ehresmann connections are given.

Lastly, in Section 4 we turn our attention again to bundle-valued forms and define the curvature 2-form of an Ehresmann connection. We have the covariant exterior differentiation D in the setting of principal bundles, playing the same role as d^∇ has in the setting of vector bundles. We'll also see here how some of the formulas given in Section 1 can now be rephrased, and we'll also use some gauge theory notation when convenient (for comparing two connections).

Contents

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1 A very brief review of vector bundles

Given a differentiable manifold M, one considers its *tangent bundle* TM, defined by

$$TM = \bigsqcup_{x \in M} T_x M,$$

the disjoint union of all the tangent spaces. We know that TM has a natural structure of a differentiable manifold, but the idea to be generalized here is the one of assigning to each point of M a vector space. This leads us to the:

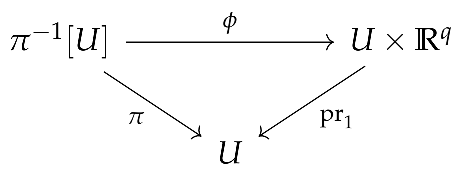
Definition 1. A *real smooth vector bundle of fiber dimension q* is a triple (E, π, M), where:

(i) E and M are differentiable manifolds;

(ii) $\pi \colon E \to M$ is smooth;

(iii) each *fiber* $E_x = \pi^{-1}(x)$ is a real vector space with dimension q, and;

(iv) for each $x_0 \in M$ there is an open neighborhood $U \subseteq M$ of x_0 and a diffeomorphism $\phi \colon \pi^{-1}[U] \to U \times \mathbb{R}^q$ such that the diagram



commutes and, for each $x \in U$, the restriction $\phi\big|_{E_x} \to \{x\} \times \mathbb{R}^q$ is an isomorphism of vector spaces.[1]

We say that E is the *total space*, M is the *base manifold* and π is the *bundle projection*. Note that π is necessarily surjective, as each fiber E_x contains its own corresponding zero vector. Each pair (U, ϕ) as above is called a VB-*chart* for E. And a collection $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \Lambda}$ for which $\{U_\alpha\}_{\alpha \in \Lambda}$ is an open cover for M is called a VB-*atlas* for E. Just like when developing manifold theory, one can also consider maximal VB-atlases.

Example 2.

(1) $\pi \colon M \times \mathbb{R}^q \to M$ is called the *trivial bundle of dimension q* over M. Each fiber is just $\{x\} \times \mathbb{R}^q$. It has one global chart (M, ϕ), where $\phi \colon M \times \mathbb{R}^q \to M \times \mathbb{R}^q$ is the identity map.

(2) The tangent bundle $\pi \colon TM \to M$, given by $\pi(x, v) = x$. A VB-chart can be constructed from a manifold-chart for M as follows: if $(U, \varphi = (x^j)_{j=1}^n)$ is given along with any $x \in U$ and $v \in T_x M$, we may write

$$v = \sum_{j=1}^n v^j \frac{\partial}{\partial x^j}\bigg|_x$$

[1]The structure in $\{x\} \times \mathbb{R}^q$ is carried from the $\mathbb{R}^q$ factor.

in a unique way in terms of coordinate vector fields, and so $\phi\colon \pi^{-1}[U] \to U \times \mathbb{R}^n$ is given by $\phi(x, \boldsymbol{v}) = (x, (v^j)_{j=1}^n)$. This is different from the manifold-chart for TM constructed in a similar process, which would be $(x, \boldsymbol{v}) \mapsto (\varphi(x), (v^j)_{j=1}^n)$.

(3) The cotangent bundle $\pi\colon T^*M \to M$, given by $\pi(x, \xi) = x$. A VB-chart can be constructed from a manifold-chart for M as follows: if $(U, \varphi = (x^j)_{j=1}^n)$ is given along with any $x \in U$ and $\xi \in T_x^*M$, we may write

$$\xi = \sum_{j=1}^n \xi_j \, \mathrm{d}x^j|_x$$

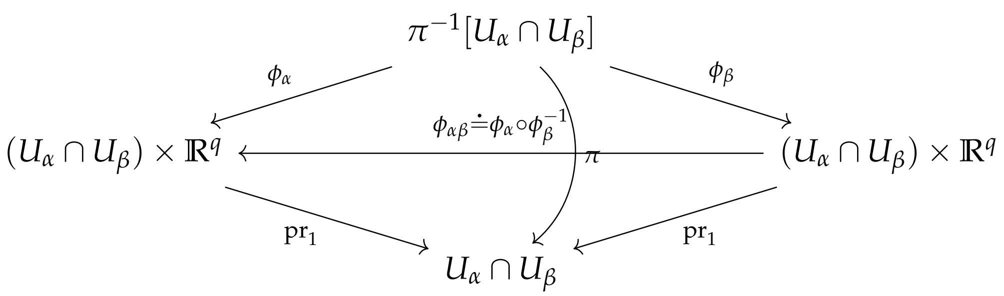
in a unique way in terms of the differentials of the coordinate functions, and so $\phi\colon \pi^{-1}[U] \to U \times \mathbb{R}^n$ is given by $\phi(x, \xi) = (x, (\xi_j)_{j=1}^n)$. Again, this is different from the manifold-chart for T^*M constructed in a similar process, which would be $(x, \xi) \mapsto (\varphi(x), (\xi_j)_{j=1}^n)$.

(4) If $M = \mathbb{R}P^n = (\mathbb{R}^{n+1} \setminus \{\boldsymbol{0}\})/_\sim$ is the real projective space, where $\boldsymbol{v} \sim \boldsymbol{w}$ if there is $\lambda \neq 0$ with $\boldsymbol{v} = \lambda \boldsymbol{w}$, we have the tautological line bundle

$$\gamma_1(\mathbb{R}) \doteq \{([\boldsymbol{v}], \boldsymbol{x}) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} \mid \boldsymbol{x} \in \mathbb{R}\boldsymbol{v}\},$$

and $\pi\colon \gamma_1(\mathbb{R}) \to \mathbb{R}P^n$ is given by $\pi([\boldsymbol{v}], \boldsymbol{x}) = [\boldsymbol{v}]$. That is, to each point $[\boldsymbol{v}]$ we assign the line $\mathbb{R}\boldsymbol{v}$. This has a natural generalization where $M = \mathrm{Gr}_k(V)$ is a Grassmannian manifold.

If $\pi\colon E \to M$ is a vector bundle with a VB-atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \Lambda}$, take $\alpha, \beta \in \Lambda$ with $U_\alpha \cap U_\beta \neq \varnothing$. Since the diagram



commutes, we know that $\phi_{\alpha\beta}\colon (U_\alpha \cap U_\beta) \times \mathbb{R}^q \to (U_\alpha \cap U_\beta) \times \mathbb{R}^q$ is of the form[2] $\phi_{\alpha\beta}(x, \boldsymbol{v}) = (x, \tau_{\alpha\beta}(x)(\boldsymbol{v}))$, where $\tau_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(q, \mathbb{R})$ is smooth (note that $\tau_{\alpha\beta}(x)(\boldsymbol{v})$ is actually smooth in the pair-variable $(x, \boldsymbol{v})$, since it is a component of a composition of diffeomorphisms). Taking a third VB-chart (U_γ, ϕ_γ) satisfying that $U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing$ and doing some yoga with a diagram slightly more elaborate than the above, we get the following three properties, for each x:

(i) $\tau_{\alpha\alpha}(x) = \mathrm{Id}_{\mathbb{R}^q}$;

[2] Play with notations. One can think that the second component of $\phi_{\alpha\beta}(x, \boldsymbol{v})$ is some $f_{\alpha\beta}(x, \boldsymbol{v})$. But since fixed x, this is linear in the variable $\boldsymbol{v}$, we define $\tau_{\alpha\beta}(x)(\boldsymbol{v}) = f_{\alpha\beta}(x, \boldsymbol{v})$, so that each $\tau_{\alpha\beta}(x)$ is linear.

(ii) $\tau_{\beta\alpha}(x) = \tau_{\alpha\beta}(x)^{-1}$;

(iii) $\tau_{\alpha\gamma}(x) = \tau_{\alpha\beta}(x) \circ \tau_{\beta\gamma}(x)$.

Such properties are essential to the structure of E, as the following result shows:

Theorem 3. *If $\{U_\alpha\}_{\alpha\in\Lambda}$ is an open cover for M and we're also given smooth functions $\tau_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(q,\mathbb{R})$, for all $\alpha, \beta \in \Lambda$ such that $U_\alpha \cap U_\beta \neq \varnothing$, and satisfying the conditions (i), (ii) and (iii) above, then there is a real vector bundle (E,π,M) with fiber dimension q and a* VB*-atlas $\{(U_\alpha,\phi_\alpha)\}_{\alpha\in\Lambda}$ for which the transition functions are precisely the prescribed $\tau_{\alpha\beta}$.*

See [3] for a proof. If all the $\tau_{\alpha\beta}$ happen to land inside some subgroup G of $\mathrm{GL}(q,\mathbb{R})$, we say that $\{(U_\alpha,\phi_\alpha)\}_{\alpha\in\Lambda}$ is a G-atlas. We proceed with sections.

Definition 4. Let $\pi\colon E \to M$ be a vector bundle. A *section* of E is a smooth map $\psi\colon M \to E$ such that $\pi \circ \psi = \mathrm{Id}_M$. We write $\psi(x) = (x, \psi_x)$, with $\psi_x \in E_x$, when needed. We also denote the collection of smooth sections of E by $\Gamma(E)$.

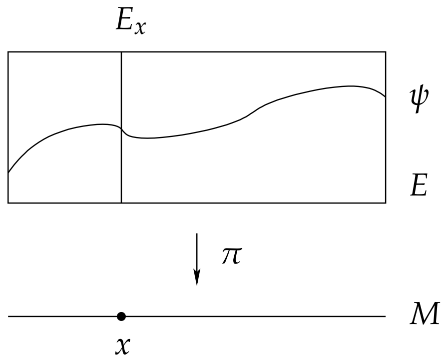


Figure 1: A section $\psi \in \Gamma(E)$.

Using VB-charts, one can show that $\Gamma(E)$ is a $\mathscr{C}^\infty(M)$-module (namely, one has to show that $\psi_1 + \psi_2$ and $f\psi$ are smooth when ψ, ψ_1 and ψ_2 are – this is not immediately trivial since the operations $+$ and $\cdot$ are also varying with the points in the base manifold).

When ψ is not defined on all of M, but only in an open subset of M, we'll say that ψ is a *local section* of E. The collection of local sections defined in some open subset U of M will be denoted by $\Gamma_U(E)$. And for $x \in M$, the collection of local sections defined in *some* neighborhood of x will be denoted by $\Gamma_x(E)$.

Example 5.

(1) $\Gamma(TM) = \mathfrak{X}(M)$ consists of vector fields on M;

(2) $\Gamma(T^*M) = \Omega^1(M)$ consists of differentiable 1-forms on M;

(3) $\Gamma(M \times \mathbb{R}^q) \cong \mathscr{C}^\infty(M, \mathbb{R}^q)$.

A VB-chart (U, ϕ) for E defines a collection $(e_a)_{a=1}^q \subseteq \Gamma_U(E)$ with the property that $(e_a(x))_{a=1}^q$ is a basis for E_x, for each $x \in U$. Such collection will be called a *frame*. Conversely, a frame defines a VB-chart. Using frames one can give new examples of vector bundles, by constructing frames for new bundles in terms of frames for a given initial bundle. In particular, one can form pull-backs, direct sums (here called *Whitney sums*), tensor products, homs and duals of vector bundles. For example, one can form the bundle

$$\mathcal{T}^r_s(M) \doteq T^*M^{\otimes r} \otimes TM^{\otimes s} \to M,$$

whose sections are called *tensor fields of type* (r, s).

Given a pseudo-Riemannian manifold (M, g), one has the so-called Levi-Civita connection ∇, which controls its geometry. The notion of connection makes sense for vector bundles as well:

Definition 6. Let $E \to M$ be a vector bundle. A *Koszul connection* in E is a $\mathbb{R}$-bilinear map $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ satisfying:

(i) $\nabla_{f\boldsymbol{X}}\psi = f\nabla_{\boldsymbol{X}}\psi$ and;

(ii) $\nabla_{\boldsymbol{X}}(f\psi) = \boldsymbol{X}(f)\psi + f\nabla_{\boldsymbol{X}}\psi$,

for each $\boldsymbol{X} \in \mathfrak{X}(M)$, $f \in \mathscr{C}^\infty(M)$ and $\psi \in \Gamma(E)$.

We'll use Einstein's convention from here on when convenient. Given coordinates (x^j) and a frame (e_a), one may write $\nabla_{\partial_j} e_a = \Gamma^b_{ja} e_b$, for some (local) functions Γ^b_{ja}. With this, we write

$$\nabla_{\boldsymbol{X}}\psi = X^j(\partial_j\psi^a + \Gamma^a_{jb}\psi^b)e_a,$$

which shows that the value of $(\nabla_{\boldsymbol{X}}\psi)_x$ depends only on $\boldsymbol{X}_x$ and on the values of ψ in a neighborhood of x. This means that we can restrict ∇ in a consistent way to open subsets of M and work with local sections when needed. We also consider the map $\nabla\psi\colon \mathfrak{X}(M) \to \Gamma(E)$ given by $\boldsymbol{X} \mapsto \nabla_{\boldsymbol{X}}\psi$, whose components will be denoted simply by $\psi^a{}_{;j} \doteq \partial_j\psi^a + \Gamma^a_{jb}\psi^b$. We will say that ψ is *parallel* if $\nabla\psi = 0$.

Example 7.

(1) In the trivial bundle $M \times \mathbb{R}^q$, we define a connection D by $\mathrm{D}_{\boldsymbol{X}}\psi \doteq \mathrm{d}\psi(\boldsymbol{X})$, where in the right-hand side we use the identification $\Gamma(M \times \mathbb{R}^q) \cong \mathscr{C}^\infty(M, \mathbb{R}^q)$ previously mentioned. This is called the *standard flat connection* in $M \times \mathbb{R}^q$.

(2) If ∇ is a connection in E and $A\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ is $\mathscr{C}^\infty(M)$-bilinear (i.e., a tensor), then $\nabla + A$ is also a connection. Conversely, the difference of two connection is such a tensor A. This says that the space of connections over E is an affine space, and the associated translation space is the space of such tensors A. Under this light, the connection symbols Γ^b_{ja} actually get a meaning: fixed a VB-chart (U, ϕ), one can use ϕ^{-1} to pull-back ∇ to a connection ∇^ϕ in the trivial bundle $U \times \mathbb{R}^q$ over U. There, one also has the standard flat connection D, and we may write $\nabla^\phi = \mathrm{D} + \Gamma$, where Γ is the so-called *Christoffel tensor* of ∇ relative to ϕ. Note, however, that the components Γ^b_{ja} do not transform like a tensor.

(3) If $E', E'' \to M$ are vector bundles with connections ∇' and ∇'', we consider the bundle $\operatorname{Hom}(E', E'')$ (whose fiber at $x \in M$ is $\operatorname{Hom}(E'_x, E''_x)$). There is a unique connection ∇ in $\operatorname{Hom}(E', E'')$ characterized by the Leibniz rule, in the following sense: given $\psi \in \Gamma(E)$ and a section F of $\operatorname{Hom}(E', E'')$, one may define a section $F(\psi)$ of E'' by $x \mapsto F_x(\psi_x)$. Then we set

$$(\nabla_{\boldsymbol{X}} F)(\psi) = \nabla''_{\boldsymbol{X}}(F(\psi)) - F\big(\nabla'_{\boldsymbol{X}}\psi\big).$$

It is straightforward to see that this indeed defines a connection. The same procedure is used to define connections in tensor bundles created from E and E^* (bearing in mind that $E^* = \operatorname{Hom}(E, M \times \mathbb{R})$).

Definition 8. Let $E \to M$ be a vector bundle. A pseudo-Euclidean fiber metric on E is a smooth section $g \in \Gamma(E^* \otimes E^*)$ such that each g_x is symmetric and non-degenerate (i.e., a pseudo-Euclidean scalar product in E_x).

When $E = TM$, g is just a pseudo-Riemannian metric on M. This has a relation with G-atlases mentioned above. For example, the existence of an Euclidean (i.e., positive-definite) fiber metric is equivalent to the existence of a $\mathrm{O}(q, \mathbb{R})$-atlas for E. Since a connection ∇ in E induces a connection on $E^* \otimes E^*$ via the Leibniz rule (which will also be denoted by ∇), it makes sense to ask whether $\nabla g = 0$ or not. In the affirmative case, we say that ∇ and g are compatible, or that ∇ is a *metric connection*. The geometric interpretation is given in terms of the *musical isomorphism*

$$\Gamma(E) \ni \psi \mapsto \psi_\flat = g(\psi, \cdot) \in \Gamma(E^*)$$

and its inverse $\sharp$, in the following result:

Proposition 9. *Let $E \to M$ be a vector bundle equipped with a pseudo-Euclidean fiber metric g and a connection ∇. So:*

(i) $\nabla_{\boldsymbol{X}}(\psi_\flat) = (\nabla_{\boldsymbol{X}}\psi)_\flat + (\nabla_{\boldsymbol{X}} g)(\psi, \cdot)$, *for all* $\boldsymbol{X} \in \mathfrak{X}(M)$ *and* $\psi \in \Gamma(E)$;

(ii) $\nabla_{\boldsymbol{X}}\xi = (\nabla_{\boldsymbol{X}}(\xi^\sharp))_\flat + (\nabla_{\boldsymbol{X}} g)(\xi^\sharp, \cdot)$, *for all* $\boldsymbol{X} \in \mathfrak{X}(M)$ *and* $\xi \in \Gamma(E^*)$.

Thus, the following are equivalent: g is a parallel section of $E^* \otimes E^*$; ∇ and $\flat$ commute; and ∇ and $\sharp$ commute.

Remark. It might be instructive to note that the relations

$$(\nabla_{\boldsymbol{X}}\flat)(\psi) = \nabla_{\boldsymbol{X}}(\psi_\flat) - (\nabla_{\boldsymbol{X}}\psi)_\flat \quad \text{and} \quad (\nabla_{\boldsymbol{X}}\sharp)(\xi) = \nabla_{\boldsymbol{X}}(\xi^\sharp) - (\nabla_{\boldsymbol{X}}\xi)^\sharp$$

hold, so that how much g deviates from being parallel directly measures the non-commutativity of ∇ with $\flat$ and $\sharp$.

Proof: Let's check only the first formula, being the second one analogous: take a section $\phi \in \Gamma(E)$ and compute

$$\begin{aligned}(\nabla_{\boldsymbol{X}}(\psi_\flat))(\phi) &= \nabla_{\boldsymbol{X}}(\psi_\flat(\phi)) - \psi_\flat(\nabla_{\boldsymbol{X}}\phi) \\ &= \boldsymbol{X}g(\psi, \phi) - g(\psi, \nabla_{\boldsymbol{X}}\phi) \\ &= g(\nabla_{\boldsymbol{X}}\psi, \phi) + (\nabla_{\boldsymbol{X}} g)(\psi, \phi) \\ &= (\nabla_{\boldsymbol{X}}\psi)_\flat(\phi) + (\nabla_{\boldsymbol{X}} g)(\psi, \phi).\end{aligned}$$

□

One construction relevant for understanding connections in principal bundles is the one of *horizontal lifts*. If $E \to M$ is a vector bundle with a connection ∇, we define the horizontal lift of a vector $\boldsymbol{v} \in T_xM$ as $\boldsymbol{v}^{\rm hor}_{(x,\phi)} \doteq \mathrm{d}\psi_x(\boldsymbol{v}) \in T_{(x,\phi)}E$, where $\psi \in \Gamma_p(E)$ is any section satisfying $(\nabla\psi)_x = 0$ and $\psi_x = \phi$. We need to check that this does not depend on the choice of ψ. This is done locally, first observing that a VB-atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in\Lambda}$ and a manifold-atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ for M (both assumed with the same domains U_α, by reducing them if necessary) together define a manifold-atlas for E via the compositions

$$\pi^{-1}[U_\alpha] \xrightarrow{\phi_\alpha} U_\alpha \times \mathbb{R}^q \xrightarrow{\varphi_\alpha \times \mathrm{Id}_{\mathbb{R}^q}} \varphi_\alpha[U_\alpha] \times \mathbb{R}^q \subseteq \mathbb{R}^{n+q},$$

which in practice say that $(x,\phi) \mapsto (x^j, \phi^a)$ and $\dim E = \dim M + q$, giving us coordinate vector fields (∂_j, ∂_a) tangent to E (note the slight abuse of notation: the ∂_j's here are fields tangent to E which project under $\mathrm{d}\pi$ to the actual ∂_j's tangent to M). In particular, the canonical isomorphism $T_\phi(E_x) \cong E_x$ consists simply of $\partial_a|_x \mapsto e_a|_x$. In any way, we have the expressions $\phi = \phi^a e_a(x)$ and $\psi = \psi^a e_a$ – so that $\psi^a(x) = \phi^a$ for all a. Also $\psi^a{}_{;j}(x) = 0$ becomes $(\partial_j\psi^a)(x) = -\Gamma^a_{jb}(x)\phi^b$, and with this we see that

$$\mathrm{d}\psi_x(\boldsymbol{v}) = v^j\partial_j\big|_{(x,\phi)} + v^j(\partial_j\psi^a)(x)\partial_a\big|_{(x,\phi)} = v^j\partial_j\big|_{(x,\phi)} - \Gamma^a_{jb}(x)v^j\phi^b\partial_a\big|_{(x,\phi)}$$

depends only on $\boldsymbol{v}$ and ϕ, but not on ψ. Indeed, we have that the components of $\mathrm{d}\psi_x(\boldsymbol{v})$ in the direction of ∂_j are just v^j in view of $\pi \circ \psi = \mathrm{Id}_U$, while the component in the direction of ∂_a is $\boldsymbol{v}(\psi^a)$. Denoting the image of the (injective) linear map $T_xM \ni \boldsymbol{v} \mapsto \boldsymbol{v}^{\rm hor}_{(x,\phi)} \in T_{(x,\phi)}E$ by $\mathrm{Hor}_{(x,\phi)}(E)$ (the ∇-*horizontal subspace* of $T_{(x,\phi)}E$) and setting $\mathrm{Ver}_{(x,\phi)}(E) = \ker \mathrm{d}\pi_{(x,\phi)}$ (the *vertical subspace* of $T_{(x,\phi)}E$, which by the way is canonical), we have that

$$T_{(x,\phi)}E = \mathrm{Hor}_{(x,\phi)}(E) \oplus \mathrm{Ver}_{(x,\phi)}(E),$$

where $\mathrm{Hor}_{(x,\phi)}(E) \cong T_xM$ via the restriction of $\mathrm{d}\pi_{(x,\phi)}$: in fact, the restriction of $\mathrm{d}\pi_{(x,\phi)}$ to any subspace of $T_{(x,\phi)}E$ complementary to $\mathrm{Ver}_{(x,\phi)}(E)$ will be an isomorphism onto the tangent space T_xM. Any vector $\boldsymbol{Z}_{(x,\phi)} \in T_{(x,\phi)}E$ can be decomposed according to this as

$$\begin{aligned}\boldsymbol{Z}_{(x,\phi)} &= Z^j\partial_j\big|_{(x,\phi)} + Z^a\partial_a\big|_{(x,\phi)} \\ &= \underbrace{\left(Z^j\partial_j\big|_{(x,\phi)} - \Gamma^a_{jb}(x)Z^j\phi^b\partial_a\big|_{(x,\phi)}\right)}_{\text{horizontal}} + \underbrace{(Z^a + \Gamma^a_{jb}(x)Z^j\phi^b)\partial_a\big|_{(x,\phi)}}_{\text{vertical}}.\end{aligned}$$

Letting (x,ϕ) range over E, we obtain a smooth distribution $\mathrm{Hor}(E) \hookrightarrow TE$. At this point, the natural question is whether $\mathrm{Hor}(E)$ is integrable. This leads us to the definition:

Definition 10. Let $E \to M$ be a vector bundle equipped with a connection ∇. The *curvature* of ∇ is the map $R^\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ given by

$$R^\nabla(\boldsymbol{X},\boldsymbol{Y})\psi \doteq \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\psi - \nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\psi - \nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\psi.$$

A straightforward computation show that R^∇ is skew-symmetric in the first two entries, and also that it is $\mathscr{C}^\infty(M)$-trilinear, and thus defines a tensor. That is to say, the value of $R^\nabla(\boldsymbol{X},\boldsymbol{Y})\psi$ at a point x depends only on the values $\boldsymbol{X}_x, \boldsymbol{Y}_x$ and ψ_x. With respect to a coordinate system (x^j) and a local frame (e_a), we may write

$$R^\nabla(\boldsymbol{X},\boldsymbol{Y})\psi = R_{jka}{}^b X^j Y^k \psi^a e_b,$$

where the components are computed as

$$\begin{aligned} R^\nabla(\partial_j,\partial_k)e_a &= \nabla_{\partial_j}\nabla_{\partial_k}e_a - \nabla_{\partial_k}\nabla_{\partial_j}e_a \\ &= \nabla_{\partial_j}(\Gamma^b_{ka}e_b) - \nabla_{\partial_k}(\Gamma^b_{ja}e_b) \\ &= (\partial_j\Gamma^b_{ka})e_b + \Gamma^b_{ka}\Gamma^c_{jb}e_c - (\partial_k\Gamma^b_{ja})e_b - \Gamma^b_{ja}\Gamma^c_{kb}e_c \\ &= (\partial_j\Gamma^b_{ka} - \partial_k\Gamma^b_{ja} + \Gamma^c_{ka}\Gamma^b_{jc} - \Gamma^c_{ja}\Gamma^b_{kc})e_b, \end{aligned}$$

so that $R_{jka}{}^b = \partial_j\Gamma^b_{ka} - \partial_k\Gamma^b_{ja} + \Gamma^c_{ka}\Gamma^b_{jc} - \Gamma^c_{ja}\Gamma^b_{kc}$.

Example 11.

(1) If D is the standard connection in $M \times \mathbb{R}^q$, then $R^{\mathrm{D}} = 0$. This can be seen in terms of a global parallel frame induced by a basis of $\mathbb{R}^q$, or simply by computing

$$R^{\mathrm{D}}(\boldsymbol{X},\boldsymbol{Y})\psi = \boldsymbol{X}(\boldsymbol{Y}(\psi)) - \boldsymbol{Y}(\boldsymbol{X}(\psi)) - [\boldsymbol{X},\boldsymbol{Y}](\psi) = 0.$$

(2) If $E', E'' \to M$ are vector bundles with connections ∇' and ∇'', and ∇ is the natural connection induced in $\mathrm{Hom}(E',E'')$, then

$$R^\nabla(\boldsymbol{X},\boldsymbol{Y})F = R^{\nabla''}(\boldsymbol{X},\boldsymbol{Y}) \circ F - F \circ R^{\nabla'}(\boldsymbol{X},\boldsymbol{Y}).$$

In particular, when $E' = E''$, we just get the usual commutator of endomorphisms.

Theorem 12. *Let $E \to M$ be a vector bundle equipped with a connection ∇. Then the distribution* $\mathrm{Hor}(E) \hookrightarrow TE$ *is integrable if and only if $R^\nabla = 0$.*

Proof: A straightforward computation gives that $\Omega^a_{jk} + R_{jkb}{}^a\phi^b = 0$, where Ω^a_{jk} is the *Levi symbol*[3] (curvature form) of $\mathrm{Hor}(E)$ in TE. □

[3]The Levi symbol of a distribution $D \hookrightarrow TM$ is the collection of maps $\Omega_x\colon \Gamma(D)\times\Gamma(D) \to TM/D$, for $x \in M$, defined by $\Omega_x(\boldsymbol{X},\boldsymbol{Y}) = \pi([\boldsymbol{X},\boldsymbol{Y}]_x)$. If coordinates (x^j, y^λ) are adapted to D in the sense that D is described by $\mathrm{d}y^\lambda = H^\lambda_j\,\mathrm{d}x^j$, then the components of Ω (relative to the local frame $e_j = \partial_j + H^\lambda_j\partial_\lambda$ tangent to D) are

$$\Omega^\lambda_{jk} = \partial_j H^\lambda_k - \partial_k H^\lambda_j + H^\mu_j\partial_\mu H^\lambda_k - H^\mu_k\partial_\mu H^\lambda_j.$$

So D is involutive (hence integrable, by the Frobenius Theorem) if and only $\Omega^\lambda_{jk} = 0$ for all choices of indices. Now, for the horizontal distribution $\mathrm{Hor}(E) \hookrightarrow TE$, we have $H^a_j = -\Gamma^a_{jb}\phi^b$.

If $E \to M$ has a connection ∇ and a pseudo-Euclidean fiber metric g, one can form totally covariant version of R^∇ by setting $R^\nabla(\boldsymbol{X},\boldsymbol{Y},\psi,\phi) = g(R^\nabla(\boldsymbol{X},\boldsymbol{Y})\psi,\phi)$, and then the condition $\nabla g = 0$ implies in skew-symmetry in the last two entries.

Connections can be used to generalize the notion of exterior derivative to bundle-valued forms. Namely, if ∇ is a connection in a vector bundle $E \to M$, we just mimic the coordinate-free description of the operator d (also known as the Palais formula) to define the *covariant exterior derivative* $\mathrm{d}^\nabla\colon \Omega^k(M;E) \to \Omega^{k+1}(M;E)$ by the formula

$$(\mathrm{d}^\nabla\omega)(\boldsymbol{X}_0,\ldots,\boldsymbol{X}_k) \doteq \sum_{i=0}^{k}(-1)^i\nabla_{\boldsymbol{X}_i}(\omega(\boldsymbol{X}_0,\ldots,\widehat{\boldsymbol{X}_i},\ldots,\boldsymbol{X}_k)) + \sum_{0\leq i<j\leq k}(-1)^{i+j}\omega([\boldsymbol{X}_i,\boldsymbol{X}_j],\ldots,\widehat{\boldsymbol{X}_i},\ldots,\widehat{\boldsymbol{X}_j},\ldots,\boldsymbol{X}_k).$$

For example, if $\omega \in \Omega^1(M,E)$ we have

$$(\mathrm{d}^\nabla\omega)(\boldsymbol{X},\boldsymbol{Y}) = \nabla_{\boldsymbol{X}}(\omega(\boldsymbol{Y})) - \nabla_{\boldsymbol{Y}}(\omega(\boldsymbol{X})) - \omega([\boldsymbol{X},\boldsymbol{Y}]).$$

When $E = M \times \mathbb{R}$ and $\nabla = \mathrm{D}$, we recover the usual exterior derivative. This captures all the relevant information about ∇ and R^∇, in the sense of the:

Proposition 13. *Given $\psi \in \Gamma(E) = \Omega^0(M;E)$, we have:*

(i) $(\mathrm{d}^\nabla\psi)(\boldsymbol{X}) = \nabla_{\boldsymbol{X}}\psi$.

(ii) $\big((\mathrm{d}^\nabla)^2\psi\big)(\boldsymbol{X},\boldsymbol{Y}) = R^\nabla(\boldsymbol{X},\boldsymbol{Y})\psi$.

(iii) $\big((\mathrm{d}^\nabla)^3\psi\big)(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}) = R^\nabla(\boldsymbol{X},\boldsymbol{Y})\nabla_{\boldsymbol{Z}}\psi + R^\nabla(\boldsymbol{Y},\boldsymbol{Z})\nabla_{\boldsymbol{X}}\psi + R^\nabla(\boldsymbol{Z},\boldsymbol{X})\nabla_{\boldsymbol{Y}}\psi$.

(iv) $$\big((\mathrm{d}^\nabla)^4\psi\big)(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z},\boldsymbol{W}) = R^\nabla(\boldsymbol{X},\boldsymbol{Y})R^\nabla(\boldsymbol{Z},\boldsymbol{W})\psi + R^\nabla(\boldsymbol{Z},\boldsymbol{X})R^\nabla(\boldsymbol{Y},\boldsymbol{W})\psi + R^\nabla(\boldsymbol{X},\boldsymbol{W})R^\nabla(\boldsymbol{Y},\boldsymbol{Z})\psi + R^\nabla(\boldsymbol{Y},\boldsymbol{Z})R^\nabla(\boldsymbol{X},\boldsymbol{W})\psi + R^\nabla(\boldsymbol{W},\boldsymbol{Y})R^\nabla(\boldsymbol{X},\boldsymbol{Z})\psi + R^\nabla(\boldsymbol{Z},\boldsymbol{W})R^\nabla(\boldsymbol{X},\boldsymbol{Y})\psi.$$

Proof: Brute force. □

Also, we may regard R^∇ as an element of $\Omega^2(M;\mathrm{End}(E))$, and use the induced connection in the latter bundle to compute $\mathrm{d}^\nabla R^\nabla = 0$. This is known as the *Second Bianchi Identity*, usually described as

$$(\nabla_{\boldsymbol{X}}R^\nabla)(\boldsymbol{X},\boldsymbol{Y}) + (\nabla_{\boldsymbol{Y}}R^\nabla)(\boldsymbol{Z},\boldsymbol{X}) + (\nabla_{\boldsymbol{Z}}R^\nabla)(\boldsymbol{X},\boldsymbol{Y}) = 0,$$

when using a torsion-free connection in TM. The operation d^∇ is also useful to express the curvature of a connection modified by a tensor. For this, we will also need a new type of operation: when we have forms taking values in an *algebra bundle* $\mathscr{A}$ (i.e., just like a vector bundle, but each fiber has a bilinear multiplication $\cdot$), we may mimic the definition of the wedge product and define, for $\omega \in \Omega^k(M;\mathscr{A})$, $\eta \in \Omega^\ell(M,\mathscr{A})$, a new element $\omega \circledast \eta \in \Omega^{k+\ell}(M,\mathscr{A})$ by

$$(\omega \circledast \eta)(\boldsymbol{X}_1,\ldots,\boldsymbol{X}_{k+\ell}) \doteq \frac{1}{k!\ell!}\sum_{\sigma\in S_{k+\ell}}\mathrm{sgn}(\sigma)\omega(\boldsymbol{X}_{\sigma(1)},\ldots,\boldsymbol{X}_{\sigma(k)})\cdot\eta(\boldsymbol{X}_{\sigma(k+1)},\ldots,\boldsymbol{X}_{\sigma(k+\ell)}).$$

A very frequent case is when the fibers are actually Lie algebras, in which case the product is denoted by $[\omega, \eta]$. For $k = \ell = 1$ we have

$$[\omega, \eta](\boldsymbol{X}, \boldsymbol{Y}) = [\omega(\boldsymbol{X}), \eta(\boldsymbol{Y})] - [\omega(\boldsymbol{Y}), \eta(\boldsymbol{X})],$$

and in particular $[\omega, \omega](\boldsymbol{X}, \boldsymbol{Y}) = 2[\omega(\boldsymbol{X}), \omega(\boldsymbol{Y})]$. While the operation $\circledast$ might not have any symmetry since we do not know whether the product structure in $\mathscr{A}$ is commutative or skew, in the Lie algebra case we do have that $[\eta, \omega] = (-1)^{k\ell+1}[\omega, \eta]$.

As a last remark about $\circledast$ before we move one, we note that this operation also relates to the usual exterior derivative via $\mathrm{d}(\omega \circledast \eta) = \mathrm{d}\omega \circledast \eta + (-1)^k \omega \circledast \mathrm{d}\eta$, as expected. Back to the curvature of a modified connection, we have the:

Proposition 14 (Palatini-like identity)**.** *Let $E \to M$ be a vector bundle with a connection ∇, and consider also a tensor $A\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$. Then*

$$R^{\nabla + A} = R^{\nabla} + \mathrm{d}^{\nabla} A + \frac{1}{2}[A, A].$$

Here we regard A as an element of $\Omega^1(M, \mathrm{End}(E))$.

Remark. When we use a VB-chart (U, ϕ) and write $\nabla^{\phi} = \mathrm{D} + \Gamma$, the above formula gives that $R^{\nabla} = \mathrm{d}^{\nabla}\Gamma + [\Gamma, \Gamma]/2$. Then the differential term $\partial_j \Gamma^b_{ka} - \partial_k \Gamma^b_{ja}$ corresponds to $\mathrm{d}^{\nabla}\Gamma$, while the term $\Gamma^c_{ka}\Gamma^b_{jc} - \Gamma^c_{ja}\Gamma^b_{kc}$ corresponds to the commutator, and so we indeed have the full expression for ${R_{jka}}^b$.

Proof: Just distribute the "products" to get

$$\begin{aligned} R^{\nabla + A}(\boldsymbol{X}, \boldsymbol{Y})\psi &= (\nabla + A)_{\boldsymbol{X}}((\nabla + A)_{\boldsymbol{Y}}\psi) - (\nabla + A)_{\boldsymbol{Y}}((\nabla + A)_{\boldsymbol{X}}\psi) - (\nabla + A)_{[\boldsymbol{X}, \boldsymbol{Y}]}\psi \\ &= R^{\nabla}(\boldsymbol{X}, \boldsymbol{Y})\psi + \nabla_{\boldsymbol{X}} A_{\boldsymbol{Y}}\psi + A_{\boldsymbol{X}} A_{\boldsymbol{Y}}\psi + A_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\psi \\ &\qquad - \nabla_{\boldsymbol{Y}} A_{\boldsymbol{X}}\psi + A_{\boldsymbol{Y}} A_{\boldsymbol{X}}\psi + A_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\psi - A_{[\boldsymbol{X}, \boldsymbol{Y}]}\psi \\ &= R^{\nabla}(\boldsymbol{X}, \boldsymbol{Y})\psi + ((\mathrm{d}^{\nabla} A)(\boldsymbol{X}, \boldsymbol{Y}))\psi + [A_{\boldsymbol{X}}, A_{\boldsymbol{Y}}]\psi, \end{aligned}$$

as wanted. □

In the particular case when $E = TM$ is equipped with a connection ∇, there is one very special TM-valued 1-form on which we can apply d^{∇}: the bundle identity $\mathrm{Id}_{TM}\colon TM \to TM$ itself. With this, one could also *define* the torsion of a connection ∇ as the map $\tau^{\nabla}\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by

$$\tau^{\nabla}(\boldsymbol{X}, \boldsymbol{Y}) \doteq (\mathrm{d}^{\nabla}\mathrm{Id})(\boldsymbol{X}, \boldsymbol{Y}) = \nabla_{\boldsymbol{X}}\boldsymbol{Y} - \nabla_{\boldsymbol{Y}}\boldsymbol{X} - [\boldsymbol{X}, \boldsymbol{Y}].$$

This is of course $\mathscr{C}^{\infty}(M)$-bilinear, and can be seen as an element $\tau^{\nabla} \in \Omega^2(M, TM)$. One proceeds and computes

$$(\mathrm{d}^{\nabla}\tau^{\nabla})(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}) = R^{\nabla}(\boldsymbol{X}, \boldsymbol{Y})\boldsymbol{Z} + R^{\nabla}(\boldsymbol{Y}, \boldsymbol{Z})\boldsymbol{X} + R^{\nabla}(\boldsymbol{Z}, \boldsymbol{X})\boldsymbol{Y}.$$

So:

Proposition 15. *Let ∇ be a torsion-free connection in TM. Then the* First Bianchi Identity *holds for all vector fields $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \mathfrak{X}(M)$:*

$$R^\nabla(\boldsymbol{X}, \boldsymbol{Y})\boldsymbol{Z} + R^\nabla(\boldsymbol{Y}, \boldsymbol{Z})\boldsymbol{X} + R^\nabla(\boldsymbol{Z}, \boldsymbol{X})\boldsymbol{Y} = \mathbf{0}.$$

Remark. In other words, the content of the First Bianchi Identity is just the algebraic statement that "the derivative of zero equals zero".

Another last geometric feature in TM worth mentioning is that connections produce covariant Hessians of smooth functions: $\mathrm{Hess}^\nabla(f)(\boldsymbol{X}, \boldsymbol{Y}) = \boldsymbol{X}(\boldsymbol{Y}(f)) - \mathrm{d}f(\nabla_{\boldsymbol{X}}\boldsymbol{Y})$, and those Hessians are symmetric tensors if and only if $\tau^\nabla = 0$.

Let's conclude the section with an alternative way to locally record information about a connection ∇ and its curvature R^∇, in a general vector bundle $E \to M$. Let (e_a) be a local frame for E. Write

$$\nabla_{\boldsymbol{X}} e_b = \omega^a{}_b(\boldsymbol{X}) e_a \quad \text{and} \quad R^\nabla(\boldsymbol{X}, \boldsymbol{Y}) e_b = \Omega^a{}_b(\boldsymbol{X}, \boldsymbol{Y}) e_b,$$

for all a. Then the $\omega^a{}_b \in \Omega^1(M)$ are called the *connection* 1-*forms* relative to (e_a), while the $\Omega^a{}_b \in \Omega^2(M)$ are called the *curvature* 2-*forms* relative to (e_a). Of course, we may regard ω and Ω as $\mathfrak{gl}(q, \mathbb{R})$-valued differential forms. So, a few remarks are in order:

- we can extend the $\wedge$ operation to such forms, by replacing the usual matrix multiplication with $\wedge$.
- if we consider the standard flat connection D in the trivial (Lie algebra) bundle $M \times \mathfrak{gl}(q, \mathbb{R})$, we can form the covariant exterior derivatives $\mathrm{d}^{\mathrm{D}}\omega$ and $\mathrm{d}^{\mathrm{D}}\Omega$, which just amount to taking the exterior derivatives of all entries of the matrix in question. Thus when writing $\mathrm{d}\omega$, we mean a matrix whose entries are 2-forms, for example.
- if $\boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{X}(M)$ and $\alpha \in \Omega^1(M, \mathfrak{gl}(n, \mathbb{R}))$, then $\boldsymbol{X}(\alpha(\boldsymbol{Y}))$ is the matrix whose entries are $\boldsymbol{X}(\alpha^i{}_j(\boldsymbol{Y}))$.

That being said, we have some relations between ω and Ω, given in the next two results:

Proposition 16. *Let $E \to M$ be a vector bundle with a connection ∇ and (e_a) a local frame for E. Then*

$$\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega] = \mathrm{d}\omega + \omega \wedge \omega.$$

In terms of matrix entries, we have $\Omega^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$.

Proof: Write $\mathfrak{e} = [e_1 \cdots e_q]$ as a row vector of local sections[4]. Then by definition of connection and curvature forms, we have $\nabla_{\boldsymbol{X}}\mathfrak{e} = \mathfrak{e}\omega(\boldsymbol{X})$ and $R^\nabla(\boldsymbol{X}, \boldsymbol{Y})\mathfrak{e} = \mathfrak{e}\Omega(\boldsymbol{X}, \boldsymbol{Y})$, where the right sides of those equations are actual matrix products, and the left sides

[4] This sort of notation using the same kernel letter in German font is used, for example, in [6].

are defined by applying the corresponding operation in each entry[5]. So now we use the definition of curvature and an obvious product rule in this setting to get

$$\begin{aligned}
\mathfrak{e}\Omega(\boldsymbol{X},\boldsymbol{Y}) &= R^{\nabla}(\boldsymbol{X},\boldsymbol{Y})\mathfrak{e} \\
&= \nabla_{\boldsymbol{X}}(\mathfrak{e}\omega(\boldsymbol{Y})) - \nabla_{\boldsymbol{Y}}(\mathfrak{e}\omega(\boldsymbol{X})) - \mathfrak{e}\omega([\boldsymbol{X},\boldsymbol{Y}]) \\
&= \mathfrak{e}\omega(\boldsymbol{X})\omega(\boldsymbol{Y}) + \mathfrak{e}\boldsymbol{X}(\omega(\boldsymbol{Y})) - \mathfrak{e}\omega(\boldsymbol{Y})\omega(\boldsymbol{X}) - \mathfrak{e}\boldsymbol{Y}(\omega(\boldsymbol{X})) - \mathfrak{e}\omega([\boldsymbol{X},\boldsymbol{Y}]) \\
&= \mathfrak{e}\,(\boldsymbol{X}(\omega(\boldsymbol{Y})) - \boldsymbol{Y}(\omega(\boldsymbol{X})) - \omega([\boldsymbol{X},\boldsymbol{Y}]) + [\omega(\boldsymbol{X}),\omega(\boldsymbol{Y})]) \\
&= \mathfrak{e}\left(\mathrm{d}\omega(\boldsymbol{X},\boldsymbol{Y}) + \frac{1}{2}[\omega,\omega](\boldsymbol{X},\boldsymbol{Y})\right).
\end{aligned}$$

The conclusion follows from linear independence of $\mathfrak{e}$ (which is then cancelled in the previous equality). □

Corollary 17 (Second Bianchi Identity)**.** *Under the same assumptions as the previous result, we have* $\mathrm{d}\Omega = \Omega\wedge\omega - \omega\wedge\Omega$.

Proof: Apply d on both sides of $\Omega = \mathrm{d}\omega + \omega\wedge\omega$ to get

$$\begin{aligned}
\mathrm{d}\Omega &= 0 + \mathrm{d}\omega\wedge\omega - \omega\wedge\mathrm{d}\omega \\
&= (\Omega - \omega\wedge\omega)\wedge\omega - \omega\wedge(\Omega - \omega\wedge\omega) \\
&= \Omega\wedge\omega - (\omega\wedge\omega)\wedge\omega - \omega\wedge\Omega + \omega\wedge(\omega\wedge\omega) \\
&= \Omega\wedge\omega - \omega\wedge\Omega,
\end{aligned}$$

since $(\omega\wedge\omega)\wedge\omega = (-1)^{2\cdot 1}\omega\wedge(\omega\wedge\omega) = \omega\wedge(\omega\wedge\omega)$, ensuring the desired calculation.

□

Remark. Applying d again to both sides of $\mathrm{d}\Omega = \Omega\wedge\omega - \omega\wedge\Omega$ actually gives $0 = 0$.

Proposition 18. *Let $E \to M$ be a vector bundle with a connection ∇ and $\mathfrak{e} = (e_a)$ a local frame for E, seen as a row of sections. Then if $\mathfrak{e}^{\top} = (e^a)$ is the coframe dual to (e_a), seen as a column of sections, we have that*

$$\mathrm{d}^{\nabla}\mathfrak{e}^{\top} + \omega\mathfrak{e}^{\top} = 0.$$

Proof: For any local section ψ in the frame domain, we have that $\psi = e^b(\psi)e_b$. Apply $\nabla_{\boldsymbol{X}}$ to get

$$\nabla_{\boldsymbol{X}}\psi = \nabla_{\boldsymbol{X}}(e^b(\psi)e_b) = \boldsymbol{X}(e^a(\psi))e_a + e^b(\psi)\nabla_{\boldsymbol{X}}e_b = (\boldsymbol{X}(e^a(\psi)) + \omega^a{}_b(\boldsymbol{X})e^b(\psi))e_a,$$

so that $e^a(\nabla_{\boldsymbol{X}}\psi) = \boldsymbol{X}(e^a(\psi)) + \omega^a{}_b(\boldsymbol{X})e^b(\psi)$. This means that

$$(\mathrm{d}^{\nabla}e^a)(\boldsymbol{X})(\psi) + \omega^a{}_b(\boldsymbol{X})e^b(\psi) = 0,$$

and we are done. □

[5]One can also omit the vector fields, writing only $\nabla\mathfrak{e} = \mathfrak{e}\omega$ and $R^{\nabla}\mathfrak{e} = \mathfrak{e}\Omega$, as functions of $\boldsymbol{X}$ and $\boldsymbol{Y}$.

In the particular case where $E = TM$ and we have a local frame $(\boldsymbol{E}_i)$, with dual coframe (θ^i), the exterior derivative $\mathrm{d}\theta^i$ is not exactly the same thing as the covariant exterior derivative $\mathrm{d}^\nabla\theta^i$ of θ^i seen as a T^*M-valued 0-form. So to cast Proposition 18 under this light, we'll need to replace the matrix product there by a $\wedge$, and a correction factor involving the torsion τ^∇ will appear.

Proposition 19. *Let ∇ be a connection in TM, $(\boldsymbol{E}_i)$ be a local frame and (θ^i) be the corresponding dual coframe. Then*

$$\mathrm{d}\theta + \omega \wedge \theta = \tau^\nabla,$$

where τ^∇ is seen as a column of 2-forms.

Proof: The current Proposition 18 applied twice gives

$$\boldsymbol{X}(\theta^i(\boldsymbol{Y})) - \theta^i(\nabla_{\boldsymbol{X}}\boldsymbol{Y}) + \omega^i{}_j(\boldsymbol{X})\theta^j(\boldsymbol{Y}) = 0 \quad \text{and} \quad \boldsymbol{Y}(\theta^i(\boldsymbol{X})) - \theta^i(\nabla_{\boldsymbol{Y}}\boldsymbol{X}) + \omega^i{}_j(\boldsymbol{Y})\theta^j(\boldsymbol{X}) = 0,$$

by definition of $\mathrm{d}^\nabla\theta^i$. Now use that $\nabla_{\boldsymbol{X}}\boldsymbol{Y} - \nabla_{\boldsymbol{Y}}\boldsymbol{X} = \tau^\nabla(\boldsymbol{X},\boldsymbol{Y}) + [\boldsymbol{X},\boldsymbol{Y}]$ and subtract the second equation from the first to get

$$\boldsymbol{X}(\theta^i(\boldsymbol{Y})) - \boldsymbol{Y}(\theta^i(\boldsymbol{X})) - \theta^i(\tau^\nabla(\boldsymbol{X},\boldsymbol{Y}) + [\boldsymbol{X},\boldsymbol{Y}]) + (\omega^i{}_j \wedge \theta^j)(\boldsymbol{X},\boldsymbol{Y}) = 0.$$

But this becomes $\tau^i(\boldsymbol{X},\boldsymbol{Y}) \doteq \theta^i(\tau^\nabla(\boldsymbol{X},\boldsymbol{Y})) = \mathrm{d}\theta^i(\boldsymbol{X},\boldsymbol{Y}) + (\omega^i{}_j \wedge \theta^j)(\boldsymbol{X},\boldsymbol{Y})$. □

Corollary 20 (First Bianchi Identity)**.** *Under the same assumptions as the previous result, we have $\mathrm{d}\tau^\nabla = \Omega \wedge \theta - \omega \wedge \tau^\nabla$.*

Proof: Apply d to $\tau^\nabla = \mathrm{d}\theta + \omega \wedge \theta$ to get

$$\begin{aligned}
\mathrm{d}\tau^\nabla &= 0 + \mathrm{d}\omega \wedge \theta - \omega \wedge \mathrm{d}\theta \\
&= (\Omega - \omega \wedge \omega) \wedge \theta - \omega \wedge (\tau^\nabla - \omega \wedge \theta) \\
&= \Omega \wedge \theta - (\omega \wedge \omega) \wedge \theta - \omega \wedge \tau^\nabla + \omega \wedge (\omega \wedge \theta) \\
&= \Omega \wedge \theta - \omega \wedge \tau^\nabla.
\end{aligned}$$

□

Remark. If $\tau^\nabla = 0$, we get $\Omega \wedge \theta = 0$. This means that $(\Omega^i{}_j \wedge \theta^j)(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}) = 0$ for any fields $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$. So $\Omega^i{}_j(\boldsymbol{X},\boldsymbol{Y})\theta^j(\boldsymbol{Z}) + \Omega^i{}_j(\boldsymbol{Y},\boldsymbol{Z})\theta^j(\boldsymbol{X}) + \Omega^i{}_j(\boldsymbol{Z},\boldsymbol{X})\theta^j(\boldsymbol{Y}) = 0$. Contract against $\boldsymbol{E}_i$ and get $R^\nabla(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z} + R^\nabla(\boldsymbol{Y},\boldsymbol{Z})\boldsymbol{X} + R^\nabla(\boldsymbol{Z},\boldsymbol{X})\boldsymbol{Y} = \boldsymbol{0}$, as usual.

Now, connection 1-forms and curvature 2-forms for a connection ∇ in a vector bundle $E \to M$ are, a priori, local objects, depending on a choice of local frame. However, a miracle happens: these are actually local manifestations of a global object living not on E, but on the *frame bundle* $\mathrm{Fr}(E)$ of E, to be discussed in Section 2. It will be useful to register how will the ω's and Ω's transform when we change from a local frame to another. So:

Proposition 21. *Assume that $E \to M$ is a vector bundle with a connection ∇, and (e_a) and $(\widetilde{e}_a)$ are local frames for E, related on their common domain via $\widetilde{e}_b = A^a{}_b e_a$, where $A = (A^a{}_b)$ is a smooth $\mathrm{GL}(q,\mathbb{R})$-valued function. In other words, $\widetilde{\mathfrak{e}} = \mathfrak{e}A$. Then:*

(i) $\widetilde{\omega} = A^{-1}\omega A + A^{-1}\,\mathrm{d}A$.

(ii) $\widetilde{\Omega} = A^{-1}\Omega A$.

Remark. The transformation law in (ii) reflects the fact that R^∇ is a tensor. The transformation law in (i), in turn, will serve as a motivation for defining an Ehresmann connection in a principal bundle, in Section 3.

Proof:

(i) On one hand, $\nabla\widetilde{\mathfrak{e}} = \widetilde{\mathfrak{e}}\widetilde{\omega} = \mathfrak{e}A\widetilde{\omega}$. On the other hand, we have

$$\nabla\widetilde{\mathfrak{e}} = \nabla(\mathfrak{e}A) = (\nabla\mathfrak{e})A + \mathfrak{e}\,\mathrm{d}A = \mathfrak{e}\omega A + \mathfrak{e}\,\mathrm{d}A = \mathfrak{e}(\omega A + \mathrm{d}A).$$

So we get that $\mathfrak{e}(A\widetilde{\omega}) = \mathfrak{e}(\omega A + \mathrm{d}A)$, and linear independence of $\mathfrak{e}$ implies the relation $A\widetilde{\omega} = \omega A + \mathrm{d}A$. Thus $\widetilde{\omega} = A^{-1}\omega A + A^{-1}\,\mathrm{d}A$.

(ii) Similarly, we have that $R^\nabla\widetilde{\mathfrak{e}} = \widetilde{\mathfrak{e}}\widetilde{\Omega} = \mathfrak{e}A\widetilde{\Omega}$, and also the linearity of R^∇ gives that $R^\nabla\widetilde{\mathfrak{e}} = R^\nabla(\mathfrak{e}A) = (R^\nabla\mathfrak{e})A = \mathfrak{e}\Omega A$. So, linear independence of $\mathfrak{e}$ gives $A\widetilde{\Omega} = \Omega A$, and so $\widetilde{\Omega} = A^{-1}\Omega A$.

□

2 Principal bundles – basic definitions

We want to repeat what was done in the last section but in a "different category". Instead of assigning to each point of M a vector space, we will assign a Lie group. With Lie groups, we have several related concepts such as actions, Lie algebras, etc.. We want to take those things into account in a suitable definition of "principal G-bundle".

Definition 22. A *principal G-bundle* is a triple (P, π, M), where:

(i) P and M are differentiable manifolds;

(ii) $\pi\colon P \to M$ is smooth;

(iii) we have a smooth right action $P \circlearrowleft G$ which preserves the fibers $P_x \doteq \pi^{-1}(x)$ and is free[6] and transitive[7] on all of them, that is, for all $x \in M$ and $p \in P_x$, the orbit map $G \ni g \mapsto p \cdot g \in P_x$ is a bijection[8].

(iv) for each $x_0 \in M$ there is an open neighborhood $U \subseteq M$ of x_0 and a diffeomorphism $\phi\colon \pi^{-1}[U] \to U \times G$ such that the diagram

$$\begin{array}{ccc} \pi^{-1}[U] & \xrightarrow{\ \phi\ } & U \times G \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & U & \end{array}$$

commutes[9] and ϕ is G-equivariant[10]. That is, $\phi(p \cdot g) = \phi(p) \cdot g$, where the right action $(U \times G) \circlearrowleft G$ is given by $(x, h) \cdot g = (x, hg)$.

Again, we'll say that P is the *total space*, M is the *base manifold*, and π is the *bundle projection*. Note that π is necessarily surjective as each fiber P_x is in bijection with the non-empty set G (it contains the neutral element). Each pair (U, ϕ) here will be called a *principal G-chart*, and a collection $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \Lambda}$ for which $\{U_\alpha\}_{\alpha \in \Lambda}$ is an open cover for M is called a *principal G-atlas* for P. It will also be convenient to denote the action maps by $\mathrm{R}_g\colon P \to P$, $\mathrm{R}_g(p) = p \cdot g$.

Before we move on to examples, there is one technicality regarding the above definition which will be convenient to clear up now:

Proposition 23. *Given a triple (P, π, M) satisfying items (i) and (ii) in Definition 22 above, and a Lie group action $P \circlearrowleft G$ which satisfies item (iv), then it automatically satisfies item (iii) as well and thus (P, π, M) is a principal G-bundle.*

[6]Each orbit map is injective. This is equivalent to the stabilizers of all points being trivial.

[7]Each orbit map is surjective. This is equivalent to the action having a single orbit.

[8]Then the Orbit-Stabilizer Theorem implies that $P_x \cong G$, for all $x \in M$. The isomorphisms will vary from point to point and *a priori* we have no control over that.

[9]This implies that $\phi|_{P_x}\colon P_x \to \{x\} \times G$. So footnote number 5 above is thus, in some sense, retconned: principal G-charts gather a bunch of isomorphisms $P_x \cong G$ in a single map, for x ranging over an open subset of M.

[10]If X and Y are (for concreteness, left) G-sets, a map $f\colon X \to Y$ is G-equivariant if $f(g \cdot x) = g \cdot f(x)$ for all $x \in X$. So G-equivariant maps are the morphisms in the category of G-sets. Copy and paste for right actions.

Proof: The action is fiber-preserving because of the commutativity of the diagrams provided by (iv). Now, let $p \in P$ be any point and set $x = \pi(p)$. We want to show that the orbit map $\mathcal{O}_p \colon G \to P_x$ given by $\mathcal{O}_p(g) = p \cdot g$ is bijective. Take a principal G-chart (U, ϕ) around x and write $\phi = (\pi, \phi_2)$.

- First we show that $\mathrm{Stab}(p)$ is trivial by assuming that $p \cdot g = p$ and showing that $g = e$: apply $\mathrm{pr}_2 \circ \phi$ on both sides to get $\phi_2(p \cdot g) = \phi_2(p)$. Since we have that ϕ is G-equivariant, so is ϕ_2 and the previous relation is rewritten as $\phi_2(p)g = \phi_2(p)$. But this last equality holds in G and, in a group, we have a cancellation law. Thus we conclude that $g = e$, and this means that $\mathcal{O}_p$ is injective.

- Now we show that $\mathcal{O}_p$ is surjective, by assuming that p' is another point in P_x, and exhibiting an element in G which moves p to p'. If we want $p \cdot g = p'$, morally g should be the "ratio" p'/p. This is achieved formally by using ϕ as follows:

$$\begin{aligned}
\phi\left(p \cdot \phi_2(p)^{-1}\phi_2(p')\right) &= \left(x, \phi_2(p \cdot \phi_2(p)^{-1}\phi_2(p'))\right) \\
&\stackrel{(*)}{=} \left(x, \phi_2(p)\phi_2(p)^{-1}\phi_2(p')\right) \\
&= (x, \phi_2(p')) \\
&= \phi(p'),
\end{aligned}$$

  where in $(*)$ we used G-equivariance of ϕ. But ϕ is a diffeomorphism, so we obtain that $p \cdot \phi_2(p)^{-1}\phi_2(p') = p'$, as wanted.

$\square$

Remark. In other words, the above proposition says that condition (iii) was essentially superfluous in Definition 22. We have included it anyway for pedagogical reasons and the geometric appeal.

Now we can move on to some examples.

Example 24.

(1) $\pi \colon M \times G \to M$ is called the *trivial principal G-bundle* over M. Each fiber is just $\{x\} \times G$, and the action $(M \times G) \circlearrowleft G$ is given by $(x, h) \cdot g = (x, hg)$. Clearly such action preserves the fibers and act freely and transitively on then. Moreover, we have a global principal G-chart (M, ϕ), where $\phi \colon M \times G \to M \times G$ is the identity map.

(2) Example (1) is a particular case of a more general situation: assume that $N \circlearrowleft G$ is a free and transitive action on a second manifold N. By the Orbit-Stabilizer Theorem, we have that G and N are G-equivariantly diffeomorphic[11], say, via $\theta \colon N \to G$. So

[11] Let's recall the proof in the context of sets and left actions (nothing really changes). Assume $G \circlearrowright X$ and fix $x_0 \in X$. Look at the map $f \colon G \to \mathrm{Orb}(x_0)$ given by $f(g) = g \cdot x_0$. This is surjective by definition of orbit, and if we say that $g \sim h$ if and only if $h^{-1}g \in \mathrm{Stab}(x_0)$, then $g \sim h$ if and only if $f(g) = f(h)$. So f passes to the quotient $f \colon G/_\sim = G/\mathrm{Stab}(x_0) \to \mathrm{Orb}(x_0)$ as a bijection, given by $\widetilde{f}(g\,\mathrm{Stab}(x_0)) = g \cdot x_0$. But G also acts on $G/\mathrm{Stab}(x_0)$ by the left in the obvious way, and $\widetilde{f}$ becomes G-equivariant.

consider $\pi\colon M \times N \to M$, where $(M \times N) \circlearrowleft G$ is defined by $(x,y)\cdot g \doteq (x, y\cdot g)$. Then this new action is free and transitive on each fiber $\{x\} \times N$, and we have one global principal G-chart (M, ϕ), where $\phi\colon M \times N \to M \times G$ is simply given by $\phi(x,y) = (x, \theta(y))$. We have

$$\begin{aligned}\phi((x,y)\cdot g) &= \phi(x, y\cdot g) = (x, \theta(y\cdot g)) \\ &= (x, \theta(y)\cdot g) = (x, \theta(y))\cdot g \\ &= \phi(x,y)\cdot g.\end{aligned}$$

The previous example is just where $N = G$ and $\theta = \mathrm{Id}_G$.

(3) Another toy model: any Lie group G naturally acts on itself by the right, via right translations. Such action is clearly free and transitive, and so we get a principal G-bundle over a one-point space (i.e., a zero-dimensional manifold): $G \to \{*\}$, with the whole total space G as the single fiber. The global principal G-chart is essentially the identity $G \to \{*\} \times G$.

(4) Let $\pi\colon P \to M$ be a principal G-bundle and $f\colon N \to M$. Let's define a principal G-bundle over N as follows: the total space will be defined as

$$f^*P \doteq \{(y,p) \in N \times P \mid p \in P_{f(y)}\}.$$

To argue that f^*P is a differentiable manifold (more precisely, an embedded submanifold of $N \times P$), one considers the smooth map $F\colon N \times P \to M \times M$ given by $F(y,p) = (f(y), \pi(p))$, verifies that $F \pitchfork \Delta$ and notes that $f^*P = F^{-1}[\Delta]$, where Δ is the diagonal in $M \times M$. Also, we have that[12]

$$T_{(y,p)}(f^*P) = \{(\boldsymbol{v}, \boldsymbol{w}) \in T_yN \times T_pP \mid \mathrm{d}f_y(\boldsymbol{v}) = \mathrm{d}\pi_p(\boldsymbol{w})\}.$$

The smooth projection $\widetilde{\pi}\colon f^*P \to N$ is the obvious one.

Next, the right action $f^*P \circlearrowleft G$ is given by $(y,p)\cdot g \doteq (y, p\cdot g)$. Such action preserves the fibers of f^*P, as the original action preserves the fibers of P (namely, $p \in (f^*P)_y = P_{f(y)}$ and $g \in G$ implies $p\cdot g \in P_{f(y)} = (f^*P)_y$). Moreover, since the orbit maps $G \ni g \mapsto p\cdot g \in P_x$ are bijections for all $p \in P_x$ and all $x \in M$, this is particular remains true for the points of the form $x = f(y)$ for $y \in N$, meaning that the action $f^*P \circlearrowleft G$ is also free and transitive on the fibers of f^*P.

Lastly, we form principal G-charts for f^*P via principal G-charts for P as follows: given (U, ϕ) for P, we consider the open set $f^{-1}[U] \subseteq N$. Then define $\widetilde{\phi}\colon \widetilde{\pi}^{-1}[f^{-1}[U]] \to f^{-1}[U] \times G$ by $\widetilde{\phi}(y,p) \doteq (y, \phi_2(p))$, where ϕ_2 is the second component of ϕ. It is clear that $\widetilde{\phi}$ is G-equivariant. Now, the inverse mapping is $\widetilde{\phi}^{-1}\colon f^{-1}[U] \times G \to \widetilde{\pi}^{-1}[f^{-1}[U]]$ given by $\widetilde{\phi}^{-1}(y,h) = (y, \phi^{-1}(f(y),h))$. Indeed, on one hand we have that

$$\widetilde{\phi}(\widetilde{\phi}^{-1}(y,h)) = \widetilde{\phi}(y, \phi^{-1}(f(y),h)) = (y, \phi_2(\phi^{-1}(f(y),h))) = (y,h),$$

[12] This is a general fact: if $F\colon M \to N$ is a smooth map, $S \subseteq N$ is a submanifold, and $F \pitchfork S$, then $F^{-1}[S]$ is an embedded submanifold of M and $T_x(F^{-1}[S]) = (\mathrm{d}F_x)^{-1}[T_{F(x)}S]$. The fast way to remember is to "differentiate" both sides of the equation $F(x) \in S$ with respect to x and evaluate at $\boldsymbol{v}$ to get $\mathrm{d}F_x(\boldsymbol{v}) \in T_{F(x)}S$.

since for all $x \in U$ and $h \in G$ (in particular when $x = f(y)$ with $y \in f^{-1}[U]$) we have that

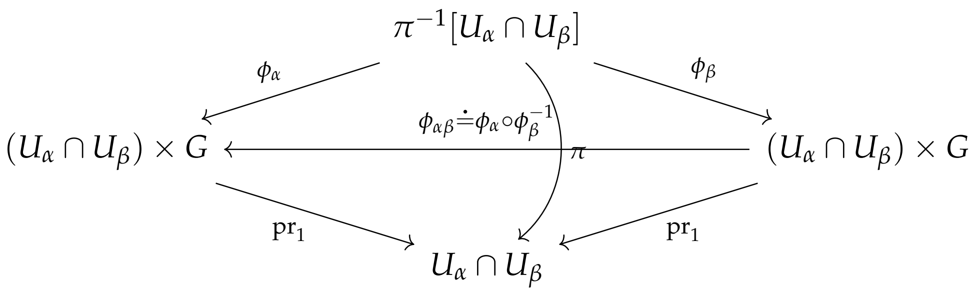
$$(x,h) = \phi(\phi^{-1}(x,h)) = (x, \phi_2(\phi^{-1}(x,h))) \implies \phi_2(\phi^{-1}(x,h)) = h.$$

On the other hand, we have

$$\widetilde{\phi}^{-1}(\widetilde{\phi}(y,p)) = \widetilde{\phi}^{-1}(y, \phi_2(p)) = (y, \phi^{-1}(\phi_2(p))) = (y,p),$$

since for all $x \in U$ and $p \in P_x$ (in particular when $x = f(y)$ with $y \in f^{-1}[U]$) we have $p = \phi^{-1}(\phi(x,p)) = \phi^{-1}(x, \phi_2(p))$.

Let's mimic what we have done before to study the transition maps of a VB-atlas. So let $\pi \colon P \to M$ be a principal G-bundle with a principal G-atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \Lambda}$. Since the diagram



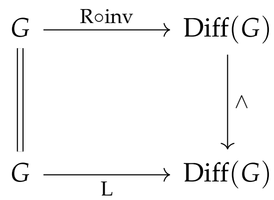
commutes, we know that $\phi_{\alpha\beta} \colon (U_\alpha \cap U_\beta) \times G \to (U_\alpha \cap U_\beta) \times G$ is of the form[13] $\phi_{\alpha\beta}(x,g) = (x, \tau_{\alpha\beta}(x)(g))$, where $\tau_{\alpha\beta} \colon U_\alpha \cap U_\beta \to G \subseteq \mathrm{Diff}(G)$ is smooth. Taking a third principal G-chart (U_γ, ϕ_γ) satisfying that $U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing$, we again obtain for each x that:

(i) $\tau_{\alpha\alpha}(x) = \mathrm{Id}_G$;

(ii) $\tau_{\beta\alpha}(x) = \tau_{\alpha\beta}(x)^{-1}$;

(iii) $\tau_{\alpha\gamma}(x) = \tau_{\alpha\beta}(x) \circ \tau_{\beta\gamma}(x)$.

[13] Again, say that the second component of $\phi_{\alpha\beta}(x,g)$ is some $f_{\alpha\beta}(x,g)$. But since fixed x, this is an equivariant map in the variable g, we define $\tau_{\alpha\beta}(x)(g) = f_{\alpha\beta}(x,g) = f_{\alpha\beta}(x,e)g$. This also justifies why $\tau_{\alpha\beta}$ takes values in $G \subseteq \mathrm{Diff}(G)$, as $\tau_{\alpha\beta}(x) \colon G \to G$ consists of left multiplication by the element $f_{\alpha\beta}(x,e)$. More precisely, there are two natural ways to inject $G \hookrightarrow \mathrm{Diff}(G)$. Either by $g \mapsto \mathrm{L}_g$ or $g \mapsto \mathrm{R}_{g^{-1}}$ (observe that $g \mapsto \mathrm{R}_g$ is *not* a homomorphism). To relate those two injections, we define a "hat" map as follows: given $\varphi \in \mathrm{Diff}(G)$, define $\hat{\varphi} \in \mathrm{Diff}(G)$ by the formula $\hat{\varphi}(a) = \varphi(a^{-1})^{-1}$. Then the diagram



commutes. Note that the "hat" map fixes the subgroup $\mathrm{Aut}(G) \subseteq \mathrm{Diff}(G)$. Of course this all works in the pure category Grp, with $\mathrm{Diff}(G)$ replaced by the group $\mathrm{Sym}(G)$ of all bijections $G \to G$.

Just like for vector bundles, those transition functions are enough to "reconstruct" the bundle P.

Theorem 25. *If $\{U_\alpha\}_{\alpha\in\Lambda}$ is an open cover for M and $\{\tau_{\alpha\beta}\colon U_\alpha\cap U_\beta\to G\subseteq \mathrm{Diff}(G)\}_{\alpha,\beta\in\Lambda}$ is a collection of smooth maps satisfying (i), (ii) and (iii) above, whenever they make sense, then*

$$P=\left(\bigsqcup_{\alpha\in\Lambda} U_\alpha\times G\right)/\sim$$

is a principal G-bundle, where $\sim$ is defined by $(\alpha,x,g)\sim(\beta,y,h)$ if $x=y$ and $g=\tau_{\alpha\beta}(x)h$.

We'd like some concrete relation between vector bundles and principal bundles. For example, if V is a vector space, one could think that item (2) of the above list of examples works to define a principal $\mathrm{GL}(V)$-bundle $M\times V\to V$. It does not, because the stabilizer of a given non-zero vector can be huge, and we have two orbits: $\{0\}$ and $V\setminus\{0\}$. This will be remediated with the construction of *frame bundles*:

Example 26 (Frame bundles)**.**

- Start considering a real vector space V, with dimension $\dim V=q<+\infty$. Let $\mathrm{Fr}(V)$ denote the collection of all ordered bases of V. So a basis of V is seen as a point $\mathfrak{v}\in\mathrm{Fr}(V)$. Write $\mathfrak{v}=[\boldsymbol{v}_1\cdots\boldsymbol{v}_q]$ as a row vector, whose entries are again vectors. Given a matrix $A=(a^i{}_j)_{i,j=1}^q\in\mathrm{GL}(q,\mathbb{R})$, we may define a new basis for V by setting, for each $j=1,\ldots,q$, the combination $\widetilde{\boldsymbol{v}}_j=\sum_{i=1}^q a^i{}_j\boldsymbol{v}_i$. So the relation between $\mathfrak{v}$ and $\widetilde{\mathfrak{v}}$ is just $\widetilde{\mathfrak{v}}=\mathfrak{v}A$. Which is to say, $\mathrm{GL}(q,\mathbb{R})$ acts by the right on $\mathrm{Fr}(V)$.

  Such action is free (by linear independence of bases) and transitive (since bases span V), and so the Orbit-Stabilizer Theorem gives a bijection $\mathrm{GL}(q,\mathbb{R})\cong\mathrm{Fr}(V)$, which we'll use to turn $\mathrm{Fr}(V)$ into a differentiable manifold, also making the chosen bijection a diffeomorphism[14]. We then call $\mathrm{Fr}(V)$ the *frame manifold* of V.

- Let $\pi\colon E\to M$ be a real vector bundle, with fiber dimension q. For every fiber E_x, we may consider the frame manifold $\mathrm{Fr}(E_x)$. Now put all of them together to form the *frame bundle* of E:

  $$\mathrm{Fr}(E)\doteq\bigsqcup_{x\in M}\mathrm{Fr}(E_x).$$

  There's an obvious projection $\pi_{\mathrm{fr}}\colon\mathrm{Fr}(E)\to M$, and from the above point we have a right action $\mathrm{Fr}(E)\circlearrowleft\mathrm{GL}(q,\mathbb{R})$ which is free and transitive on fibers. As a matter of fact, frame bundles are the prototypes of principal bundles, and we can see this construction as a justification for using right actions instead of left actions.

[14]Rigorously, we have a bijection $\phi_{\mathfrak{v}}\colon\mathrm{GL}(q,\mathbb{R})\to\mathrm{Fr}(V)$, for each $\mathfrak{v}\in\mathrm{Fr}(V)$. They're given by $\phi_{\mathfrak{v}}(B)=\mathfrak{v}B$. We have that $\phi_{\mathfrak{v}A}=\phi_{\mathfrak{v}}\circ\mathrm{L}_A$, where $\mathrm{L}_A\colon\mathrm{GL}(q,\mathbb{R})\to\mathrm{GL}(q,\mathbb{R})$ denotes the left translation by A. Since L_A is a diffeomorphism, we conclude that the manifold structures of $\mathrm{Fr}(V)$ defined by $\phi_{\mathfrak{v}}$ and $\phi_{\mathfrak{v}A}$ coincide.

If we define smoothly compatible "formal" principal $\mathrm{GL}(q,\mathbb{R})$-charts, the manifold structure on $\mathrm{Fr}(E)$ comes for free[15]. So, fix the standard basis $(\boldsymbol{e}_a)$ for $\mathbb{R}^q$ and start with a VB-chart (U,ϕ) for E. We'll define a principal $\mathrm{GL}(q,\mathbb{R})$-chart (U,ϕ_{fr}) for $\mathrm{Fr}(E)$ by $\phi_{\mathrm{fr}}(\mathfrak{v}) = (x,A)$, where $x = \pi_{\mathrm{fr}}(\mathfrak{v})$ and $A = (a^i{}_j)_{i,j=1}^q$ is the non-singular matrix characterized by

$$\boldsymbol{v}_j = \sum_{i=1}^q a^i{}_j \phi^{-1}(x,\boldsymbol{e}_i),$$

where $\mathfrak{v} = [\boldsymbol{v}_1 \cdots \boldsymbol{v}_q]$ is a basis for E_x. It is clear that ϕ_{fr} is bijective, so let's check that it is $\mathrm{GL}(q,\mathbb{R})$-equivariant: take a non-singular matrix $B = (b^i{}_j)_{i,j=1}^q$. Then $\mathfrak{v}B = [\boldsymbol{w}_1 \cdots \boldsymbol{w}_q]$, where $\boldsymbol{w}_j = \sum_{k=1}^q b^k{}_j \boldsymbol{v}_k$. One more step gives

$$\boldsymbol{w}_j = \sum_{k=1}^q b^k{}_j \left(\sum_{i=1}^q a^i{}_k \phi^{-1}(x,\boldsymbol{e}_i) \right) = \sum_{i=1}^q \left(\sum_{k=1}^q a^i{}_k b^k{}_j \right) \phi^{-1}(x,\boldsymbol{e}_i),$$

and so $\phi_{\mathrm{fr}}(\mathfrak{v}B) = (x,AB) = (x,A)\cdot B = \phi_{\mathrm{fr}}(\mathfrak{v})\cdot B$, as wanted[16]. That said, the only thing left to check is the smooth compatibility between charts for $\mathrm{Fr}(E)$ constructed in such a way. So, assume we're given VB-charts (U_α,ϕ_α) and (U_β,ϕ_β) for E with $U_\alpha \cap U_\beta \neq \varnothing$. For simplicity, denote $e_{\alpha,i}(x) = \phi_\alpha^{-1}(x,\boldsymbol{e}_i)$, and similarly for $e_{\beta,i}$'s. Note that if we write $\phi_{\alpha\beta}(x,\boldsymbol{v}) = (x,\tau_{\alpha\beta}(x)(\boldsymbol{v}))$, then

$$e_{\beta,j}(x) = \sum_{i=1}^q \tau_{\alpha\beta}(x)^i{}_j e_{\alpha,i}(x).$$

With this, let's compute the transition $\phi_{\alpha\beta,\mathrm{fr}} \doteq \phi_{\alpha,\mathrm{fr}} \circ \phi_{\beta,\mathrm{fr}}^{-1}$ as follows: given a pair $(x,A) \in (U_\alpha \cap U_\beta) \times \mathrm{GL}(q,\mathbb{R})$, $\phi_{\beta,\mathrm{fr}}^{-1}(x,A) = \mathfrak{v} = [\boldsymbol{v}_1 \cdots \boldsymbol{v}_q]$, where for all j we have $\boldsymbol{v}_j = \sum_{k=1}^q a^k{}_j e_{\beta,k}(x)$. To apply $\phi_{\alpha,\mathrm{fr}}$, we need to write those vectors as combinations of the $e_{\alpha,i}$'s, and that's where the original transitions $\tau_{\alpha\beta}$ come in. Since

$$\boldsymbol{v}_j = \sum_{k=1}^q a^k{}_j \left(\sum_{i=1}^q \tau_{\alpha\beta}(x)^i{}_k e_{\alpha,i}(x) \right) = \sum_{i=1}^q \left(\sum_{k=1}^q \tau_{\alpha\beta}(x)^i{}_k a^k{}_j \right) e_{\alpha,i}(x),$$

we get that $\phi_{\alpha\beta,\mathrm{fr}}(x,A) = (x,\tau_{\alpha\beta}(x)A)$, which is smooth. Thus, $\pi_{\mathrm{fr}}\colon \mathrm{Fr}(E) \to M$ is indeed a principal $\mathrm{GL}(q,\mathbb{R})$-bundle. Also note that the transitions between these principal charts for $\mathrm{Fr}(E)$ indeed take values in the subgroup $\mathrm{GL}(q,\mathbb{R})$ of $\mathrm{Diff}(\mathrm{GL}(q,\mathbb{R}))$, where the embedding is given by left-multiplication. We will see the "inverse" of this construction in the end of this section.

[15] This is a general fact about fiber bundles, see Theorem 4.3.3 in [2].

[16] A more efficient use of notation is as follows: $\phi_{\mathrm{fr}}(\mathfrak{v}) = (x,A)$, where A is the unique non-singular matrix for which $\mathfrak{v} = \mathfrak{e}A$, where $\mathfrak{e} \in \mathrm{Fr}(E_x)$ is defined via ϕ. Then $\mathfrak{v}B = (\mathfrak{e}A)B = \mathfrak{e}(AB)$ reads precisely as $\phi_{\mathrm{fr}}(\mathfrak{v}B) = \phi_{\mathrm{fr}}(\mathfrak{v})\cdot B$, bypassing all summations and indices.

- If $(V, \langle\cdot,\cdot\rangle)$ is an Euclidean vector space, one may consider the manifold $\mathrm{Fr}_{\mathrm{O}}(V)$ (or $\mathrm{Fr}(V, \langle\cdot,\cdot\rangle)$) of all ordered orthonormal bases of V. And then repeating the previous construction, we get a right action on $\mathrm{Fr}_{\mathrm{O}}(V)$ by $\mathrm{O}(q, \mathbb{R})$. If $E \to M$ is a vector bundle equipped with an given Euclidean fiber metric g, one also obtains a principal $\mathrm{O}(q, \mathbb{R})$-bundle $\mathrm{Fr}_{\mathrm{O}}(E) \to M$. This all works when the products considered are no longer positive-definite, once an agreement is made on, for example, placing all timelike fiber elements last. The structure group is then the pseudo-orthogonal group $\mathrm{O}_\nu(q, \mathbb{R})$, where ν is the index of the scalar product or fiber metric.
- If (V, Ω) is a symplectic vector space, we repeat the construction to get the manifold $\mathrm{Fr}_{\mathrm{Sp}}(V)$ (or $\mathrm{Fr}(V, \Omega)$) of all Darboux bases for V. The symplectic group $\mathrm{Sp}(2q)$ acts by the right on $\mathrm{Fr}_{\mathrm{Sp}}(V)$. In the vector bundle level, we assume that $E \to M$ is equipped with a symplectic structure ω (i.e., a non-degenerate section of $E^* \wedge E^*$), and we obtain a principal $\mathrm{Sp}(2q)$-bundle $\mathrm{Fr}_{\mathrm{Sp}}(E) \to M$.

Definition 27. Let $\pi\colon P \to M$ be a principal G-bundle.

(i) A *section* of P is a mapping $\psi\colon M \to P$ such that $\pi \circ \psi = \mathrm{Id}_M$. We'll also say that ψ is a *gauge* for P. Similarly we talk about local sections (local gauges).

(ii) If $\pi'\colon P' \to M$ is another principal G-bundle, a *gauge transformation* $\Phi\colon P \to P'$ is a G-equivariant bundle isomorphism.

Remark. We will denote the space of gauge transformations of a principal G-bundle $P \to M$ by $\mathscr{G}(P)$.

Example 28.

(1) Gauges of the trivial G-bundle $M \times G \to M$ are clearly identified with smooth functions $M \to G$. Also, gauge transformations of $M \times G$ are also identified with smooth functions $M \to G$, but this requires some argument (actually very similar to when we described the transition maps for a general principal atlas): assume that $\Phi\colon M \times G \to M \times G$ is of the form $\Phi(x, g) = (x, \varphi(x, g))$, for some smooth map $\varphi\colon M \times G \to G$ (here we already used that Φ is fiber-preserving). Now the equivariance condition $\Phi(x, gh) = \Phi(x, g)h$ reads $(x, \varphi(x, gh)) = (x, \varphi(x, g)h)$. This means that we may identify Φ with the smooth map $f\colon M \to G$ given by $f(x) = \varphi(x, e)$, so that $\varphi(x, g) = \varphi(x, eg) = \varphi(x, e)g = f(x)g$. In particular, it also follows from that argument that a fiber-preserving G-equivariant smooth map is automatically a diffeomorphism (and hence a gauge transformation), as $\Phi^{-1}\colon M \times G \to M \times G$ can be given explicitly as $\Phi^{-1}(x, g) = (x, f(x)^{-1}g)$.

(2) Consider the "toy" bundle $G \to \{*\}$ over a one-point space. Then gauges are constant functions, meaning that the space of gauges is identified with G itself. In the same fashion, $\mathscr{G}(G) \cong G$, since every gauge transformation $\Phi\colon G \to G$ actually is given by $\Phi = \mathrm{L}_{\Phi(e)}$, in view of the G-equivariance condition.

(3) Local frames for a vector bundle $E \to M$ are the same thing as local gauges for the frame bundle $\mathrm{Fr}(E)$. Now, let's understand what a gauge transformation

$\Phi\colon \mathrm{Fr}(E) \to \mathrm{Fr}(E)$ is. Given $\mathfrak{v} \in \mathrm{Fr}(E_x)$, we have that $\Phi(\mathfrak{v}) \in \mathrm{Fr}(E_x)$ as well. So there is a matrix relating both bases, call it $A_\Phi(\mathfrak{v}) \in \mathrm{GL}(q,\mathbb{R})$. In other words, we have $\Phi(\mathfrak{v}) = \mathfrak{v}A_\Phi(\mathfrak{v})$. Smoothness of the gauge transformation Φ means that the matrix map $A_\Phi\colon \mathrm{Fr}(E) \to \mathrm{GL}(q,\mathbb{R})$ is also smooth. Now, the equivariance condition $\Phi(\mathfrak{v}B) = \Phi(\mathfrak{v})B$ reads as $\mathfrak{v}BA_\Phi(\mathfrak{v}B) = \mathfrak{v}A_\Phi(\mathfrak{v})B$, and linear independence of frames implies that $A_\Phi(\mathfrak{v}B) = B^{-1}A_\Phi(\mathfrak{v})B$. Now, note that $\mathrm{GL}(q,\mathbb{R})$ acts on itself on the right by conjugation (where the inverse comes in the left). This means that gauge transformations of $\mathrm{Fr}(E)$ are the same thing as $\mathrm{GL}(q,\mathbb{R})$-equivariant smooth maps $\mathrm{Fr}(E) \to \mathrm{GL}(q,\mathbb{R})$. This is a particular instance of a more general phenomenon, to be described in Proposition 29.

(4) Local orthonormal frames for a vector bundle $E \to M$ equipped with an Euclidean fiber metric are the same thing as local gauges for the orthonormal frame bundle $\mathrm{Fr}_{\mathrm{O}}(E)$. As before, gauge transformations of $\mathrm{Fr}_{\mathrm{O}}(E)$ are the same thing as $\mathrm{O}(q,\mathbb{R})$-equivariant smooth maps $\mathrm{Fr}_{\mathrm{O}}(E) \to \mathrm{O}(q,\mathbb{R})$.

(5) Local Darboux frames for a vector bundle $E \to M$ equipped with a fiber symplectic structure are the same thing as local gauges for the symplectic frame bundle $\mathrm{Fr}_{\mathrm{Sp}}(E)$. As before, gauge transformations of $\mathrm{Fr}_{\mathrm{Sp}}(E)$ are the same thing as $\mathrm{Sp}(2q,\mathbb{R})$-equivariant smooth maps $\mathrm{Fr}_{\mathrm{Sp}}(E) \to \mathrm{Sp}(2q,\mathbb{R})$.

Proposition 29. *Let $P \to M$ be a principal G-bundle. Let*

$$\mathscr{C}^\infty(P,G)^G \doteq \{f \in \mathscr{C}^\infty(P,G) \mid f(p\cdot g) = gf(p)g^{-1} \text{ for all } p \in P, g \in G\}.$$

Then the map $\sigma\colon \mathscr{G}(P) \to \mathscr{C}^\infty(P,G)^G$ characterized by $\Phi(p) = p\cdot\sigma_\Phi(p)$, for all $p \in P$, is a well-defined group isomorphism. Here, the group structure on $\mathscr{C}^\infty(P,G)^G$ is given by pointwise multiplication, and the neutral element is the constant function $e \in G$.

Remark.

- Note that the condition $f(p\cdot g) = gf(p)g^{-1}$ is equivalent to $f \circ \mathrm{R}_g = \mathrm{C}_{g^{-1}} \circ f$ where, for any $g \in G$, the conjugation $\mathrm{C}_g\colon G \to G$ is defined by $\mathrm{C}_g(a) = gag^{-1}$. In other words, the upper G in $\mathscr{C}^\infty(P,G)^G$ again stands for G-equivariance.
- In item (1) of the previous example, for the trivial G-bundle $M \times G \to M$ we indeed have that $\mathscr{C}^\infty(M\times G,G)^G \cong \mathscr{C}^\infty(M,G)$, where the isomorphism is the evaluation $f \mapsto f(\cdot,e)$.

Proof: Let's work in three steps:

- That σ_Φ is well-defined follows from the action of G on P being free and transitive on fibers. To see that σ_Φ is smooth, take a principal G-chart (U,ϕ) for P. If we have $\phi(p) = (x,\phi_2(p))$, where $p \in P_x$ and $\phi_2\colon \pi^{-1}[U] \to G$ is smooth and G-equivariant, applying ϕ_2 to the relation $\Phi(p) = p\cdot\sigma_\Phi(p)$ gives that

$$\phi_2(\Phi(p)) = \phi_2(p\cdot\sigma_\Phi(p)) = \phi_2(p)\sigma_\Phi(p),$$

so that $\sigma_\Phi(p) = \phi_2(p)^{-1}\phi_2(\Phi(p))$ and hence σ_Φ is smooth on $\pi^{-1}[U]$. Since the principal G-chart was arbitrary, σ_Φ is globally smooth. Now from the equality $\Phi(p\cdot g) = \Phi(p)\cdot g$ we compute

$$(p\cdot g)\cdot\sigma_\Phi(p\cdot g) = (p\cdot\sigma_\Phi(p))\cdot g \implies p\cdot(g\sigma_\Phi(p\cdot g)) = p\cdot(\sigma_\Phi(p)g),$$

and from freeness of the action we get that $g\sigma_\Phi(p\cdot g) = \sigma_\Phi(p)g$. It follows that $\sigma_\Phi(p\cdot g) = g^{-1}\sigma_\Phi(p)g$, so that indeed $\sigma_\Phi \in \mathscr{C}^\infty(P,G)^G$.

- The next step is checking that σ is a group homomorphism. So let $\Phi_1,\Phi_2 \in \mathscr{G}(P)$. We have that

$$\begin{aligned}(\Phi_1\circ\Phi_2)(p) &= \Phi_1(\Phi_2(p)) = \Phi_2(p)\cdot\sigma_{\Phi_1}(\Phi_2(p)) \\ &= (p\cdot\sigma_{\Phi_2}(p))\cdot\sigma_{\Phi_1}(p\cdot\sigma_{\Phi_2}(p)) = p\cdot(\sigma_{\Phi_2}(p)\sigma_{\Phi_2}(p)^{-1}\sigma_{\Phi_1}(p)\sigma_{\Phi_2}(p)) \\ &= p\cdot(\sigma_{\Phi_1}(p)\sigma_{\Phi_2}(p)) = p\cdot((\sigma_{\Phi_1}\cdot\sigma_{\Phi_2})(p)),\end{aligned}$$

so that freeness of the action and arbitrariety of p gives that $\sigma_{\Phi_1\circ\Phi_2} = \sigma_{\Phi_1}\cdot\sigma_{\Phi_2}$, as wanted.

- Finally, let's exhibit the inverse for σ. Given $f \in \mathscr{C}^\infty(P,G)^G$, define a map $\Phi^f\colon P\to P$ by $\Phi^f(p) \doteq p\cdot f(p)$. Clearly Φ^f is smooth and fiber-preserving. It is G-equivariant since

$$\begin{aligned}\Phi^f(p\cdot g) &= (p\cdot g)\cdot f(p\cdot g) = p\cdot(gg^{-1}f(p)g) \\ &= p\cdot(f(p)g) = (p\cdot f(p))\cdot g \\ &= \Phi^f(p)\cdot g.\end{aligned}$$

And it is a gauge transformation since its inverse automorphism is explictly given by $(\Phi^f)^{-1}(p) = p\cdot f(p)^{-1}$. To wit, we have that

$$\begin{aligned}(\Phi^f\circ(\Phi^f)^{-1})(p) &= \Phi^f(p\cdot f(p)^{-1}) = (p\cdot f(p)^{-1})\cdot f(p\cdot f(p)^{-1}) \\ &= p\cdot(f(p)^{-1}f(p)f(p)f(p)^{-1}) = p,\end{aligned}$$

and $(\Phi^f)^{-1}\circ\Phi = \mathrm{Id}_P$ follows from the exact same computation repeated switching the roles of f and its pointwise inverse. We will conclude the proof by noting that since $\Phi^f(p) = p\cdot f(p)$, the definition of σ says that $\sigma_{\Phi^f} = f$, while on the other hand we have that $\Phi^{\sigma_\Phi}(p) = p\cdot\sigma_\Phi(p) = \Phi(p)$.

□

We proceed to describe one of the most striking differences between vector bundles and principal bundles: while vector bundles always admit global sections (at least the zero section), this is no longer true for principal bundles. The situation is much more extreme:

Proposition 30. *Let $\pi\colon P\to M$ be a principal G-bundle. Then P admits a global gauge if and only if P is G-equivariantly diffeomorphic to the trivial G-bundle $M\times G\to M$.*

Proof: We start with the easy direction. If $\Psi\colon P \to M \times G$ is a G-equivariant diffeomorphism, define $\psi\colon M \to P$ by $\psi(x) \doteq \Psi^{-1}(x, e)$. To see that ψ is a gauge for P, first note that Ψ has necessarily the form $\Psi(p) = (\pi(p), \Psi_2(p))$ for some smooth $\Psi_2\colon P \to G$. Then for all $(x, g) \in M \times G$ we have that

$$(x, g) = \Psi\Psi^{-1}(x, g) = (\pi(\Psi^{-1}(x, g)), \Psi_2(\Psi^{-1}(x, g))) \implies \pi(\Psi^{-1}(x, g)) = x.$$

In particular, we get that $\pi(\psi(x)) = \pi(\Psi^{-1}(x, e)) = x$, as wanted.

Conversely, assume now that we have a global gauge $\psi\colon M \to P$. Let's define a G-equivariant diffeomorphism between P and $M \times G$. We will exhibit both the map and its inverse. First we define $\zeta\colon M \times G \to P$ by $\zeta(x, g) = \psi(x)g$. Indeed, we have that ζ is G-equivariant, as

$$\zeta((x, g) \cdot h) = \zeta(x, gh) = \psi(x) \cdot (gh) = (\psi(x) \cdot g) \cdot h = \zeta(x, g) \cdot h.$$

To define the inverse Ψ of ζ, we use that the action $P \circlearrowleft G$ is free and transitive on fibers. Since $p, \psi(\pi(p)) \in P_{\pi(p)}$ for all $p \in P$, there is a unique $g_p \in G$ such that $\psi(\pi(p)) = p \cdot g_p$. By a similar argument to the one done in the proof of Proposition 29, we have that g_p depends smoothly on p. In particular, when we already have $p = \psi(x)$ for some $x \in M$, we get that $g_{\psi(x)} = e$. Also, if $g \in G$ is arbitrary, we have that

$$p \cdot g_p = \psi(\pi(p)) = \psi(\pi(p \cdot g)) = (p \cdot g) \cdot g_{p \cdot g} = p \cdot (g g_{p \cdot g}),$$

and freeness of the action on fibers gives that $g_{p \cdot g} = g^{-1} g_p$. With this, define now $\Psi\colon P \to M \times G$ by $\Psi(p) = (\pi(p), g_p^{-1})$. The inverse on g_p is needed to ensure that Ψ is G-equivariant, as the following computation shows:

$$\begin{aligned}
\Psi(p \cdot g) &= (\pi(p \cdot g), g_{p \cdot g}^{-1}) = (\pi(p), (g^{-1} g_p)^{-1}) \\
&= (\pi(p), g_p^{-1} g) = (\pi(p), g_p^{-1}) g \\
&= \Psi(p) g.
\end{aligned}$$

Lastly, we check that Ψ and ζ are indeed inverses. On one hand we have that

$$\zeta(\Psi(p)) = \zeta(\pi(p), g_p^{-1}) = \psi(\pi(p)) \cdot g_p^{-1} = p,$$

and on the other hand that

$$\Psi(\zeta(x, g)) = \Psi(\psi(x) \cdot g) = (\pi(\psi(x)), g_{\psi(x) \cdot g}^{-1}) = (x, (g^{-1} g_{\psi(x)})^{-1}) = (x, g),$$

recalling that $g_{\psi(x)} = e$. We are done. □

Remark. It might be worth noting that even if $E \to M$ is just a *fiber bundle* (i.e., the projection is a surjective submersion, all the fibers E_x are diffeomorphic to a fixed *typical fiber* F and there are local trivializations – with no extra structure), $\Gamma(E) = \varnothing$ is a perfectly possible situation. For instance, consider the slit tangent bundle of $\mathbb{S}^2$, $T^\circ \mathbb{S}^2 \to \mathbb{S}^2$, where $T^\circ \mathbb{S}^2 = T\mathbb{S}^2 \setminus 0_{T\mathbb{S}^2}$. Then $\Gamma(T^\circ \mathbb{S}^2) = \varnothing$ in view of the hairy ball theorem.

Example 31.

(1) Let $\pi\colon P \to M$ be a principal G-bundle. We can pull this back to P itself via π, obtaining another principal G-bundle $\pi^*P \to P$. Observe that in this case the total space is given by $\pi^*P = \{(p_1, p_2) \in P \times P \mid \pi(p_1) = \pi(p_2)\}$. That being understood, we have an obvious global gauge $\psi\colon P \to \pi^*P$ given by $\psi(p) = (p, p)$. Thus $\pi^*P \to P$ is trivial.

(2) If M is any smooth manifold, when is $\mathrm{Fr}(TM)$ trivial? This happens precisely when M is *parallelizable*.

The idea given in the previous proof allows us to establish the version of "VB-charts are equivalent to local frames" in the setting of principal G-bundles:

Theorem 32. *Let $\pi\colon P \to M$ be a surjective submersion and $P \circlearrowleft G$ a smooth right action which is free and transitive on the fibers of π. Then $\pi\colon P \to M$ is a principal G-bundle if and only if there is an open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of M together with local gauges $\psi_\alpha\colon U_\alpha \to P$.*

The proof consists mainly in applying the previous result for every principal chart domain, i.e., to the restrictions $P_U = \pi^{-1}[U] \to U$. With this in hand, we get a few more examples:

Example 33.

(1) If P is a manifold and we have a free right action $P \circlearrowleft G$ for which the map

$$P \times G \ni (p, g) \mapsto (p, p \cdot g) \in P \times P$$

is closed (such actions are called *principal*, for good reason), then P/G is a smooth manifold[17], and the canonical projection $P \to P/G$ is a surjective submersion. The action is automatically principal and we get this conclusion, for example, if G is compact. By the rank theorem, surjective submersions always admit local sections. Hence $P \to P/G$ is a principal G-bundle.

(2) As a particular case of the previous item, if G is a Lie group and $H \leq G$ is a closed subgroup (hence a Lie subgroup), then the action $G \circlearrowleft H$ is principal and so the canonical projection $G \to G/H$ defines a principal H-bundle. The fiber over a point $aH \in G/H$ is the coset $aH \subseteq G$ itself.

- When $H = G$, we recover the "toy" bundle over a one-point space.
- A more interesting example arises from considering the Grassmannian $\mathrm{Gr}_k(\mathbb{R}^n)$ of k-dimensional subspaces of $\mathbb{R}^n$. For instance, we have that $\mathrm{O}(n, \mathbb{R})$ acts

[17] The general theorem about the structure of quotient manifolds is:

Theorem (Godement). *Let R be an equivalence relation on a manifold M. Suppose that R is a closed embedded submanifold of $M \times M$ and $\mathrm{pr}_1|_R\colon R \to M$ is a surjective submersion. Then M/R has a unique structure of a smooth manifold such that the canonical projection $M \to M/R$ is a surjective submersion.*

The proof is very technical and can be found in [2].

transitively on $\mathrm{Gr}_k(\mathbb{R}^n)$ via direct images and the stabilizer of a fixed subspace is isomorphic to $\mathrm{O}(k,\mathbb{R}) \times \mathrm{O}(n-k,\mathbb{R})$, so that the orbit-stabilizer theorem gives that

$$\mathrm{Gr}_k(\mathbb{R}^n) \cong \frac{\mathrm{O}(n,\mathbb{R})}{\mathrm{O}(k,\mathbb{R}) \times \mathrm{O}(n-k,\mathbb{R})}.$$

With this, the quotient projection on the right side (since the denominator is a closed subgroup) establishes $\mathrm{O}(n,\mathbb{R})$ as a $(\mathrm{O}(k,\mathbb{R}) \times \mathrm{O}(n-k,\mathbb{R}))$-bundle over the Grassmannian $\mathrm{Gr}_K(\mathbb{R}^n)$.

(3) Consider a sphere $\mathbb{S}^{2n+1} \subseteq \mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$. We have an action $\mathbb{S}^{2n+1} \circlearrowleft \mathbb{S}^1$ given by $z \cdot \lambda \doteq \lambda z$. This action is free, since given any $z \in \mathbb{S}^{2n+1}$ and $\lambda \in \mathbb{S}^1$ with $\lambda z = z$, we multiply both sides by $\overline{z}$ to get $\lambda = 1$. Since $\mathbb{S}^1$ is compact, we automatically get a principal $\mathbb{S}^1$-bundle $\mathbb{S}^{2n+1} \to \mathbb{S}^{2n+1}/\mathbb{S}^1 \cong \mathbb{C}P^n$. This is called the *Hopf fibration*.

Let's proceed. Given a principal G-bundle $P \to M$, we'd like to construct a vector bundle from that. However, we need an extra tool for that. Namely, a representation $\rho\colon G \to \mathrm{GL}(V)$ of G in some vector space V, which will be the typical fiber of the vector bundle to be constructed.

Lemma 34. *Let $P \to M$ be a principal G-bundle and $\rho\colon G \to \mathrm{GL}(V)$ be a representation of G in a vector space V. Then the action*

$$(P \times V) \times G \ni ((p, v), g) \mapsto (p \cdot g, \rho(g^{-1})v) \in P \times V$$

is principal. Thus the quotient $P \times_\rho V \doteq (P \times V)/G$ is a smooth manifold, and the projection $P \times V \to P \times_\rho V$ is a surjective submersion.

Remark. It is also instructive to note the alternative formulation via an equivalence relation: say that $(p, v) \sim (p', v')$ if and only if there is $g \in G$ with $p' = p \cdot g$ and $v' = \rho(g^{-1})v$. We see that the inverse in $\rho(g^{-1})$ is needed when regarding the representation ρ as a left action $V \circlearrowleft G$ instead of a right action. Indeed $\sim$ is reflexive (take $g = e$), symmetric (given g, consider g^{-1}) and transitive (take the product). So $P \times_\rho V = (P \times V)/_\sim$.

Proof: See [2]. □

Proposition 35. *Let $P \to M$ be a principal G-bundle and $\rho\colon G \to \mathrm{GL}(V)$ be a representation of G in a vector space V. Then $P \times_\rho V \to M$ is a vector bundle, whose fibers are isomorphic to V. This is called the* associated vector bundle to P via ρ.

Proof: In view of the previous result, the only thing left to do is to define the vector space operations on the fibers, and find VB-charts for $P \times_\rho V$ using principal G-charts for P. Fixed $x \in M$ and $[p, v], [q, w] \in (P \times_\rho V)_x$, we define the addition by

$$[p, v] + [q, w] \doteq [p, v + \rho(g)w] = [q, \rho(g^{-1})v + w],$$

where $g \in G$ is the unique element such that $q = p \cdot g$, and also $\lambda[p, v] = [p, \lambda v]$, for $\lambda \in \mathbb{R}$. The only thing to check here is that the addition above is well-defined. So assume that $(p', v') \sim (p, v)$ and $(q', w') \sim (q, w)$, and then take $a, b \in G$ such that

$$p' = p \cdot a, \quad v' = \rho(a^{-1})v, \quad q' = q \cdot b \quad \text{and} \quad w' = \rho(b^{-1})w.$$

If $g \in G$ is such that $q = p \cdot g$, then we immediately have $q' = p' \cdot (a^{-1}gb)$. And finally we conclude that $(p, \boldsymbol{v} + \rho(g)\boldsymbol{w}) \sim (p', \boldsymbol{v}' + \rho(a^{-1}gb)\boldsymbol{w}')$, as

$$\rho(a)\big(\boldsymbol{v}' + \rho(a^{-1}gb)\boldsymbol{w}'\big) = \rho(a)\boldsymbol{v}' + \rho(g)\rho(b)\boldsymbol{w}' = \boldsymbol{v} + \rho(g)\boldsymbol{w}.$$

This shows that the addition operation on each fiber of $P \times_\rho V$ is well-defined, as claimed. And fixed $p \in P_x$, $V \ni \boldsymbol{v} \mapsto [p, \boldsymbol{v}] \in (P \times_\rho V)_x$ is an isomorphism.

Now, regarding trivializations, it will be convenient to denote both bundle projections in the discussion by $\pi\colon P \to M$ and $\pi_\rho\colon P \times_\rho V \to M$. Assume that we start with a principal G-chart $\phi\colon \pi^{-1}[U] \to U \times G$ for P, written as $\phi(p) = (\pi(p), \phi_2(p))$. We then define a VB-chart $\phi_\rho\colon \pi_\rho^{-1}[U] \to U \times V$ for $P \times_\rho V$ simply by setting

$$\phi_\rho([p, \boldsymbol{v}]) = (\pi(p), \rho(\phi_2(p))\boldsymbol{v}).$$

The condition that $\phi_2(p \cdot g) = \phi_2(p)g$ is exactly what we need to ensure that ϕ_ρ is well-defined, as $(p \cdot g, \rho(g^{-1})\boldsymbol{v})$ gets sent to

$$(\pi(p \cdot g), \rho(\phi_2(p \cdot g))\rho(g^{-1})\boldsymbol{v}) = (\pi(p \cdot g), \rho(\phi_2(p)gg^{-1})\boldsymbol{v}) = (\pi(p), \rho(\phi_2(p))\boldsymbol{v})$$

as well. Smoothness of ϕ_ρ follows from smoothness of the remaining ingredients defining it. The (smooth) inverse $\phi_\rho^{-1}\colon U \times V \to \pi_\rho^{-1}[U]$ is written, as expected, by using the inverse $\phi^{-1}\colon U \times G \to \pi^{-1}[U]$: we have that $\phi_\rho^{-1}(x, \boldsymbol{v}) = [\phi^{-1}(x, e), \boldsymbol{v}]$. $\square$

Remark. Keeping the notation from the previous proof, it is interesting to see the relation between the transition maps $\tau_{\alpha\beta}$ associated to a principal G-atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \Lambda}$ for P, and the transitions $\tau_{\alpha\beta,\rho}$ associated to the VB-atlas $\{(U_\alpha, \phi_{\alpha,\rho})\}_{\alpha \in \Lambda}$ for $P \times_\rho V$ constructed as above. We just compute that (with suggestive notation):

$$\begin{aligned}\phi_{\alpha\beta,\rho}(x, \boldsymbol{v}) &\doteq \phi_{\alpha,\rho} \circ \phi_{\beta,\rho}^{-1}(x, \boldsymbol{v}) \\ &= \phi_{\alpha,\rho}([\phi_\beta^{-1}(x, e), \boldsymbol{v}]) \\ &= (x, \rho(\phi_{\alpha,2}(\phi_\beta^{-1}(x, e)))\boldsymbol{v}) \\ &= (x, \rho(\tau_{\alpha\beta}(x))\boldsymbol{v}).\end{aligned}$$

This means that $\tau_{\alpha\beta,\rho} = \rho \circ \tau_{\alpha\beta}$.

Example 36.

(1) When $\pi\colon P \to M$ is a principal G-bundle, V is any vector space, and we take $\rho\colon G \to \mathrm{GL}(V)$ to be the trivial representation (i.e., ρ is constant and given by $\rho(g) = \mathrm{Id}_V$ for all $g \in G$), then $P \times_{\mathrm{Id}_V} V$ is a trivial bundle. Indeed, in this case we have $(p, \boldsymbol{v}) \sim (p \cdot g, \boldsymbol{v})$ for all $g \in G$, which says that the smooth map

$$P \times_{\mathrm{Id}_V} V \ni [p, \boldsymbol{v}] \mapsto (\pi(p), \boldsymbol{v}) \in M \times V$$

is a bijection[18].

[18] Smoothness of the inverse is more subtle.

(2) Recall that the conjugation $\mathrm{C}_g\colon G\to G$ is given by $\mathrm{C}_g(a)=gag^{-1}$. The derivative $\mathrm{Ad}(g)=\mathrm{d}(\mathrm{C}_g)_e\colon\mathfrak{g}\to\mathfrak{g}$ defines the *adjoint representation*[19] $\mathrm{Ad}\colon G\to\mathrm{GL}(\mathfrak{g})$. Using that, we may form the associated vector bundle $\mathrm{Ad}(P)\doteq P\times_{\mathrm{Ad}}\mathfrak{g}$.

(3) Start with a vector bundle $E\to M$. From this, we have constructed the frame bundle $\mathrm{Fr}(E)\to M$, which is a principal $\mathrm{GL}(q,\mathbb{R})$-bundle. We have a canonical representation of $\mathrm{GL}(q,\mathbb{R})$ in $\mathbb{R}^q$ itself, namely, the identity $\mathrm{Id}\colon\mathrm{GL}(q,\mathbb{R})\to\mathrm{GL}(\mathbb{R}^q)$. With this, we may form the associated vector bundle $\mathrm{Fr}(E)\times_{\mathrm{Id}}\mathbb{R}^q$. One would expect this to be naturally isomorphic to the original vector bundle itself E. If $\mathfrak{v}=[\boldsymbol{v}_1\cdots\boldsymbol{v}_q]\in\mathrm{Fr}(E)$ and $\boldsymbol{a}=(a^1,\ldots,a^q)\in\mathbb{R}^q$, the isomorphism is given by

$$\mathrm{Fr}(E)\times_{\mathrm{Id}}\mathbb{R}^q\ni[\mathfrak{v},\boldsymbol{a}]\mapsto\sum_{i=1}^q a^i\boldsymbol{v}_i\in E.$$

This is well defined because $(\mathfrak{v},\boldsymbol{a})\sim(\mathfrak{v}A,A^{-1}\boldsymbol{a})$ for all $A\in\mathrm{GL}(q,\mathbb{R})$. Clearly such map is smooth and fiber-preserving, and linear on fibers. The inverse is defined by taking an element of E, writing it as a combination of *any* basis of the corresponding fiber, and mapping the original element to the chosen basis and the coefficients of the combination.

So, we have seen that each fiber of $P\times_\rho V$ is isomorphic to V itself. But a priori those isomorphisms are independent of each other. A choice of local gauge uniformizes that:

Proposition 37. *Let $\pi\colon P\to M$ be a principal G-bundle, $\rho\colon G\to\mathrm{GL}(V)$ a representation in some vector space V, and $\psi\colon U\subseteq M\to P$ a local gauge. Then we have a bijective correspondence between local sections $\widetilde{\psi}\colon U\subseteq M\to P\times_\rho V$ and smooth functions $f\colon U\to V$, given by $\widetilde{\psi}(x)=[\psi(x),f(x)]$, for all $x\in U$.*

Proof: If we start with a smooth function f, the local section $\widetilde{\psi}$ defined as above is smooth because the map $U\ni x\mapsto(\psi(x),f(x))\in P\times V$ is smooth itself. Conversely, if we start with a local section $\widetilde{\psi}$, there is a unique $f(x)\in V$ such that $\widetilde{\psi}(x)=[\psi(x),f(x)]$ for all $x\in U$, since the action of G on P is free and transitive on all the fibers. That said, the only thing left to check is that such function $f\colon U\to V$ is indeed smooth. For that, we will combine the constructions of propositions 30 (p. 23) and 35 (p. 26) as follows: ψ induces a principal G-chart $\Psi\colon\pi^{-1}[U]\to U\times G$ by $\Psi(p)=(\pi(p),g_p^{-1})$, where $g_p\in G$ is the unique element such that $\psi(\pi(p))=p\cdot g_p$ (and recall that g_p depends smoothly on p) – then we have the associated VB-chart $\Psi_\rho\colon\pi_\rho^{-1}[U]\to U\times V$ given by $\Psi_\rho([p,\boldsymbol{v}])=(\pi(p),\rho(g_p^{-1})\boldsymbol{v})$. Now, for every $x\in U$, we have that $g_{\psi(x)}=e$, and thus

$$\Psi_\rho\circ\widetilde{\psi}(x)=\Psi_\rho([\psi(x),f(x)])=(\pi(\psi(x)),\rho(g_{\psi(x)}^{-1})f(x))=(x,f(x)),$$

showing that f is smooth (since the composition $\Psi_\rho\circ\widetilde{\psi}$ is). $\square$

[19] Its derivative $\mathrm{ad}=\mathrm{d}(\mathrm{Ad})_e\colon\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$, in turn, is a Lie algebra representation.

3 Ehresmann connections

Let $\pi\colon P \to M$ be a principal G-bundle. Given $p \in P$, we define the *vertical space* of P at p by $\mathrm{Ver}_p(P) \doteq \ker \mathrm{d}\pi_p \subseteq T_pP$. That is, if $x = \pi(p) \in M$, then $\mathrm{Ver}_p(P) = T_p(P_x)$. These vertical spaces are canonical and can be characterized in terms of the action of G on P itself. For that, we need to recall action fields:

Definition 38. Let $P \to M$ be a principal G-bundle. Given $X \in \mathfrak{g}$, define the *action field* $X^\# \in \mathfrak{X}(P)$ generated by X, by

$$X^\#_p \doteq \mathrm{d}(\mathcal{O}_p)_e(X) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} p\cdot \exp(tX),$$

where $\mathcal{O}_p\colon G \to P$, $\mathcal{O}_p(g) = p\cdot g$, is the orbit-map of p.

Remark.

- Smoothness of $X^\#$ follows from suitable compositions and the smoothness of the full action map $P \times G \to P$.
- The map $\mathfrak{g} \ni X \mapsto X^\# \in \mathfrak{X}(P)$ is a homomorphism of Lie algebras[20], where the Lie bracket in $\mathfrak{g}$ is induced by left-invariant extensions of elements of $\mathfrak{g}$.
- The following shorthand notation will be useful: if $g \in G$ and $\boldsymbol{w} \in T_aG$, we let $g\boldsymbol{w}$ denote the image $\mathrm{d}(L_g)_a(\boldsymbol{w}) \in T_{ga}G$. In other words, the only thing that makes sense. Similarly we talk about $\boldsymbol{w}g$, or $g\boldsymbol{w}g^{-1}$, etc.. Usual rules like $e\boldsymbol{w} = \boldsymbol{w}$ and $g(h\boldsymbol{w}) = (gh)\boldsymbol{w}$ hold, due to the chain rule.

Proposition 39. *Let $\pi\colon P \to M$ be a principal G-bundle. Then for every fixed $p \in P$, the map $\mathfrak{g} \ni X \mapsto X^\#_p \in \mathrm{Ver}_p(P)$ is an isomorphism of vector spaces.*

Proof: Let $\mathcal{O}_p$ denote the orbit-map of p, again. Since the action of G on P preserves fibers, we have that $\pi(\mathcal{O}_p(g)) = \pi(p)$, for all $g \in G$. Differentiating with respect to the variable g at e and evaluating at X, we get $\mathrm{d}\pi_p(X^\#_p) = 0$, so that $X^\#_p$ is indeed vertical. The conclusion follows as $\mathcal{O}_p\colon G \to P_{\pi(p)}$ is a diffeomorphism: for any $q \in P_{\pi(p)}$ there is a unique $g_q \in G$ such that $q = p \cdot g_q$, and g_q depends smoothly on q, as is it easily seen by using a principal G-chart, just like in Proposition 29 (p. 22). ☐

Proposition 40. *Let $\pi\colon P \to M$ be a principal G-bundle, $p \in P$ and $X \in \mathfrak{g}$. Then $X^\#_p = \mathbf{0}$ if and only if $X \in \mathfrak{stab}(p)$ (the Lie algebra of the stabilizer $\mathrm{Stab}(p) \subseteq G$).*

Proof: If $X \in \mathfrak{stab}(p)$, then $\exp(tX) \in \mathrm{Stab}(p)$ for all $t \in \mathbb{R}$, so differentiating both sides of $p = p \cdot \exp(tX)$ at $t = 0$ gives $\mathbf{0} = X^\#_p$. Conversely, assume that $X^\#_p = \mathbf{0}$. This means that both $t \mapsto p$ and $t \mapsto p \cdot \exp(tX)$ are integral curves[21] of $X^\#$ starting at p, and so they must be equal. Now $p = p \cdot \exp(tX)$ says that $\exp(tX) \in \mathrm{Stab}(p)$ for all $t \in \mathbb{R}$, and so $X \in \mathfrak{stab}(p)$, as wanted. ☐

[20] It would be an anti-homomorphism if G acted on P by the left.

[21] We have that $\alpha(t) = p\cdot\exp(tX)$ is an integral curve of $X^\#$ passing through p because $\alpha(0) = p$ and

$$\alpha'(t) = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \alpha(t+s) = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} p\cdot\exp((t+s)X) = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} p\cdot(\exp(tX)\exp(sX)) = X^\#_{p\cdot\exp(tX)},$$

as claimed.

Remark. The general principle of describing a Lie algebra by differentiating at the identity the equation describing the Lie group is useful again. $g \cdot p = p$ becomes $X^\#_p = 0$.

One more quick relation:

Lemma 41. *Let $P \to M$ be a principal G-bundle, $\Phi \in \mathscr{G}(P)$ a gauge transformation, and $X \in \mathfrak{g}$. Then $\mathrm{d}\Phi_p(X^\#_p) = X^\#_{\Phi(p)}$.*

Proof: $\mathrm{d}\Phi_p(X^\#_p) = \mathrm{d}\Phi_p(\mathrm{d}(\mathcal{O}_p)_e(X)) = \mathrm{d}(\Phi \circ \mathcal{O}_p)_e(X) = \mathrm{d}(\mathcal{O}_{\Phi(p)})_e(X) = X^\#_{\Phi(p)}$. □

Example 42.

(1) For the trivial G-bundle $M \times G \to M$ and $X \in \mathfrak{g}$, we have that $X^\#_{(x,a)} = (\mathbf{0}, aX)$.

(2) For the toy bundle over a one-point space $G \to \{*\}$ and $X \in \mathfrak{g}$, we have $X^\#_a = aX$.

(3) If $H \leq G$ is a closed subgroup, we have seen that with the action $G \circlearrowleft H$ the projection $G \to G/H$ becomes a principal H-bundle. As the previous item, if $X \in \mathfrak{h}$, then $X^\#_a = aX$.

(4) If $f\colon N \to M$ is smooth and $P \to M$ is a principal G-bundle, consider the pull-back bundle $f^*P \to N$. For $X \in \mathfrak{g}$, we have that $\widetilde{X^\#}_{(y,p)} = (\mathbf{0}, X^\#_p)$, which shows that the action field does not really change, just like the original action.

One more useful property is given in the:

Lemma 43. *Let $\pi\colon P \to M$ be a principal G-bundle, and $X \in \mathfrak{g}$. Then for any $g \in G$ and $p \in P$ we have $\mathrm{d}(\mathrm{R}_g)_p(X^\#_p) = (g^{-1}Xg)^\#_{p \cdot g}$.*

Proof: Compute directly:

$$\begin{aligned} \mathrm{d}(\mathrm{R}_g)_p(X^\#_p) &= \mathrm{d}(\mathrm{R}_g)_p(\mathrm{d}(\mathcal{O}_p)_e(X)) = \mathrm{d}(\mathrm{R}_g \circ \mathcal{O}_p)_e(X) \\ &= \mathrm{d}(\mathcal{O}_p \circ \mathrm{R}_g)_e(X) = \mathrm{d}(\mathcal{O}_p)_g(\mathrm{d}(\mathrm{R}_g)_e(X)) \\ &= \mathrm{d}(\mathcal{O}_p)_g(Xg) = \mathrm{d}(\mathcal{O}_p)_g(g(g^{-1}Xg)) \\ &= \mathrm{d}(\mathcal{O}_{p \cdot g})_e(g^{-1}Xg) = (g^{-1}Xg)^\#_{p \cdot g}. \end{aligned}$$

□

Now that we have understood a little about vertical spaces, what about horizontal spaces? It turns out that there is no canonical way to choose them. But there are two equivalent ways of describing this concept:

Definition 44. Let $P \to M$ be a principal G-bundle. An *Ehresmann connection* in P is a distribution $\mathrm{Hor}(P) \hookrightarrow TP$ such that:

(i) $T_pP = \mathrm{Hor}_p(P) \oplus \mathrm{Ver}_p(P)$, for all $p \in P$;

(ii) $\mathrm{d}(\mathrm{R}_g)_p[\mathrm{Hor}_p(P)] = \mathrm{Hor}_{p \cdot g}(P)$, for all $p \in P$ and $g \in G$.

In other words, a complementary and right-invariant distribution to $\mathrm{Ver}(P)$. We'll say that the connection is *flat* if $\mathrm{Hor}(P) \hookrightarrow TP$ is integrable.

Remark. By linear algebra, if $x \in M$ and $p \in P_x$, $\mathrm{Hor}_p(P)$ being complementary to $\mathrm{Ver}_p(P)$ implies that the restriction $\mathrm{d}\pi_p \colon \mathrm{Hor}_p(P) \to T_xM$ is an isomorphism.

Example 45.

(1) Consider the trivial G-bundle $M \times G \to M$. Clearly we have that

$$\mathrm{Ver}_{(x,a)}(M \times G) = \ker \mathrm{d}\pi_{(x,a)} = \{\mathbf{0}\} \oplus T_aG.$$

Thus, a natural choice of horizontal spaces is $\mathrm{Hor}_{(x,a)}(M \times G) = T_xM \oplus \{\mathbf{0}\}$. This is clearly complementary to the vertical distribution, and so we need to check right-invariance. This is done as follows: given $g \in G$, we have that the action map of g is given by $\mathrm{R}_g(x,a) = (x, ag)$, and so $\mathrm{d}(\mathrm{R}_g)_{(x,a)} = \mathrm{Id}_{T_xM} \oplus \mathrm{d}(\mathrm{R}_g)_a$. Here we use R_g to denote two different maps $M \times G \to M \times G$ and $G \to G$, the distinction being made by context. Thus we have that

$$\begin{aligned} \mathrm{d}(\mathrm{R}_g)_a[\mathrm{Hor}_{(x,a)}(M \times G)] &= (\mathrm{Id}_{T_xM} \oplus \mathrm{d}(\mathrm{R}_g)_a)[T_xM \oplus \{\mathbf{0}\}] \\ &= \mathrm{Id}_{T_xM}[T_xM] \oplus \mathrm{d}(\mathrm{R}_g)_a[\{\mathbf{0}\}] \\ &= T_xM \oplus \{\mathbf{0}\} \\ &= \mathrm{Hor}_{(x,ag)}(M \times G). \end{aligned}$$

This is called the *canonical flat connection* of $M \times G$.

(2) Consider the toy bundle over a one-point space, $G \to \{*\}$. Since the projection is constant, for all $a \in G$ we have $\mathrm{Ver}_a(G) = T_aG$, and so we necessarily have $\mathrm{Hor}_a(G) = \{\mathbf{0}\}$, which is indeed right-invariant, thus defining an Ehresmann connection.

(3) Consider the more general case where $H \leq G$ is a closed subgroup and the principal H-bundle $\pi\colon G \to G/H$. Since $\mathrm{d}\pi_e \colon \mathfrak{g} \to T_{eH}(G/H)$ is a surjective submersion, this derivative is surjective. Let's show that $\ker \mathrm{d}\pi_e = \mathfrak{h}$. For all $X \in \mathfrak{h}$, we have that $\mathrm{e}^{tX} \in H$ for small t and so $\pi(\mathrm{e}^{tX}) = eH$, leading to $\mathrm{d}\pi_e(X) = 0$. This means that $\mathfrak{h} \subseteq \ker \mathrm{d}\pi_e$. For the reverse inclusion[22], we use the commutatitivity of

$$\begin{array}{ccc} G & \xrightarrow{\ \mathrm{L}_g\ } & G \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G/H & \xrightarrow{\ \mathrm{L}_g\ } & G/H \end{array}$$

[22] I'd like to thank Matheus Manzatto for bringing this short argument to my attention.

for arbitrary $g \in G$, as follows: if $X \in \ker \mathrm{d}\pi_e$, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\pi(\mathrm{e}^{tX}) &= \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0}\pi(\mathrm{e}^{(t+s)X}) = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0}\pi(\mathrm{e}^{tX}\mathrm{e}^{sX}) \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0}\pi\circ \mathrm{L}_{\mathrm{e}^{tX}}(\mathrm{e}^{sX}) = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0}\mathrm{L}_{\mathrm{e}^{tX}}\circ\pi(\mathrm{e}^{sX}) \\ &= \mathrm{d}\big(\mathrm{L}_{\mathrm{e}^{tX}}\big)_{eH}(\mathrm{d}\pi_e(X)) = \mathbf{0},\end{aligned}$$

meaning that $\mathrm{e}^{tX} \in H$ for all t and thus $X \in \mathfrak{h}$. This means that $\mathrm{Ver}_e(G) = \mathfrak{h}$ and also that $T_{eH}(G/H) \cong \mathfrak{g}/\mathfrak{h}$. In general, if $a \in G$, we have that $\mathrm{Ver}_a(G) = \mathrm{d}(\mathrm{L}_a)_e\mathfrak{h}$ by taking derivatives in the previous diagram to obtain

$$\mathrm{d}\pi_a = \mathrm{d}(\mathrm{L}_a)_{eH}\circ \mathrm{d}\pi_e \circ \mathrm{d}(\mathrm{L}_a)_e^{-1},$$

and noting that the first map on the right side is an isomorphism, a given vector $\boldsymbol{v} \in T_aG$ has $\mathrm{d}\pi_a(\boldsymbol{v}) = \mathbf{0}$ if and only if $\mathrm{d}\pi_e(\mathrm{d}(\mathrm{L}_a)_e^{-1}(\boldsymbol{v})) = \mathbf{0}$, which is to say that $\mathrm{d}(\mathrm{L}_a)_e^{-1}(\boldsymbol{v}) \in \ker \mathrm{d}\pi_e = \mathfrak{h}$ – equivalent in turn to $\boldsymbol{v} \in \mathrm{d}(\mathrm{L}_a)_e\mathfrak{h}$. In particular, if $a \in H$ we may write $\mathrm{Ver}_a(G) = T_aH$.

To define an Ehresmann connection, we need something else: assume that G/H is *reductive*, i.e., that there is a vector subspace $\mathfrak{m} \subseteq \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $\mathrm{Ad}(H)\mathfrak{m} \subseteq \mathfrak{m}$. This condition is natural in the sense that if $\mathfrak{h}$ is playing the role of the vertical distribution (up to translations back in G), then something complementary to it should play the role of the horizontal distribution. Moreover, the invariance condition $\mathrm{Ad}(H)\mathfrak{m} \subseteq \mathfrak{m}$ will correspond to the invariance of the horizontal distribution to be defined in what follows.

More precisely, we make the only definition we can: $\mathrm{Hor}_a(G) = \mathrm{d}(\mathrm{L}_a)_e\mathfrak{m}$. Of course that applying $\mathrm{d}(\mathrm{L}_a)_e$ to $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ we get that $T_aG = \mathrm{Ver}_a(G) \oplus \mathrm{Hor}_a(G)$. To show that given $a \in G$ and $h \in H$, the equality $\mathrm{Hor}_{ah}(G) = \mathrm{d}(\mathrm{R}_h)_a[\mathrm{Hor}_a(G)]$ holds, it suffices to show one inclusion, since both sides have the same dimension. So, let $\boldsymbol{v} \in \mathrm{Hor}_a(G)$. Our goal is to show that $\mathrm{d}(\mathrm{R}_h)_a(\boldsymbol{v}) \in \mathrm{Hor}_{ah}(G)$. Write $\boldsymbol{v} = \mathrm{d}(\mathrm{L}_a)_e(X)$ with $X \in \mathfrak{m}$. The invariance condition says that

$$Y \doteq \mathrm{Ad}(h^{-1})X = \mathrm{d}(\mathrm{L}_{h^{-1}}\circ \mathrm{R}_h)_e(X) = \mathrm{d}(\mathrm{L}_{h^{-1}})_h \circ \mathrm{d}(\mathrm{R}_h)_e(X) \in \mathfrak{m}.$$

So $\mathrm{d}(\mathrm{R}_h)_e(X) = \mathrm{d}(\mathrm{L}_h)_e(Y)$ implies that

$$\begin{aligned}\mathrm{d}(\mathrm{R}_h)_a(\boldsymbol{v}) &= \mathrm{d}(\mathrm{R}_h)_a\circ \mathrm{d}(\mathrm{L}_a)_e(X) = \mathrm{d}(\mathrm{R}_h\circ \mathrm{L}_a)_e(X) \\ &= \mathrm{d}(\mathrm{L}_a\circ \mathrm{R}_h)_e(X) = \mathrm{d}(\mathrm{L}_a)_h\circ \mathrm{d}(\mathrm{R}_h)_e(X) \\ &= \mathrm{d}(\mathrm{L}_a)_h\circ \mathrm{d}(\mathrm{L}_h)_e(Y) = \mathrm{d}(\mathrm{L}_a\circ \mathrm{L}_h)_e(Y) \\ &= \mathrm{d}(\mathrm{L}_{ah})_e(Y).\end{aligned}$$

Since $Y \in \mathfrak{m}$, by definition this means that $\mathrm{d}(\mathrm{R}_h)_a(\boldsymbol{v}) \in \mathrm{Hor}_{ah}(G)$, as wanted.

(4) Let $\pi\colon P \to M$ be a principal G-bundle, $f\colon N \to M$ be a smooth function, and consider the pull-back bundle $\widetilde{\pi}\colon f^*P \to N$. We have seen in Example 24 (p. 16) that

$$T_{(y,p)}(f^*P) = \{(\boldsymbol{v},\boldsymbol{w}) \in T_yN \times T_pP \mid \mathrm{d}f_y(\boldsymbol{v}) = \mathrm{d}\pi_p(\boldsymbol{w})\},$$

and so, setting $\boldsymbol{v} = \mathbf{0}$ (as $\mathrm{d}\widetilde{\pi}_{(y,p)}(\boldsymbol{v},\boldsymbol{w}) = \boldsymbol{v}$) we get that

$$\widetilde{\mathrm{Ver}}_{(y,p)}(f^*P) = \{(\mathbf{0},\boldsymbol{w}) \in T_yN \times T_pP \mid \mathrm{d}\pi_p(\boldsymbol{w}) = \mathbf{0}\} = \{\mathbf{0}\} \oplus \mathrm{Ver}_p(P),$$

a very intuitive result. Now, if $\mathrm{Hor}(P) \hookrightarrow TP$ is an Ehresmann connection for P, let's define an Ehresmann connection for f^*P by

$$\widetilde{\mathrm{Hor}}_{(y,p)}(f^*P) = \{(\boldsymbol{v},\boldsymbol{w}) \in T_{(y,p)}(f^*P) \mid \boldsymbol{w} \in \mathrm{Hor}_p(P)\}.$$

Let's check that:

- $T_{(y,p)}(f^*P) = \widetilde{\mathrm{Hor}}_{(y,p)}(f^*P) \oplus \widetilde{\mathrm{Ver}}_{(y,p)}(f^*P)$: given $(\boldsymbol{v},\boldsymbol{w}) \in T_{(y,p)}(f^*P)$, we write $(\boldsymbol{v},\boldsymbol{w}) = (\boldsymbol{v},\boldsymbol{w}^{\mathrm{hor}}) + (\mathbf{0},\boldsymbol{w}^{\mathrm{ver}})$. And such decomposition is unique, since $(\boldsymbol{v},\boldsymbol{w}) \in \widetilde{\mathrm{Hor}}_{(y,p)}(f^*P) \cap \widetilde{\mathrm{Ver}}_{(y,p)}(f^*P)$ implies that $\boldsymbol{v} = \mathbf{0}$ and finally that $\boldsymbol{w} \in \mathrm{Hor}_p(P) \cap \mathrm{Ver}_p(P) = \{\mathbf{0}\}$. In particular, we see that the dimension of all these new horizontal subspaces is always the same.
- $\widetilde{\mathrm{Hor}}(f^*P) \hookrightarrow T(f^*P)$ is right-invariant: our goal is to show that if the action on f^*P is given by $\widetilde{\mathrm{R}}_g(y,p) = (y,\mathrm{R}_g(p))$, then

$$\mathrm{d}(\widetilde{\mathrm{R}}_g)_{(y,p)}[\widetilde{\mathrm{Hor}}_{(y,p)}(f^*P)] = \widetilde{\mathrm{Hor}}_{(y,p\cdot g)}(f^*P).$$

  This fortunately is clear from the formula $\mathrm{d}(\widetilde{\mathrm{R}}_g)_{(y,p)} = \mathrm{Id}_{T_yN} \oplus \mathrm{d}(\mathrm{R}_g)_p$ together with $\mathrm{d}(\mathrm{R}_g)_p[\mathrm{Hor}_p(P)] = \mathrm{Hor}_{p\cdot g}(P)$.

(5) Assume that $P \to M$ is a principal G-bundle, but that we have a Riemannian metric $\langle\cdot,\cdot\rangle$ on P which is G-invariant, that is, each $\mathrm{R}_g\colon P \to P$ is an isometry for $(P,\langle\cdot,\cdot\rangle)$. Then for $p \in P$, we have that $\mathrm{Hor}_p(P) \doteq \mathrm{Ver}_p(P)^\perp$ defines an Ehresmann connection (item (i) is obvious while item (ii) follows from the fact that if a linear isometry in an Euclidean vector space leaves a subspace invariant, it also leaves its orthogonal complement invariant).

(6) The same construction as above also works if P has a symplectic form with symplectic fibers, by replacing orthogonal complements with symplectic complements (if the fibers are not symplectic, condition (i) fails).

Describing connections in, for example, frame bundles $\mathrm{Fr}(E) \to M$, is easier using the second notion:

Definition 46. Let $P \to M$ be a principal G-bundle. A *connection 1-form* (or *gauge field*) for P is a 1-form $\omega \in \Omega^1(P,\mathfrak{g})$ satisfying:

(i) $\omega(X^\#) = X$, for all $X \in \mathfrak{g}$, and;

(ii) $(\mathrm{R}_g)^*\omega = \mathrm{Ad}(g^{-1}) \circ \omega$, for all $g \in G$.

Remark.

- Another common notation (from Physics) for the connection 1-form is A instead of ω.

- Axiom (ii) just says that the 1-form regarded as the "full map" $\omega\colon TP \to \mathfrak{g}$ is G-equivariant, seeing G act on TP via derivative, and by the right on $\mathfrak{g}$ via the adjoint representation. Furthermore, axiom (i) implies that axiom (ii) is true for vertical vectors, as we have

$$\begin{aligned}\mathrm{d}(\mathrm{R}_g)_p(X^\#_p) &= \mathrm{d}(\mathrm{R}_g)_p(\mathrm{d}(\mathcal{O}_p)_e(X)) = \mathrm{d}(\mathrm{R}_g\circ\mathcal{O}_p)_e(X) \\ &= \mathrm{d}(\mathcal{O}_p\circ\mathrm{R}_g)_e(X) = \mathrm{d}(\mathcal{O}_p)_g(Xg) \\ &= \mathrm{d}(\mathcal{O}_p)_g(\mathrm{d}(\mathrm{L}_g)_e(g^{-1}Xg)) = \mathrm{d}(\mathcal{O}_p\circ\mathrm{L}_g)_e(g^{-1}Xg) \\ &= \mathrm{d}(\mathcal{O}_{p\cdot g})_e(g^{-1}Xg) = (g^{-1}Xg)^\#_{p\cdot g},\end{aligned}$$

  and so

$$((\mathrm{R}_g)^*\omega)_p(X^\#_p) = \omega_{p\cdot g}(\mathrm{d}(\mathrm{R}_g)_p(X^\#_p)) = \omega_{p\cdot g}((g^{-1}Xg)^\#_{p\cdot g}) = g^{-1}Xg = g^{-1}\omega_p(X^\#_p)g,$$

  as wanted. So, while axiom (i) might seem more natural than axiom (ii) if one does not notice the G-equivariance of the full map, we can see axiom (ii) of a natural extension of a property that automatically holds for vertical vectors, in view of (i).

- Principal bundles always have connection 1-forms. It is the usual argument using a partition of unity for P and that convex combinations of connection 1-forms are again connection 1-forms.

- When G is abelian, axiom (ii) says that ω is G-invariant.

Example 47.

(1) Recall that if G is any Lie group, the Maurer-Cartan form of G is $\Theta \in \Omega^1(G,\mathfrak{g})$ given by $\Theta_g(\boldsymbol{w}) = g^{-1}\boldsymbol{w}$. So, consider the trivial G-bundle $M\times G \to M$. There, define $\omega \in \Omega^1(M\times G,\mathfrak{g})$ by $\omega_{(x,a)}(\boldsymbol{v},\boldsymbol{w}) = \Theta_a(\boldsymbol{w})$. Let's check that this ω is a connection 1-form for $M\times G$. First, if $X\in\mathfrak{g}$, we have that

$$\omega_{(x,a)}(X^\#_{(x,a)}) = \omega_{(x,a)}(\mathbf{0},aX) = \Theta_a(aX) = a^{-1}aX = X.$$

Now, if $g\in G$, we compute

$$\begin{aligned}((\mathrm{R}_g)^*\omega)_{(x,a)}(\boldsymbol{v},\boldsymbol{w}) &= \omega_{(x,ag)}(\mathrm{d}(\mathrm{R}_g)_{(x,a)}(\boldsymbol{v},\boldsymbol{w})) = \omega_{(x,ag)}(0,\boldsymbol{w}g) \\ &= \Theta_{ag}(\boldsymbol{w}g) = (ag)^{-1}\boldsymbol{w}g \\ &= g^{-1}a^{-1}\boldsymbol{w}g = g^{-1}\Theta_a(\boldsymbol{w})g \\ &= g^{-1}\omega_{(x,a)}(\boldsymbol{v},\boldsymbol{w})g = \mathrm{Ad}(g^{-1})\big(\omega_{(x,a)}(\boldsymbol{v},\boldsymbol{w})\big),\end{aligned}$$

as wanted. This is called the Maurer-Cartan connection of $M\times G$.

(2) For the toy bundle over a one-point space $G\to\{*\}$, the Maurer-Cartan form $\Theta\in\Omega^1(G,\mathfrak{g})$ itself is a connection 1-form, as the calculations from the above item also show. The non-trivial content of this example is that if $\omega\in\Omega^1(G,\mathfrak{g})$ is a

connection 1-form, then necessarily $\omega = \Theta$. Given $a \in G$ and $\boldsymbol{v} \in T_aG$, we have that

$$\omega_a(\boldsymbol{v}) = \omega_a((\boldsymbol{v}a^{-1})a) \overset{(ii)}{=} \mathrm{Ad}(a^{-1})\big(\omega_e(\boldsymbol{v}a^{-1})\big) \overset{(i)}{=} a^{-1}(\boldsymbol{v}a^{-1})a = a^{-1}\boldsymbol{v} = \Theta_a(\boldsymbol{v}).$$

Note that $\omega = \Theta$ is always injective and compare this with item (2) of Example 45 (p. 31).

(3) Consider again the case where $H \leq G$ is a closed subgroup and we have a reductive principal H-bundle $\pi\colon G \to G/H$. As before, write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ where $\mathfrak{m}$ is a $\mathrm{Ad}(H)$-invariant vector subspace of $\mathfrak{g}$. The connection 1-form in this case has to be a form $\omega \in \Omega^1(G,\mathfrak{h})$. We'll project the Maurer-Cartan form onto $\mathfrak{h}$ and define $\omega = \mathrm{pr}_{\mathfrak{h}} \circ \Theta$, where $\mathrm{pr}_{\mathfrak{h}}\colon \mathfrak{g} \to \mathfrak{h}$ is the direct sum projection. If $X \in \mathfrak{h}$ and $a \in G$, we have $\omega_a(X^\#_a) = \mathrm{pr}_{\mathfrak{h}}(\Theta_a(X^\#_a)) = \mathrm{pr}_{\mathfrak{h}}(X) = X$. For the invariance condition, note that for any $h \in H$ we have

$$\begin{aligned} \mathrm{R}_h^*\omega = \mathrm{R}_h^*(\mathrm{pr}_{\mathfrak{h}} \circ \Theta) &\overset{(1)}{=} \mathrm{pr}_{\mathfrak{h}} \circ \mathrm{R}_h^*\Theta \\ &\overset{(2)}{=} \mathrm{pr}_{\mathfrak{h}} \circ \mathrm{Ad}(h^{-1}) \circ \Theta \overset{(3)}{=} \mathrm{Ad}(h^{-1}) \circ \mathrm{pr}_{\mathfrak{h}} \circ \Theta \\ &= \mathrm{Ad}(h^{-1}) \circ \omega, \end{aligned}$$

where in (1) we use that $\mathrm{pr}_{\mathfrak{h}}$ is linear, in (2) previous computations, and in (3) the $\mathrm{Ad}(H)$-invariance assumption.

(4) If $f\colon N \to M$ is smooth and $P \to M$ is a principal G-bundle, consider the pull-back bundle $f^*P \to N$. Assume that we're given a connection 1-form $\omega \in \Omega^1(P,\mathfrak{g})$. Define $\widetilde{\omega} \in \Omega^1(f^*P,\mathfrak{g})$ by $\widetilde{\omega}_{(y,p)}(\boldsymbol{v},\boldsymbol{w}) = \omega_p(\boldsymbol{w})$. Let's verify that $\widetilde{\omega}$ is also a connection 1-form. If $X \in \mathfrak{g}$, we have that

$$\widetilde{\omega}_{(y,p)}(\widetilde{X^\#}_{(y,p)}) = \widetilde{\omega}_{(y,p)}(\mathbf{0}, X^\#_p) = \omega_p(X^\#_p) = X,$$

and if $g \in G$ is given, we compute

$$\begin{aligned} ((\widetilde{\mathrm{R}}_g)^*\widetilde{\omega})_{(y,p)}(\boldsymbol{v},\boldsymbol{w}) &= \widetilde{\omega}_{(y,p\cdot g)}(\mathrm{d}(\widetilde{\mathrm{R}}_g)_{(y,p)}(\boldsymbol{v},\boldsymbol{w})) \\ &= \widetilde{\omega}_{(y,p\cdot g)}(\mathbf{0}, \mathrm{d}(\mathrm{R}_g)_p(\boldsymbol{w})) \\ &= \omega_{p\cdot g}(\mathrm{d}(\mathrm{R}_g)_p(\boldsymbol{w})) \\ &= ((\mathrm{R}_g)^*\omega)_p(\boldsymbol{w}) \\ &= \mathrm{Ad}(g^{-1})\big(\omega_p(\boldsymbol{w})\big) \\ &= \mathrm{Ad}(g^{-1})\big(\widetilde{\omega}_{(y,p)}(\boldsymbol{v},\boldsymbol{w})\big), \end{aligned}$$

concluding the proof. Note that $\widetilde{\omega} = \mathrm{pr}_2^*\omega$, where $\mathrm{pr}_2\colon f^*P \to P$ is the projection in the second factor[23].

[23] This gives another proof that $\widetilde{\omega}$ satisfies axiom (ii) in Definition 46 (p. 33):

$$(\widetilde{\mathrm{R}}_g)^*\widetilde{\omega} = (\widetilde{\mathrm{R}}_g)^*\mathrm{pr}_2^*\omega = (\mathrm{pr}_2 \circ \widetilde{\mathrm{R}}_g)^*\omega = (\mathrm{R}_g \circ \mathrm{pr}_2)^*\omega = \mathrm{pr}_2^*\mathrm{R}_g^*\omega = \mathrm{pr}_2^*(\mathrm{Ad}(g^{-1}) \circ \omega) = \mathrm{Ad}(g^{-1}) \circ \widetilde{\omega}.$$

(5) Let $E \to M$ be a vector bundle equipped with a Koszul connection ∇. Let's define a connection 1-form $\omega^\nabla \in \Omega^1(\mathrm{Fr}(E), \mathfrak{gl}(k,\mathbb{R}))$ from it. Given a point $\mathfrak{v} \in \mathrm{Fr}(E)$ and a tangent vector $\dot{\mathfrak{v}} \in T_{\mathfrak{v}}\mathrm{Fr}(E)$, we need to see what $\omega^\nabla_{\mathfrak{v}}(\dot{\mathfrak{v}}) \in \mathfrak{gl}(k,\mathbb{R})$ should be. We regard $\mathfrak{v}$ as a row vector whose entries form a basis for $E_{\pi_{\mathrm{fr}}(\mathfrak{v})}$, and such a basis may be seen as an isomorphism $\mathbb{R}^k \to E_{\pi_{\mathrm{fr}}(\mathfrak{v})}$ that takes the i-th canonical vector in $\mathbb{R}^k$ to the i-th element of $\mathfrak{v}$. Thus it makes sense to consider $\mathfrak{v}^{-1}$. If we consider any curve $t \mapsto \mathfrak{v}(t)$ in $\mathrm{Fr}(E)$ with $\mathfrak{v}(0) = \mathfrak{v}$ and $\mathfrak{v}'(0) = \dot{\mathfrak{v}}$ and we write $\mathfrak{v}(t) = [v_1(t) \cdots v_k(t)]$, then it makes sense to set

$$\frac{\mathrm{D}\mathfrak{v}}{\mathrm{d}t}(t) \doteq \left[\frac{\mathrm{D}v_1}{\mathrm{d}t}(t) \cdots \frac{\mathrm{D}v_k}{\mathrm{d}t}(t)\right],$$

where $\mathrm{D}/\mathrm{d}t$ is the connection induced in the pull-back bundle $(\pi_{\mathrm{fr}} \circ \mathfrak{v})^*(E) \to M$. For $t = 0$, this depends only on $\mathfrak{v}$, $\dot{\mathfrak{v}}$ and on ∇ itself (via Christoffel symbols), but not on the values of $\mathfrak{v}(t)$ or $\mathfrak{v}'(t)$ for $t \neq 0$. Since $\mathfrak{v}$ is a basis for $E_{\pi_{\mathrm{fr}}(\mathfrak{v})}$, we may write

$$\frac{\mathrm{D}\mathfrak{v}}{\mathrm{d}t}(0) = \mathfrak{v} \cdot A$$

for some matrix $A \in \mathfrak{gl}(k,\mathbb{R})$. Thus, our dfinition is born as follows: we write

$$\omega^\nabla_{\mathfrak{v}}(\dot{\mathfrak{v}}) = \mathfrak{v}^{-1}\frac{\mathrm{D}\mathfrak{v}}{\mathrm{d}t}(0)$$

where $t \mapsto \mathfrak{v}(t)$ is any curve in $\mathrm{Fr}(E)$ with $\mathfrak{v}(0) = \mathfrak{v}$ and $\mathfrak{v}'(0) = \dot{\mathfrak{v}}$. Now, let's prove that ω^∇ is indeed a connection 1-form. If $X \in \mathfrak{gl}(k,\mathbb{R})$, then $X^\sharp_{\mathfrak{v}} = \mathfrak{v} \cdot X$, since

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathfrak{v} \cdot \mathrm{e}^{tX} = \mathfrak{v}X.$$

With this, we compute $\omega^\nabla_{\mathfrak{v}}(X^\sharp_{\mathfrak{v}}) = \omega^\nabla_{\mathfrak{v}}(\mathfrak{v}X) = \mathfrak{v}^{-1}(\mathfrak{v}X) = X$, as wanted. For the equivariance condition, fix $A \in \mathrm{GL}(k,\mathbb{R})$. Fix any curve realizing $\dot{\mathfrak{v}} \in T_{\mathfrak{v}}\mathrm{Fr}(E)$. Compute

$$\begin{aligned}
(\mathrm{R}_A^*\omega^\nabla)_{\mathfrak{v}}(\dot{\mathfrak{v}}) &= \omega^\nabla_{\mathfrak{v}A}(\mathrm{d}(\mathrm{R}_A)_{\mathfrak{v}}(\dot{\mathfrak{v}})) = \omega^\nabla_{\mathfrak{v}A}(\dot{\mathfrak{v}}A) \\
&= (\mathfrak{v}A)^{-1}\left.\frac{\mathrm{D}}{\mathrm{d}t}\right|_{t=0}(\mathfrak{v}(t)A) = (\mathfrak{v}A)^{-1}\frac{\mathrm{D}\mathfrak{v}}{\mathrm{d}t}(0)A \\
&= A^{-1}\mathfrak{v}^{-1}\frac{\mathrm{D}\mathfrak{v}}{\mathrm{d}t}(0)A = A^{-1}\omega^\nabla_{\mathfrak{v}}(\dot{\mathfrak{v}})A \\
&= \mathrm{Ad}(A^{-1})(\omega^\nabla_{\mathfrak{v}}(\dot{\mathfrak{v}})),
\end{aligned}$$

as wanted.

Proposition 48. *Let $P \to M$ be a principal G-bundle, $\omega \in \Omega^1(P,\mathfrak{g})$ a connection 1-form, and $\psi, \widetilde{\psi}\colon U \subseteq M \to P$ be two local gauges for P. There is a smooth map $A\colon U \to G$ such that $\widetilde{\psi}(x) = \psi(x) \cdot A(x)$, for all $x \in U$. Then:*

(i) $\mathrm{d}\widetilde{\psi}_x(v) = \mathrm{d}(\mathrm{R}_{A(x)})_{\psi(x)}(\mathrm{d}\psi_x(v)) + \mathrm{d}(\mathfrak{G}_{\psi(x)})_{A(x)}(\mathrm{d}A_x(v))$, *for all $x \in M$ and vectors $v \in T_xM$;*

(ii) $\mathrm{d}(\mathcal{O}_{\psi(x)})_{A(x)}(\mathrm{d}A_x(\boldsymbol{v})) = (A(x)^{-1}\mathrm{d}A_x(\boldsymbol{v}))^{\#}_{\widetilde{\psi}(x)}$, *for all* $x \in M$ *and* $\boldsymbol{v} \in T_xM$;

(iii) $\widetilde{\psi}^*\omega = \mathrm{Ad}(A^{-1})(\psi^*\omega) + A^{-1}\,\mathrm{d}A$.

Remark. Item (ii) may be rephrased as $\big(A^*(\mathcal{O}_{\psi(x)})\big)_x(\boldsymbol{v}) = ((A^*\Theta)_x(\boldsymbol{v}))^{\#}_{\widetilde{\psi}(x)}$, while item (iii) may be rephrased as $\widetilde{\psi}^*\omega = \mathrm{Ad}(A^{-1})(\psi^*\omega) + A^*\Theta$.

Proof:

(i) Just a product rule (e.g., define $f(x,y) = \psi(x)\cdot A(y)$, note that $\widetilde{\psi}(x) = f(x,x)$ and compute $\mathrm{d}\widetilde{\psi}_x(\boldsymbol{v}) = (\partial_1 f)_{(x,x)}(\boldsymbol{v}) + (\partial_2 f)_{(x,x)}(\boldsymbol{v})$).

(ii) The idea is to drag $\mathrm{d}A_x(\boldsymbol{v})$ back to $\mathfrak{g}$ and use the chain rule, to get:

$$\begin{aligned}\mathrm{d}(\mathcal{O}_{\psi(x)})_{A(x)}(\mathrm{d}A_x(\boldsymbol{v})) &= \mathrm{d}(\mathcal{O}_{\psi(x)})_{A(x)}(\mathrm{d}(\mathrm{R}_{A(x)})_e(A(x)^{-1}\mathrm{d}A_x(\boldsymbol{v})))\\ &= \mathrm{d}(\mathcal{O}_{\psi(x)}\circ \mathrm{R}_{A(x)})_e(A(x)^{-1}\mathrm{d}A_x(\boldsymbol{v}))\\ &= \mathrm{d}(\mathcal{O}_{\widetilde{\psi}(x)})_e(A(x)^{-1}\mathrm{d}A_x(\boldsymbol{v}))\\ &= (A(x)^{-1}\mathrm{d}A_x(\boldsymbol{v}))^{\#}_{\widetilde{\psi}(x)}.\end{aligned}$$

(iii) Compute directly that

$$\begin{aligned}(\widetilde{\psi}^*\omega)_x(\boldsymbol{v}) &= \omega_{\widetilde{\psi}(x)}(\mathrm{d}\widetilde{\psi}_x(\boldsymbol{v}))\\ &= \omega_{\widetilde{\psi}(x)}(\mathrm{d}(\mathrm{R}_{A(x)})_{\psi(x)}(\mathrm{d}\psi_x(\boldsymbol{v})) + \mathrm{d}(\mathcal{O}_{\psi(x)})_{A(x)}(\mathrm{d}A_x(\boldsymbol{v})))\\ &= \omega_{\psi(x)\cdot A(x)}(\mathrm{d}(\mathrm{R}_{A(x)})_{\psi(x)}(\mathrm{d}\psi_x(\boldsymbol{v}))) + \omega_{\widetilde{\psi}(x)}(\mathrm{d}(\mathcal{O}_{\psi(x)})_{A(x)}(\mathrm{d}A_x(\boldsymbol{v})))\\ &= (\mathrm{R}^*_{A(x)}\omega)_{\psi(x)}(\mathrm{d}\psi_x(\boldsymbol{v})) + \omega_{\widetilde{\psi}(x)}((A(x)^{-1}\mathrm{d}A_x(\boldsymbol{v}))^{\#}_{\widetilde{\psi}(x)})\\ &= \mathrm{Ad}(A(x)^{-1})\big((\psi^*\omega)_x(\boldsymbol{v})\big) + A(x)^{-1}\mathrm{d}A_x(\boldsymbol{v}),\end{aligned}$$

as wanted.

□

Remark. The map $A\colon U \to G$ is to be understood as a "change of basis", just like when two frames $\widetilde{\mathfrak{v}}$ and $\mathfrak{v}$ are related like $\widetilde{\mathfrak{v}} = \mathfrak{v}A$ in $\mathrm{Fr}(E)$. So, compare this with the expression obtained in Proposition 21 (p. 13).

The above result also tells us how to glue local connection 1-forms. Here's another related result:

Proposition 49. *Let* $P \to M$ *be a principal* G*-bundle,* $\omega \in \Omega^1(P,\mathfrak{g})$ *be a connection* 1*-form, and* $\Phi \in \mathcal{G}(P)$ *be a gauge transformation. Then* $\Phi^*\omega \in \Omega^1(P,\mathfrak{g})$ *is also a connection* 1*-form. Moreover, it is actually related to* ω *via:*

$$(\Phi^*\omega)_p = \mathrm{Ad}(\sigma_\Phi(p)^{-1})\circ\omega_p + \sigma_\Phi(p)^{-1}\mathrm{d}(\sigma_\Phi)_p,$$

where $\sigma_\Phi\colon P \to G$ *corresponds to* Φ *as in Proposition 29 (p. 22), satisfying the relation* $\Phi(p) = p\cdot\sigma_\Phi(p)$*, for all* $p \in P$.

Remark. This relation may be also written as $(\Phi^*\omega)_p = \mathrm{Ad}\left(\sigma_\Phi(p)^{-1}\right) \circ \omega_p + (\sigma_\Phi^*\Theta)_p$.

Proof: The two axioms are verified almost instantly:

(i) By Lemma 41 (p. 30), $(\Phi^*\omega)_p(X_p^\#) = \omega_{\Phi(p)}(\mathrm{d}\Phi_p(X_p^\#)) = \omega_{\Phi(p)}(X^\#_{\Phi(p)}) = X$;

(ii) Since Φ and R_g commute, so will the pull-backs. Thus we obtain

$$\mathrm{R}_g^*\Phi^*\omega = \Phi^*\mathrm{R}_g^*\omega = \Phi^*(\mathrm{Ad}(g^{-1}) \circ \omega) = \mathrm{Ad}(g^{-1}) \circ \Phi^*\omega.$$

The explicit formula for $\Phi^*\omega$ follows from noting that $\Phi(p) = p \cdot \sigma_\Phi(p)$ implies that

$$\mathrm{d}\Phi_p(\boldsymbol{v}) = \mathrm{d}(\mathrm{R}_{\sigma_\Phi(p)})_p(\boldsymbol{v}) + \mathrm{d}(\mathscr{O}_p)_{\sigma_\Phi(p)}(\mathrm{d}(\sigma_\Phi)_p(\boldsymbol{v})),$$

and repeating the calculations from the previous proof, dragging $\mathrm{d}(\sigma_\Phi)_p(\boldsymbol{v}) \in T_{\sigma_\Phi(p)}G$ back to $\sigma_\Phi(p)^{-1}\mathrm{d}(\sigma_\Phi)_p(\boldsymbol{v}) \in \mathfrak{g}$. □

Theorem 50. *Let $\pi\colon P \to M$ be a principal G-bundle.*

(a) Let $\mathrm{Hor}(P) \hookrightarrow TP$ be a right-invariant horizontal distribution on P. Then $\omega \in \Omega^1(P, \mathfrak{g})$ defined by $\omega(\boldsymbol{X}) = X$, where $X \in \mathfrak{g}$ is the unique element such that $\boldsymbol{X} = \boldsymbol{X}^h + X^\#$, with $\boldsymbol{X}^h$ a field along $\mathrm{Hor}(P)$, is a connection 1-form.

(b) Let $\omega \in \Omega^1(P, \mathfrak{g})$ be a connection 1-form. Then $\mathrm{Hor}(P) = \ker\omega$ is a right-invariant horizontal distribution on P.

Furthermore, constructions (a) and (b) are inverses. Thus, we'll say that either $\mathrm{Hor}(P)$ or ω itself is an Ehresmann connection *for $P \to M$.*

Proof:

(a) Start with $\mathrm{Hor}(P)$. Smoothness of ω follows from the smoothness of the distribution, and it clearly is $\mathscr{C}^\infty(P)$-linear. So we have to check that ω defined in this way satisfies both axioms of the definition of a connection 1-form. Axiom (i) is trivial from the definition, while for axiom (ii) we argue as follows: take $g \in G$, $\boldsymbol{v} \in T_pP$, and write $\boldsymbol{v} = \boldsymbol{v}^h + X_p^\#$ for some $X \in \mathfrak{g}$ and $\boldsymbol{v}^h \in \mathrm{Hor}_p(P)$. Then by Proposition 43 (p. 30) we have that

$$\mathrm{d}(\mathrm{R}_g)_p(\boldsymbol{v}) = \mathrm{d}(\mathrm{R}_g)_p(\boldsymbol{v}^h) + \mathrm{d}(\mathrm{R}_g)_p(X_p^\#) = \mathrm{d}(\mathrm{R}_g)_p(\boldsymbol{v})^h + (g^{-1}Xg)^\#_{p\cdot g}.$$

Indeed, $\mathrm{d}(\mathrm{R}_g)_p(\boldsymbol{v}^h)$ has no vertical components, since we have the right-invariance $\mathrm{d}(\mathrm{R}_g)_p[\mathrm{Hor}_p(P)] = \mathrm{Hor}_{p\cdot g}(P)$. Thus, we conclude that

$$((\mathrm{R}_g)^*\omega)_p(\boldsymbol{v}) = \omega_{p\cdot g}(\mathrm{d}(\mathrm{R}_g)_p(\boldsymbol{v})) = g^{-1}Xg = \mathrm{Ad}(g^{-1})(\omega_p(\boldsymbol{v})),$$

as wanted.

(b) Start with $\omega \in \Omega^1(P, \mathfrak{g})$. Smoothness of $\ker\omega$ follows from smoothness of ω and the fact ω never vanishes (hence $\ker\omega$ has constant rank). Now, since for every $g \in G$ the map $\mathrm{Ad}(g^{-1})$ is an isomorphism, the relation

$$\omega_{p\cdot g}(\mathrm{d}(\mathrm{R}_g)_p(\boldsymbol{v})) = (\mathrm{R}_g^*\omega)_p(\boldsymbol{v}) = \mathrm{Ad}(g^{-1})(\omega_p(\boldsymbol{v}))$$

for all $\boldsymbol{v} \in T_pP$ says that $\boldsymbol{v} \in \mathrm{Hor}_p(P)$ if and only if $\mathrm{d}(\mathrm{R}_g)_p(\boldsymbol{v}) \in \mathrm{Hor}_{p\cdot g}(P)$, and thus $\mathrm{d}(\mathrm{R}_g)_p[\mathrm{Hor}_p(P)] = \mathrm{Hor}_{p\cdot g}(P)$ as wanted.

□

In view of the above result, we have a (somewhat clear) rephrasing of Proposition 49 in terms of horizontal distributions:

Proposition 51. *Let $P \to M$ be a principal G-bundle, $\omega \in \Omega^1(P, \mathfrak{g})$ be a connection 1-form, and $\Phi \in \mathscr{G}(P)$ be a gauge transformation. Then*

$$\mathrm{d}\Phi_p[\mathrm{Hor}_p^{\Phi^*\omega}(P)] = \mathrm{Hor}^{\omega}_{\Phi(p)}(P) \quad \textit{and} \quad \mathrm{d}\Phi_p[\mathrm{Ver}_p(P)] = \mathrm{Ver}_{\Phi(p)}(P).$$

Proof: Clearly $\mathrm{d}\Phi_p$ preserves dimensions (since it is an isomorphism), so $\pi \circ \Phi = \pi$ gives the result for the vertical distribution, and it suffices to verify one inclusion for the horizontal distribution. Take $\boldsymbol{v} \in \mathrm{Hor}_p^{\Phi^*\omega}(P)$. Then

$$\omega_{\Phi(p)}(\mathrm{d}\Phi_p(\boldsymbol{v})) = (\Phi^*\omega)_p(\boldsymbol{v}) = 0 \implies \mathrm{d}\Phi_p(\boldsymbol{v}) \in \mathrm{Hor}^{\omega}_{\Phi(p)}(P)$$

as wanted. □

The idea of seeing when $\mathrm{Hor}(P)$ is integrable in terms of ω naturally translates as $\omega \wedge \mathrm{d}\omega = 0$ (here we use the standard flat connection in the trivial bundle $P \times \mathfrak{g}$ to define $\mathrm{d}\omega$), which leads us to the idea of curvature form, to be discussed in the next section.

4 Tensorial and Curvature forms

Definition 52. Let $P \to M$ be a principal G-bundle equipped with a connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ and a representation $\rho\colon G \to \mathrm{GL}(V)$. We'll say that a k-form $\alpha \in \Omega^k(P, V)$ is:

(i) *pseudo-tensorial of type* ρ if $\mathrm{R}_g^*\alpha = \rho(g^{-1}) \circ \alpha$, for all $g \in G$.

(ii) *horizontal* if $\alpha(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k) = \boldsymbol{0}$ whenever at least one of the arguments $\boldsymbol{X}_i$ is vertical;

(iii) *tensorial of type* ρ if it is both pseudo-tensorial of type ρ and horizontal.

We'll set $\Omega^k_\rho(P, V) = \{\alpha \in \Omega^k(P, V) \mid \alpha \text{ is tensorial of type } \rho\}$.

Remark.

- All the representations here will be assumed finite-dimensional.
- Pseudo-tensoriality can be understood as equivariance: a k-form $\alpha \in \Omega^k(P, V)$ can be seen as a full map $\alpha\colon TP^{\oplus k} \to V$. But the action $P \circlearrowleft G$ induces a diagonal derivative action $TP^{\oplus k} \circlearrowleft G$, and we also have $V \circlearrowleft G$ given via the representation by $(v, g) \mapsto \rho(g^{-1})v$. So, α is pseudo-tensorial of type ρ if this full map $TP^{\oplus k} \to V$ is G-equivariant with respect to these two actions just described.
- Essentially, α is horizontal if it only cares about horizontal inputs.

Example 53. Any connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ is pseudo-tensorial of type Ad, but not tensorial (in fact, a connection 1-form is pretty much the opposite of being horizontal)! In contrast, if $\alpha \in \Omega^1_{\mathrm{Ad}}(P, \mathfrak{g})$, then $\omega + \alpha$ is again a connection 1-form, for the following reason: axiom (i) of Definition 46 (p. 33) holds since α is pseudo-tensorial, while axiom (ii) holds because α is horizontal. Conversely, the difference between two connection 1-forms is tensorial of type Ad. Thus, the space of connection 1-forms in a principal bundle is an affine space, and the underlying translation space is $\Omega^1_{\mathrm{Ad}}(P, \mathfrak{g})$.

Example 54. A cheap way to construct tensorial forms is just by considering pseudo-tensorial ones and taking horizontal components of all the arguments. More precisely, if $\alpha \in \Omega^k(P, V)$ is pseudo-tensorial, then $\widetilde{\alpha}$ defined by $\widetilde{\alpha}_p(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k) \doteq \alpha_p(\boldsymbol{v}_1^h, \ldots, \boldsymbol{v}_k^h)$, for $p \in P$ and vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in T_pP$, is tensorial. Of course, for every $i = 1, \ldots, k$, $\boldsymbol{v}_i^h \in \mathrm{Hor}_p(P)$ is the horizontal component of $\boldsymbol{v}_i \in T_pP$.

This new object $\Omega^k_\rho(P, V)$ is not completely alien, though:

Theorem 55. *Consider $P \to M$ be a principal G-bundle equipped with a connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ and a representation $\rho\colon G \to \mathrm{GL}(V)$. Then there is an isomorphism*

$$\Omega^k_\rho(P, V) \ni \alpha \overset{\cong}{\longmapsto} \alpha^\sharp \in \Omega^k(M, P \times_\rho V),$$

given by $\alpha^\sharp_x(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k) \doteq [p, \alpha_p(\widetilde{\boldsymbol{v}}_1, \ldots, \widetilde{\boldsymbol{v}}_k)]$, where the right side does not depend on the chosen point $p \in P_x$ and lifts $\widetilde{\boldsymbol{v}}_i \in T_pP$ of the $\boldsymbol{v}_i \in T_xM$ (i.e., $\mathrm{d}\pi_p(\widetilde{\boldsymbol{v}}_i) = \boldsymbol{v}_i$, for all indices $i = 1, \ldots, k$).

Proof: See [6]. □

Example 56. We have seen that for any principal G-bundle $P \to M$ with the trivial representation $G \to \mathrm{GL}(V)$, with $\rho(g) = \mathrm{Id}_V$ for all $g \in G$, we have $P \times_{\mathrm{Id}_V} V \cong M \times V$, and so

$$\Omega^k_{\mathrm{Id}_V}(P, V) = \{\alpha \in \Omega^k(P, V) \mid \mathrm{R}_g^*\alpha = \alpha\} \cong \Omega^k(M, V).$$

And what can we do with tensorial forms? If $\alpha \in \Omega^k(P.V)$ is pseudo-tensorial of type ρ, then $\mathrm{d}\alpha \in \Omega^k(P, V)$ is also pseudo-tensorial of type ρ, since d commutes with pull-backs and every $\rho(g^{-1})\colon V \to V$ is linear, by definition. The problem is that even if α is horizontal, $\mathrm{d}\alpha$ need not be. This begs for the cheap correction mentioned in Example 54 above:

Definition 57. Consider $P \to M$ be a principal G-bundle equipped with a connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$, and a vector space V. The *exterior covariant derivative* of a k-form $\alpha \in \Omega^k(P, V)$ is $\mathrm{D}\alpha \in \Omega^{k+1}(P, V)$ defined by

$$\mathrm{D}\alpha_p(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1}) \doteq \mathrm{d}\alpha_p(\boldsymbol{v}_1^h, \ldots, \boldsymbol{v}_{k+1}^h),$$

where, again, for each $i = 1, \ldots, k$ we have that $\boldsymbol{v}_i^h \in \mathrm{Hor}_p(P)$ is the horizontal component of $\boldsymbol{v}_i \in T_pP$.

Remark. The definition of D does not depend on any choice of representation. But if we're given a representation $\rho\colon G \to \mathrm{GL}(V)$, then we clearly have that this exterior covariant derivative restricts to a map $\mathrm{D}\colon \Omega^k_\rho(P, V) \to \Omega^{k+1}_\rho(P, V)$. But in a fashion similar to what happened in Proposition 13 (p. 9), it is not generally true that $\mathrm{D}^2 = 0$. This deviation will be measured by curvature (see Corollary 64 soon).

Definition 58. Let $P \to M$ be a principal G-bundle, and $\omega \in \Omega^1(P, \mathfrak{g})$ be a connection 1-form. The *curvature* (or *field strength*) of ω is $\Omega \in \Omega^2(P, \mathfrak{g})$ defined by $\Omega \doteq \mathrm{D}\omega$. We say that ω is *flat* if $\Omega = 0$.

Remark. In Physics, it is usual to denote connection 1-forms (gauge fields) by A and the curvature 2-form (field strength) by F_{A}. Once a local gauge $\psi\colon U \subseteq M \to P$ has been chosen, we can consider the pull-backs $\psi^*\mathsf{A} \in \Omega^1(U, \mathfrak{g})$ and $\psi^*(F_{\mathsf{A}}) \in \Omega^2(U, \mathfrak{g})$, which can be expressed in terms of local coordinates (x^j) for M on U and a basis (X_a) for $\mathfrak{g}$, by (omitting ψ^* and) writing $\mathsf{A} = A^a \otimes X_a$ and $F_{\mathsf{A}} = F^a \otimes X_a$, where A^a and F^a are real-valued local 1 and 2-forms, respectively. Then we have that

$$A_j = \mathsf{A}(\partial_j) = A^a_j X_a \in \mathfrak{g} \quad \text{and} \quad F_{jk} = F_{\mathsf{A}}(\partial_j, \partial_k) = F^a_{jk} X_a \in \mathfrak{g},$$

where $A^a_j = A^a(\partial_j)$ and $F^a_{jk} = F^a(\partial_j, \partial_k)$ are real-valued functions on U. The exterior covariant derivative is also sometimes denoted by d_{A} or ∇_{A}.

Proposition 59. *Let $P \to M$ be a principal G-bundle equipped with a connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$. Then $\mathrm{Hor}(P)$ is integrable if and only if $\Omega = 0$.*

Proof: Take $\boldsymbol{X}^h, \boldsymbol{Y}^h$ horizontal fields on P. So $\omega(\boldsymbol{X}^h) = \omega(\boldsymbol{Y}^h) = \boldsymbol{0}$ in $\mathfrak{g}$ and by definition of exterior derivative we get $\Omega(\boldsymbol{X}^h, \boldsymbol{Y}^h) = -\omega([\boldsymbol{X}^h, \boldsymbol{Y}^h])$, meaning that $\Omega = 0$ if and only if $\mathrm{Hor}(P)$ is closed under Lie brackets. □

Example 60.

(1) Indeed, for the trivial bundle $M \times G \to M$ and the toy-bundle $G \to \{*\}$ equipped with the Maurer-Cartan connection, we immediately get that $\mathrm{D}\omega = 0$ and $\mathrm{D}\Theta = 0$.

(2) Let $H \leq G$ be a closed subgroup and consider the $\mathfrak{m}$-reductive H-principal bundle $\pi\colon G \to G/H$, where $\mathfrak{m} \subseteq \mathfrak{g}$ is a $\mathrm{Ad}(H)$-invariant vector space complementary to $\mathfrak{h}$. The canonical connection is flat if and only if $\mathfrak{m}$ is in fact a subalgebra of $\mathfrak{g}$. In this case, if $M \leq G$ is a Lie subgroup with Lie algebra $\mathfrak{m}$, for all $a \in M$ we have $\mathrm{Hor}_a(G) = T_a M$.

Proposition 61 (Structure equation)**.** *Let $P \to M$ be a principal G-bundle equipped with a connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$. Then we have*

$$\Omega = \mathrm{D}\omega = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega].$$

Remark.

- Compare this with Proposition 16 (p. 11).
- In terms of the gauge theory notation introduced in the last remark, pulling this back to a relation between local forms on M, we have that

$$F_{jk} = \partial_j A_k - \partial_k A_j + [A_j, A_k],$$

with $[A_j, A_k] = 0$ if the structure group G is abelian (since this is a Lie bracket and not a commutator).

Proof: We can prove it by cases:

- Two horizontal fields: Since ω annihilates horizontal fields, there's nothing to check.
- Two vertical fields: the left side vanishes by definition of D. For the right side, note that for all $X, Y \in \mathfrak{g}$, we have $X^\#(\omega(Y^\#)) = X^\#(Y) = 0$, as Y is constant. Thus

$$\begin{aligned} \mathrm{d}\omega(X^\#, Y^\#) + \frac{1}{2}[\omega,\omega](X^\#, Y^\#) &= -\omega([X^\#, Y^\#]) + [\omega(X^\#), \omega(Y^\#)] \\ &= -\omega([X, Y]^\#) + [X, Y] \\ &= 0. \end{aligned}$$

- One horizontal field and one vertical field: The left side vanishes again. And for any horizontal field $\boldsymbol{X}^h$ and any $Y \in \mathfrak{g}$ we have $\boldsymbol{X}^h(\omega(Y^\#)) = \boldsymbol{X}^h(Y) = 0$ and $Y^\#(\omega(\boldsymbol{X}^h)) = Y^\#(0) = 0$, so that

$$\begin{aligned} \mathrm{d}\omega(\boldsymbol{X}^h, Y^\#) + \frac{1}{2}[\omega,\omega](\boldsymbol{X}^h, Y^\#) &= -\omega([\boldsymbol{X}^h, Y^\#]) + [\omega(\boldsymbol{X}^h), \omega(Y^\#)] \\ &= -\omega(\mathbf{0}) + [0, Y] \\ &= 0, \end{aligned}$$

as the bracket of a horizontal vector and a vertical vector is again vertical.

□

Corollary 62. *Let $P \to M$ be a principal G-bundle equipped with a connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$. Then $\mathrm{d}\Omega = [\Omega, \omega]$.*

Remark. Compare this with Corollary 17 (p. 12).

Proof: It is a direct computation:

$$\begin{aligned}\mathrm{d}\Omega &= \mathrm{d}\left(\mathrm{d}\omega + \frac{1}{2}[\omega,\omega]\right) = \frac{1}{2}\mathrm{d}[\omega,\omega] \\ &= \frac{1}{2}([\mathrm{d}\omega,\omega] - [\omega,\mathrm{d}\omega]) = [\mathrm{d}\omega,\omega] \\ &= \left[\Omega - \frac{1}{2}[\omega,\omega],\omega\right] = [\Omega,\omega],\end{aligned}$$

since[24] $[[\omega,\omega],\omega] = 0$. □

The above results begs the following question: when we have a representation $\rho\colon G \to \mathrm{GL}(V)$, how does D act in terms of d and ρ? To answer this, we'll slightly modify the operation $\circledast$ defined for algebra-bundle valued forms, in Section 1. We have the infinitesimal representation $\rho_* = \mathrm{d}\rho_e\colon \mathfrak{g} \to \mathfrak{gl}(V)$, which means that for $X \in \mathfrak{g}$ and $v \in V$, we have that $\rho_*(X)v \in V$. So, for $\alpha \in \Omega^k(P,\mathfrak{g})$ and $\beta \in \Omega^\ell(P,V)$, we define $\alpha \circledast \beta \in \Omega^{k+\ell}(P,V)$ by

$$(\alpha \circledast \beta)(\boldsymbol{X}_1,\ldots,\boldsymbol{X}_{k+\ell}) = \frac{1}{k!\ell!}\sum_{\sigma\in S_{k+\ell}} \mathrm{sgn}(\sigma)\rho_*\big(\alpha(\boldsymbol{X}_{\sigma(1)},\ldots,\boldsymbol{X}_{\sigma(k)})\big)\beta(\boldsymbol{X}_{\sigma(k+1)},\ldots,\boldsymbol{X}_{\sigma(k+\ell)}).$$

Theorem 63. *Let $P \to M$ be a principal G-bundle equipped with a connection 1-form $\omega \in \Omega^1(P,\mathfrak{g})$ and a representation $\rho\colon G \to \mathrm{GL}(V)$. If $\alpha \in \Omega^k_\rho(P,V)$, then*

$$\mathrm{D}\alpha = \mathrm{d}\alpha + \omega \circledast \alpha.$$

Proof: The proof is long and deals with several cases. See [6]. □

Remark. The fact that α is tensorial is crucial. For instance, with $\rho = \mathrm{Ad}$ in Proposition 61 above we see that in the formula for $\Omega = \mathrm{D}\omega$ there is a $1/2$ factor which is missing above. This is precisely because, as mentioned before, ω itself is not tensorial. But since Ω itself is tensorial, it follows from Corollary 62 (p. 43) that

$$\mathrm{D}\Omega = \mathrm{d}\Omega + [\omega,\Omega] = [\Omega,\omega] + [\omega,\Omega] = 0.$$

So, just like we have seen $\mathrm{d}^\nabla R^\nabla = 0$ for vector bundles, $\mathrm{D}\Omega = 0$ also deserves to be called a Bianchi identity. In gauge theory notation, we have $\mathrm{d}_A F_A = 0$.

[24] Computing it via the definition as a sum over $\sigma \in S_3$ gives us six terms, three of them corresponding to even permutations, and the remaining three terms to odd permutations: the Jacobi identity for the bracket in $\mathfrak{g}$ appears twice, and $0 - 0 = 0$. The fact that ω is a connection 1-form is irrelevant here, this holds for any $\mathfrak{g}$-valued 1-form.

The above theorem also allows us to precisely state how the curvature Ω measures the difference between D^2 and 0:

Corollary 64. *Let $P \to M$ be a principal G-bundle equipped with a connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ and a representation $\rho\colon G \to \mathrm{GL}(V)$. Then, for any $\alpha \in \Omega^k_\rho(P, V)$ we have that*

$$\mathrm{D}^2\alpha = \Omega \circledast \alpha.$$

In particular, ω is flat if and only if $\mathrm{D}^2 = 0$.

Remark. Compare this with the relation $((\mathrm{d}^\nabla)^2\psi)(X, Y) = R^\nabla(X, Y)\psi$ given in Proposition 13 (p. 9), in the setting of vector bundles.

Proof: We combine the expressions obtained for D so far to compute

$$\begin{aligned}
\mathrm{D}^2\alpha &= \mathrm{D}(\mathrm{d}\alpha + \omega \circledast \alpha) \\
&= \mathrm{d}(\mathrm{d}\alpha + \omega \circledast \alpha) + \omega \circledast (\mathrm{d}\alpha + \omega \circledast \alpha) \\
&= 0 + \mathrm{d}\omega \circledast \alpha - \omega \circledast \mathrm{d}\alpha + \omega \circledast \mathrm{d}\alpha + \omega \circledast (\omega \circledast \alpha) \\
&= \mathrm{d}\omega \circledast \alpha + \frac{1}{2}[\omega, \omega] \circledast \alpha \\
&= \Omega \circledast \alpha.
\end{aligned}$$

The step $2\omega \circledast (\omega \circledast \alpha) = [\omega, \omega] \circledast \alpha$ is true in general but easily verified in the case $k = 0$ (where $\alpha\colon P \to V$ is a smooth function), which in particular justifies the need for the factor $1/2$. ☐

Trying to follow the same train of thought as in Section 1, the next thing would be to recall Example 53 and establish a Palatini-like identity (Proposition 14, p. 10).

Proposition 65. *Let $P \to M$ be a principal G-bundle equipped with a connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ and a representation $\rho\colon G \to \mathrm{GL}(V)$. For any $\alpha \in \Omega^1_{\mathrm{Ad}}(P, \mathfrak{g})$, the curvature 2-forms F_ω and $F_{\omega+\alpha}$ are related via*

$$F_{\omega+\alpha} = F_\omega + \mathrm{D}\alpha + \frac{1}{2}[\alpha, \alpha].$$

Proof: The idea is to use the structure equation for curvature 2-forms together with Theorem 63 (p. 43) correctly, to recover the $\mathrm{D}\alpha$ term in the computation to be done. Noting that $[\omega, \alpha] = [\alpha, \omega]$ (since $(-1)^{1\cdot 1+1} = 1$), we have that

$$\begin{aligned}
F_{\omega+\alpha} &= \mathrm{d}(\omega + \alpha) + \frac{1}{2}[\omega + \alpha, \omega + \alpha] \\
&= \mathrm{d}\omega + \mathrm{d}\alpha + \frac{1}{2}([\omega, \omega] + [\omega, \alpha] + [\alpha, \omega] + [\alpha, \alpha]) \\
&= \mathrm{d}\omega + \frac{1}{2}[\omega, \omega] + \mathrm{d}\alpha + [\omega, \alpha] + \frac{1}{2}[\alpha, \alpha] \\
&= F_\omega + \mathrm{D}\alpha + \frac{1}{2}[\alpha, \alpha],
\end{aligned}$$

as wanted. ☐

Note that in the above calculation, the symbol D was actually used with two meanings, denoting both the covariant exterior derivative defined from ω and the one defined for $\omega + \alpha$. In situations like this, using gauge theory notation does have an advantage. To illustrate this, let's consider gauge transformations again:

Proposition 66 (Naturality). *Let $P \to M$ be a principal G-bundle, $\omega \in \Omega^1(P, \mathfrak{g})$ be a connection 1-form, and $\Phi \in \mathscr{G}(P)$ be a gauge transformation. If V is a vector space and $\alpha \in \Omega^k(P, V)$, then*

$$\Phi^*(\mathrm{D}^\omega \alpha) = \mathrm{D}^{\Phi^*\omega}(\Phi^*\alpha).$$

In particular, it follows that $\Phi^*(F_\omega) = F_{\Phi^*\omega}$.

Remark. Of course, this should be thought as "pull-backs under gauge transformations commute with D". By definition of $\Phi^*\omega$, of course $\Phi\colon (P, \Phi^*\omega) \to (P, \omega)$ becomes connection-preserving (and so it makes perfect sense for it to pull-back one curvature form to the other).

Proof: The key is to use Proposition 51 (p. 39) to conclude that $\mathrm{d}\Phi_p$ intertwines horizontal projections. And of course, there's no loss of generality in proving the result just for $k = 1$. If $p \in P$ and $\boldsymbol{v} \in T_pP$, compute:

$$\begin{aligned}(\Phi^*(\mathrm{D}^\omega \alpha))_p(\boldsymbol{v}) &= (\mathrm{D}^\omega \alpha)_{\Phi(p)}(\mathrm{d}\Phi_p(\boldsymbol{v})) = \mathrm{d}\alpha_{\Phi(p)}(\mathrm{d}\Phi_p(\boldsymbol{v})^h) \\ &= \mathrm{d}\alpha_{\Phi(p)}(\mathrm{d}\Phi_p(\boldsymbol{v}^h)) = \mathrm{d}(\Phi^*\alpha)_p(\boldsymbol{v}^h) \\ &= (\mathrm{D}^{\Phi^*\omega}(\Phi^*\alpha))_p(\boldsymbol{v}),\end{aligned}$$

as wanted. □

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