Projective spaces, the Fubini-Study metric and a little bit more

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This is yet another set of notes written to organize my thoughts (it's likely that I have written too much details at some points and/or have been repetitive, but experience says that shortest path is not always the simplest to follow). I have tried to keep this self-contained, within reason. The title and the section names should be self-explanatory this time. We will fix throughout this text (unless explicitly mentioned otherwise) a finite-dimensional complex vector space V, with $\dim_{\mathbb{C}} V = n \geq 1$. Some things to be done here work over $\mathbb{R}$ instead of $\mathbb{C}$, and some fact are true for spheres of any dimension (as opposed to odd-dimensional sphere as in Section 3). But enough talking, and let's get to action.

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1 Projective spaces

1.1 General features

Definition 1. The *projective space* of V is defined as

$$\mathrm{P}V \doteq \{L \subseteq V \mid L \text{ is a subspace of } V \text{ with } \dim L = 1\}.$$

One may also write $\mathrm{P}V = (V \setminus \{0\})/_\sim$, where the equivalence relation $\sim$ is defined by saying that $x \sim y$ if there is $\lambda \in \mathbb{C} \setminus \{0\}$ such that $y = \lambda x$. Or in other words, it is the orbit-space of the action $\mathbb{C} \setminus \{0\} \circlearrowright V \setminus \{0\}$. In particular, we have the canonical projection $\Pi\colon V \setminus \{0\} \to \mathrm{P}V$ (explictly given by $\Pi(x) = \mathbb{C}x$), which allows us to equip $\mathrm{P}V$ with the quotient topology. Properties of this topology can be easily established with the aid of a hermitian inner product on V.

Proposition 2. $\mathrm{P}V$ *is a compact and connected Hausdorff space.*

Proof: Fix a hermitian inner product $\langle \cdot, \cdot \rangle$ on V. Since every line in V may be spanned by a unit vector, if $\Sigma = \{x \in V \mid \|v\| = 1\}$ denotes the unit sphere of $(V, \langle \cdot, \cdot \rangle)$, we have that the restriction $\Pi|_\Sigma\colon \Sigma \to \mathrm{P}V$ is surjective – so that Σ being compact and connected implies that $\mathrm{P}V$ is also compact and connected. For Hausdorffness, we will show that $\sim \subseteq (V \setminus \{0\}) \times (V \setminus \{0\})$ is closed. So, assume that $x, y \in V \setminus \{0\}$ and that we have sequences $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$ in $V \setminus \{0\}$ with $x_n \sim y_n$ for all $n \geq 0$, with $x_n \to x$ and $y_n \to y$. Our goal is to show that $x \sim y$. So, for every $n \geq 0$, write $y_n = \lambda_n x_n$. Then $x \neq 0$ allows us to write

$$\lambda_n = \frac{\langle y_n, x_n \rangle}{\|x_n\|^2} \to \lambda \doteq \frac{\langle y, x \rangle}{\|x\|^2}.$$

So $(\lambda_n)_{n \geq 0} \subseteq \mathbb{C} \setminus \{0\}$ converges. But making $n \to +\infty$ in $y_n = \lambda_n x_n$ leads to $y = \lambda x$, so that $y \neq 0$ implies that $\lambda \neq 0$. Thus $x \sim y$, as wanted. $\square$

With this in place, $\mathrm{P}V$ could only be nicer if it turned out to be a manifold as well. Some things are still good in this world:

Proposition 3. $\mathrm{P}V$ *is a complex manifold (hence it is orientable), the quotient projection* $\Pi\colon V \setminus \{0\} \to \mathrm{P}V$ *becomes a holomorphic submersion, and* $\dim_{\mathbb{C}} \mathrm{P}V = \dim_{\mathbb{C}} V - 1$.

Proof: For every linear functional $f \in V^*$ let $U_f = \{L \in \mathrm{P}V \mid f[L] = \mathbb{C}\}$. Note that $\Pi^{-1}[U_f] = \{x \in V \setminus \{0\} \mid f(x) \neq 0\}$ is open in V, so that $U_f \subseteq \mathrm{P}V$ is open, by definition of quotient topology. This is true even if $f = 0$, in which case $U_f = \varnothing$. Then, for $f \neq 0$, define $\varphi_f\colon U_f \to f^{-1}(1)$ by setting

$$\varphi_f(L) = \frac{x}{f(x)},$$

where $x \in V \setminus \{0\}$ is any element with $L = \mathbb{C}x$. Since f is linear, this does not depend on the choice of x. In fact, let's see that φ_f is continuous. This is done by noting that since the subset $V \setminus \ker f \subseteq V \setminus \{0\}$ is open and saturated[1], the restricted projection $\Pi|_{V\setminus \ker f} \colon V \setminus \ker f \to U_f$ is also a quotient map. Thus, by the characteristic property of the quotient topology, φ_f is continuous because the map

$$V \setminus \ker f \ni x \mapsto \frac{x}{f(x)} \in f^{-1}(1)$$

is. Next, checking that φ_f is bijective is simple:

- If $L_1, L_2 \in \mathrm{P}V$ are such that $\varphi_f(L_1) = \varphi_f(L_2)$, choose $x_1, x_2 \in V \setminus \{0\}$ such that $L_1 = \mathbb{C}x_1$ and $L_2 = \mathbb{C}x_2$. So $x_1/f(x_1) = x_2/f(x_2)$ leads to $x_1 = (f(x_1)/f(x_2))x_2$, and thus $L_1 = L_2$, so φ_f is injective.

- Given $u \in f^{-1}(1)$, note that $u \neq 0$ and consider $\mathbb{C}u \in \mathrm{P}V$. Clearly $\varphi_f(\mathbb{C}u) = u$, so φ_f is surjective.

Then, we conclude[2] that φ_f is a homeomorphism. At this point, the manifold-atlas $\{(U_f, \varphi_f)\}_{f \in V^* \setminus \{0\}}$ shows that $\mathrm{P}V$ is a topological manifold. To proceed, we need to look at the transition maps. So, let $f, g \in V^* \setminus \{0\}$. Note that the transition map $\varphi_f \circ \varphi_g^{-1} \colon \varphi_g[U_f \cap U_g] \to \varphi_f[U_f \cap U_g]$ is given by

$$\varphi_f \circ \varphi_g^{-1}(u) = \varphi_f(\mathbb{C}u) = \frac{u}{f(u)},$$

which is manifestly holomorphic. Hence $\mathrm{P}V$ is a complex manifold and, by definition of this complex structure, every φ_f is now holomorphic. The last thing to check is that $\Pi \colon V \setminus \{0\} \to \mathrm{P}V$ becomes a holomorphic submersion, but this is surprisingly simple: Π is holomorphic because the local expressions for Π are holomorphic. To wit, considering the identity chart for the domain $V \setminus \{0\}$ and a chart (U_f, φ_f) for $\mathrm{P}V$, we just take $u \in V \setminus \{0\}$ and just compute

$$(\varphi_f \circ \Pi)(u) = \varphi_f(\mathbb{C}u) = \frac{u}{f(u)},$$

which is holomorphic. These charts were *meant* to work like this. And Π is a submersion because around every $L \in \mathrm{P}V$ there is a local holomorphic section for Π, namely, some φ_f. And if we replace $\mathbb{C}$ by $\mathbb{R}$ in everything done so far, we obtain a smooth manifold instead and Π becomes a smooth submersion. $\square$

Remark. When $V = \mathbb{C}^{n+1}$, we write $\mathbb{C}\mathrm{P}^n = \mathrm{P}(\mathbb{C}^{n+1})$. Using the also standard notation $\Pi(z^0, \ldots, z^n) = [z^0 : \cdots : z^n]$ for the quotient projection (to obtain the so-called *homogeneous coordinates* of $(z^0, \ldots, z^n)$), for each $0 \leq i \leq n$ we may consider the linear projection $f_i \colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}$ given by $f_i(z^0, \ldots, z^n) = z^i$. The associated

[1] This is a general fact about topology: if $\mathrm{p} \colon X \to Y$ is a quotient map, given $A \subseteq X$ it is not necessarily true that $\mathrm{p}|_A \colon A \to \mathrm{p}[A]$ is also a quotient map. It is true provided that A is saturated (i.e., $A = \mathrm{p}^{-1}[\mathrm{p}[A]]$) and open (or saturated and closed).

[2] Any continuous bijection from a compact space onto a Hausdorff space is automatically a homeomorphism.

chart domain is then $U_i \doteq U_{f_i} = \{[z^0 : \cdots : z^n] \in \mathbb{C}P^n \mid z^i \neq 0\}$, and so the chart $\varphi_i \doteq \varphi_{f_i} \colon U_i \to f_i^{-1}(1) \cong \mathbb{C}^n$ is given by

$$\varphi_i([z^0 : \cdots : z^n]) = \left(\frac{z^0}{z^i}, \ldots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \ldots, \frac{z^n}{z^i}\right).$$

Note that the identification $f_i^{-1}(1) \cong \mathbb{C}^n$ is given by deleting the i-th natural coordinate of $\mathbb{C}^{n+1}$. Again, everything adapts without issues to $\mathbb{R}P^n = P(\mathbb{R}^{n+1})$ in the real case (except $\mathbb{R}P^n$ is no longer orientable if n is even).

Example 4 (Projective groups)**.** Let $T \colon V \to V$ be an automorphism. Then T induces a *projective automorphism* $PT \colon PV \to PV$ by setting $PT(L) \doteq T[L]$. We claim that PT is holomorphic. Fix charts (U_f, φ_f) and (U_g, φ_g) for PV and compute

$$\varphi_f \circ PT \circ \varphi_g^{-1}(u) = \varphi_f \circ PT(\mathbb{C}u) = \varphi_f(T[\mathbb{C}u]) = \varphi_f(\mathbb{C}T(u)) = \frac{T(u)}{f(T(u))}.$$

Since the map $u \mapsto T(u)/f(T(u))$ is holomorphic, so is PT. Clearly we have that $P\mathrm{Id}_V = \mathrm{Id}_{PV}$, while if $S \colon V \to V$ is another automorphism, we also have the relation $P(T \circ S) = PT \circ PS$. This in particular implies that PT is biholomorphic with inverse $(PT)^{-1} = P(T^{-1})$, and we have a functor[3] $P \colon \mathsf{core}(\mathsf{Vect}_{\mathbb{C}}) \to \mathsf{PVect}_{\mathbb{C}}$, where $\mathsf{PVect}_{\mathbb{C}}$ is the category whose objects are complex projective spaces and the morphisms are holomorphic maps between them. We then define the *projective general linear group* as $\mathrm{PGL}(V) \doteq \{PT \mid T \in \mathrm{GL}(V)\}$. For one more concrete characterization of this group, assume that $PT = PS$, for $T, S \in \mathrm{GL}(V)$. This means that for every $x \in V \setminus \{0\}$ there is $\lambda(x) \in \mathbb{C}$ such that $T(x) = \lambda(x)S(x)$, since $T(x) \in T[\mathbb{C}x] = S[\mathbb{C}x]$. This gives us a function $\lambda \colon V \setminus \{0\} \to \mathbb{C} \setminus \{0\}$. Now we'll argue that λ is constant (and in particular may be extended to the whole of V). For $z \in \mathbb{C} \setminus \{0\}$, we have that

$$T(zx) = \lambda(zx)S(zx) \implies z\lambda(x)S(x) = z\lambda(zx)S(x) \implies \lambda(zx) = \lambda(x).$$

So, λ is constant on complex rays. On the other hand, given $x, y \in V \setminus \{0\}$, *linearly independent*, we have that

$$T(x+y) = \lambda(x+y)S(x+y) \Longrightarrow \lambda(x)S(x) + \lambda(y)S(y) = \lambda(x+y)S(x) + \lambda(x+y)S(y),$$

leading to $\lambda(x+y) = \lambda(x) = \lambda(y)$ since $\{S(x), S(y)\}$ is also linearly independent. This implies that λ is a constant, as wanted. We conclude that $PT = PS$ if and only if $T = \lambda S$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ (the converse is clear). Writing the center of $\mathrm{GL}(V)$ as $\mathrm{ZGL}(V) \doteq \{\lambda\,\mathrm{Id}_V \mid \lambda \in \mathbb{C} \setminus \{0\}\} \subseteq \mathrm{GL}(V)$, we conclude that

$$\mathrm{PGL}(V) = \frac{\mathrm{GL}(V)}{\mathrm{ZGL}(V)}.$$

Similarly, we may restrict ourselves subgroups of $\mathrm{GL}(V)$:

[3] Recall that for any category C, the category $\mathsf{core}(\mathsf{C})$ is defined by setting $\mathsf{Obj}(\mathsf{core}(\mathsf{C})) = \mathsf{Obj}(\mathsf{C})$ and $\mathrm{Hom}_{\mathsf{core}(\mathsf{C})}(A, B) = \{f \in \mathrm{Hom}_{\mathsf{C}}(A, B) \mid f \text{ is an isomorphism}\}$, for any objects $A, B \in \mathsf{Obj}(\mathsf{C})$.

- Consider the special linear group $\mathrm{SL}(V)$, and the *projective special linear group* $\mathrm{PSL}(V) \doteq \{\mathrm{P}T \mid T \in \mathrm{SL}(V)\}$. Noting that $\det(\lambda\,\mathrm{Id}_V) = \lambda^n$ and imposing that $\det T = \det S = 1$ in the above discussion, we may also write that
$$\mathrm{PSL}(V) = \frac{\mathrm{SL}(V)}{\mathrm{ZSL}(V)},$$
where $\mathrm{ZSL}(V) \doteq \{\lambda\,\mathrm{Id}_V \mid \lambda^n = 1\}$ is identified with the group of n-th roots of unity.

- Consider the orthogonal group $\mathrm{O}(V,Q) \doteq \{T \in \mathrm{GL}(V) \mid T^*Q = Q\}$ associated to a given quadratic form $Q\colon V \times V \to \mathbb{C}$. Then the associated *projective orthogonal group* is $\mathrm{PO}(V,Q) \doteq \{\mathrm{P}T \mid T \in \mathrm{O}(V,Q)\}$. This time $\mathrm{ZO}(V,Q) \doteq \{\pm\mathrm{Id}_V\}$ and so
$$\mathrm{PO}(V,Q) = \frac{\mathrm{O}(V,Q)}{\{\pm\mathrm{Id}_V\}}.$$

- Consider the unitary group $\mathrm{U}(V,\langle\cdot,\cdot\rangle) \doteq \{T \in \mathrm{GL}(V) \mid T^*\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle\}$ associated to a given hermitian inner product $\langle\cdot,\cdot\rangle$ on V. By now you should expect the *projective unitary group* to be $\mathrm{PU}(V,\langle\cdot,\cdot\rangle) \doteq \{\mathrm{P}T \mid T \in \mathrm{U}(V,\langle\cdot,\cdot\rangle)\}$ and, since $\mathrm{ZU}(V,\langle\cdot,\cdot\rangle) = \{\lambda\,\mathrm{Id}_V \mid |\lambda| = 1\} \cong \mathbb{S}^1 \cong \mathrm{U}(1)$, we may write
$$\mathrm{PU}(V,\langle\cdot,\cdot\rangle) = \frac{\mathrm{U}(V,\langle\cdot,\cdot\rangle)}{\mathrm{U}(1)}.$$

Remark. When T is not necessarily injective, it will not be defined on the whole $\mathrm{P}V$. But an inclusion of vector spaces induces an inclusion of the corresponding projective spaces (see Example 6 below), and so we may still consider the induced projective map $\mathrm{P}T\colon \mathrm{P}V \setminus \mathrm{P}\ker T \to \mathrm{P}V$ defined in the same way as above. This gives a functor $\mathrm{P}\colon \mathsf{Vect}_{\mathbb{C}} \to \mathsf{PVect}_{\mathbb{C}}$, if now we consider the morphisms on $\mathsf{PVect}_{\mathbb{C}}$ to be *partial*[4] holomorphic maps between complex projective spaces.

Example 5 (A taste of duality)**.** Consider a hyperplane $H \subseteq V^*$. Since we have $\dim H = \dim V - 1$, the associated polar space[5] $H_0 = \{x \in V \mid H(x) = 0\}$ has $\dim H_0 = 1$. Thus $H_0 \in \mathrm{P}V$. Conversely, given any line $L \in \mathrm{P}V$, the polar space $L^0 = \{f \in V^* \mid f|_L = 0\}$ is a hyperplane in V^*. Moreover, $(L^0)_0 = L$ and $(H_0)^0 = H$, which shows that $\mathrm{P}V$ is with bijective correspondence with the collection of all hyperplanes in V^*. When discussing Grassmannians in Section 4, we will see in Example 62 that in general $\mathrm{Gr}_k(V) \cong \mathrm{Gr}_{n-k}(V^*)$, where $n = \dim V$. In general, we have that for any subspace $W \subseteq V$, $\mathrm{P}W \subseteq \mathrm{P}V$ corresponds to $\mathrm{P}(W^0) \subseteq \mathrm{P}(V^*)$, and the relation $\dim W + \dim W^0 = \dim V$ translates into $\dim \mathrm{P}W + \dim \mathrm{P}(W^0) = \dim \mathrm{P}V - 1$ on the projective level.

Example 6 (About incidence relations)**.** Let $W \subseteq V$ be a subspace. Then every line contained in W is also contained in V, so that $\mathrm{P}W \subseteq \mathrm{P}V$. We will call $\mathrm{P}W$ a *projective subspace* of $\mathrm{P}V$. We will say that $\mathrm{P}W$ is a *projective line* or a *projective plane* according to whether $\dim W = 2$ or $\dim W = 3$ (because then $\dim \mathrm{P}W = 1$ or $\dim \mathrm{P}W = 2$, respectively[6]). Some axioms of projective geometry may be verified using this language. For

[4] That is, only defined in subsets of the full domain.
[5] Here, writing $H(x) = 0$ is a shorthand for "$f(x) = 0$ for all $f \in H$".
[6] One can go further and say that lines in V are "projective points" in $\mathrm{P}V$.

instance, let's show that any two distinct projective points $L_1, L_2 \in \mathrm{P}V$ are contained in a unique projective line. Choose vectors $x_1, x_2 \in V \setminus \{0\}$ (necessarily linearly independent) such that $L_1 = \mathbb{C}x_1$ and $L_2 = \mathbb{C}x_2$, and let $W = \mathbb{C}x_1 \oplus \mathbb{C}x_2$. Then $\mathrm{P}W$ is a projective line passing through both L_1 and L_2. If $\mathrm{P}W'$ is another projective line passing through L_1 and L_2, then $x_1, x_2 \in W'$ leads to $W \subseteq W'$. Now $\dim W = \dim W'$ forces $W = W'$. This idea will be further explored in Example 8 below.

Example 7 (Cellular decomposition)**.** Let

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V_n = V$$

be a flag of subspaces of V, i.e., $\dim_{\mathbb{C}} V_r = r$ for $0 \leq r \leq n$. Writing $V_{n-1} = \ker f$ for some non-zero linear functional $f \in V^*$, we see that a line in V is either contained in V_{n-1} or hits (in a single point) the affine hyperplane $f^{-1}(1)$, which is diffeomorphic to V_{n-1} itself (via a translation). This shows that $\mathrm{P}V \cong V_{n-1} \sqcup \mathrm{P}V_{n-1}$. Proceeding inductively, this shows that $\mathrm{P}V \cong V_{n-1} \sqcup V_{n-2} \sqcup \cdots \sqcup V_1 \sqcup \{0\}$. This turns out to be an actual cellular decomposition for $\mathrm{P}V$, which allows us to compute the homology of the cellular complex with alternating $\mathbb{Z}$'s and 0's,

$$\mathbb{Z} \to 0 \to \mathbb{Z} \to \cdots \to \mathbb{Z} \to 0 \to \mathbb{Z},$$

to conclude that $H_{2k}(\mathrm{P}V) = \mathbb{Z}$ and $H_{2k-1}(\mathrm{P}V) = 0$, for $0 \leq k \leq n$ (bearing in mind that $\dim_{\mathbb{R}} \mathrm{P}V = 2n$). In particular, we see that $\mathbb{C}\mathrm{P}^1 \cong \mathbb{C} \sqcup \{\infty\} \cong \mathbb{S}^2$.

Example 8 (Comparisons with Linear Algebra)**.** If $W_1, W_2 \subseteq V$ are subspaces, then we have seen that $\mathrm{P}W_1, \mathrm{P}W_2 \subseteq \mathrm{P}V$. Another subspace is the intersection $W_1 \cap W_2$, and we have seen that $\mathrm{P}(W_1 \cap W_2) = \mathrm{P}W_1 \cap \mathrm{P}W_2$. This is a particular instance of a more general phenomenon, namely, that if $(W_i)_{i \in I}$ is any family of subspaces of V, then

$$\mathrm{P}\bigg(\bigcap_{i \in I} W_i\bigg) = \bigcap_{i \in I} \mathrm{P}W_i.$$

One particular consequence of this is that given any subset $S \subseteq \mathrm{P}V$, there is the smallest projective subspace of $\mathrm{P}V$ containing S:

$$\langle S \rangle_{\mathrm{proj}} = \bigcap \{\mathrm{P}W \mid W \text{ is a subspace of } V \text{ and } B \subseteq \mathrm{P}W\}.$$

The inverse image $\Pi^{-1}[S] \subseteq V$ also has a linear span $\langle \Pi^{-1}[S] \rangle_{\mathrm{lin}}$, and the relation $\langle S \rangle_{\mathrm{proj}} = \mathrm{P}\big(\langle \Pi^{-1}[S] \rangle_{\mathrm{lin}}\big)$ holds. Similarly, noting that $W_1 + W_2 = \mathrm{span}(W_1 \cup W_2)$ gives that $\mathrm{P}(W_1 + W_2) = \langle \mathrm{P}W_1 \cup \mathrm{P}W_2 \rangle_{\mathrm{proj}}$. Although $\mathrm{P}W_1 + \mathrm{P}W_2$ does not make sense (as addition is not defined in $\mathrm{P}V$), we may *define* this to mean $\langle \mathrm{P}W_1 \cup \mathrm{P}W_2 \rangle_{\mathrm{proj}}$. With this in place, since projective dimensions are just the usual vector space dimensions subtracted by 1, we recover the following formula for any two subspaces $W_1, W_2 \subseteq V$:

$$\dim(\mathrm{P}W_1 + \mathrm{P}W_2) + \dim(\mathrm{P}W_1 \cap \mathrm{P}W_2) = \dim \mathrm{P}W_1 + \dim \mathrm{P}W_2,$$

provided we set $\dim \varnothing = -1$ (i.e., the projective dimension of $\varnothing$ is $0 - 1 = -1$). This gives another proof that there is a unique projective line passing through two distinct projective points $L_1, L_2 \in \mathrm{P}V$: for $\dim(\{L_1\} \cap \{L_2\}) = -1$, $\dim\{L_1\} = \dim\{L_2\} = 0$ and so $\dim(\{L_1\} + \{L_2\}) = 1$.

All of those examples above are meant to help us gain some intuition on basic projective geometry, but they barely touch the manifold structure of $\mathrm{P}V$. The next step is understanding the tangent spaces to $\mathrm{P}V$. A key object will be the derivative $\mathrm{d}\Pi_x\colon V \to T_{\mathbb{C}x}(\mathrm{P}V)$.

Proposition 9. *Let $L \in \mathrm{P}V$. Then the map*

$$\mathrm{Hom}(L, V/L) \ni H \mapsto \mathrm{d}\Pi_x(\widetilde{H}x) \in T_L(\mathrm{P}V)$$

is an isomorphism, where $\widetilde{H}\colon L \to V$ is any linear lift of H and $x \in V$ is such that $L = \mathbb{C}x$. In short, we have that $T_L(\mathrm{P}V) = \mathrm{Hom}(L, V/L)$.

Proof: Since $\dim \mathrm{Hom}(L, V/L) = \dim \mathrm{P}V = \dim V - 1$, it suffices to show that the above map is well-defined and surjective.

- It is independent of the choice of linear lift $\widetilde{H}$. For if $\widetilde{H}_1, \widetilde{H}_2\colon L \to V$ are two linear lifts of H, we have that $\widetilde{H}_2 = \widetilde{H}_1 + B$, where $B\colon L \to L$ is linear. This implies that $\mathrm{d}\Pi_x(\widetilde{H}_2 x) = \mathrm{d}\Pi_x(\widetilde{H}_1 x) + \mathrm{d}\Pi_x(Bx)$, but we have that

  $$\mathrm{d}\Pi_x(Bx) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Pi(x + tBx) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} L = 0,$$

  since for t small enough we have that $x + tBx \neq 0$ and $x + tBx$ spans L.

- It is independent of the choice of non-zero $x \in L$. For if $y \in V \setminus \{0\}$ is another vector such that $L = \mathbb{C}y$, then $y = \lambda x$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. The multiplication map by this λ, which we'll also denote by $\lambda\colon V \to V$, is linear. So $\Pi \circ \lambda = \Pi$. The chain rule implies that $\mathrm{d}\Pi_{\lambda x} \circ \lambda = \mathrm{d}\Pi_x$. Evaluating at $\widetilde{H}x$ gives that

  $$\mathrm{d}\Pi_x(\widetilde{H}x) = \mathrm{d}\Pi_{\lambda x}(\lambda \widetilde{H}x) = \mathrm{d}\Pi_y(\widetilde{H}y)$$

  as wanted, since $\widetilde{H}$ is linear.

It remains to show that this assignment is surjective, and this indirectly follows from the Π being a surjective submersion, as follows: given $\boldsymbol{v} \in T_L(\mathrm{P}V)$ and choosing non-zero $x \in V$ with $L = \mathbb{C}x$, there is $v \in V$ with $\boldsymbol{v} = \mathrm{d}\Pi_x(v)$. Now define $H\colon L \to V/L$ by $Hx = v + L$ and extend linearly. One linear lift clearly is $\widetilde{H}\colon L \to V$ given by extending $\widetilde{H}(x) = v$ linearly. Now compute $\mathrm{d}\Pi_x(\widetilde{H}x) = \mathrm{d}\Pi_x(v) = \boldsymbol{v}$ as wanted. $\square$

Remark.

- In practice, if $L\colon (-\epsilon, \epsilon) \to \mathrm{P}V$ is a smooth curve of lines such that $L(0) = L$ and $L'(0) = H$, then $Hx = x'(0) + L$, where $x\colon (-\epsilon, \epsilon) \to V$ is a curve with $x(0) = x$ and $x(t) \in L(t)$ for all $t \in (-\epsilon, \epsilon)$. This does not depend on the choice of curve, and the idea is as follows: write $W(t) = \ker A(t)$ where each $A(t)\colon V \to V$ is a linear map for each t, and differentiate $A(t)x(t) = 0$ at $t = 0$ to get $A'(0)x + A(0)x'(0) = 0$, so that $x'(0)$ is determined from x up to elements in $\ker A(0) = W$. Thus the map $x \mapsto x'(0)$ is well-defined and linear. Also, when V has some extra structure (e.g., a hermitian inner product), we may identify V/L with something else and avoid quotients entirely (such an identification would correspond to a choice of "canonical" lift, meaning that $T_L(\mathrm{P}V) \cong \mathrm{Hom}(L, L^\perp)$; for instance, we'll do this in Example 11 in what follows). See [10] for more details.

- This sort of reasoning allows us to compute differentials as well. Let N be any smooth manifold and $f\colon \mathrm{P}V \to N$ be a smooth map. Define $\widetilde{f}\colon V\setminus\{0\} \to N$ by $\widetilde{f}(x) \doteq f(\mathbb{C}x)$ and note that $\widetilde{f}$ is homogeneous of degree 0 (conversely, any degree 0 homogeneous map defined on $V\setminus\{0\}$ passes to the quotient as a map on $\mathrm{P}V$). Moreover, since f is smooth, so is $\widetilde{f}$. Given $L \in \mathrm{P}V$, our goal in several examples to follow will be to compute $\mathrm{d}f_L\colon T_L(\mathrm{P}V) \to T_{f(L)}N$. For this, we will use the relation $\widetilde{f} = f\circ\Pi$ together with the chain rule. Namely, given $H \in T_L(\mathrm{P}V)$, the relation $\mathrm{d}f_L(H) = \mathrm{d}\widetilde{f}_x(\widetilde{H}x)$ holds, where $\widetilde{H}\colon L \to V$ is any linear lift of H and $x \in V\setminus\{0\}$ is any vector such that $L = \mathbb{C}x$. Indeed, H corresponds to $\mathrm{d}\Pi_x(\widetilde{H}x)$, so that differentiating $\widetilde{f} = f\circ\Pi$ and evaluating at $\widetilde{H}x$ gives that $\mathrm{d}\widetilde{f}_x(\widetilde{H}x) = \mathrm{d}f_L(\mathrm{d}\Pi_x(\widetilde{H}x)) = \mathrm{d}f_L(H)$ as wanted, and this is automatically independent of the choices of $\widetilde{H}$ and x. If V is equipped with a hermitian inner product one may repeat the strategy with the restriction $\Pi|_\Sigma\colon \Sigma \to \mathrm{P}V$, where Σ is the unit sphere, we take any $x \in L\cap\Sigma$ and take $\widetilde{H} = H\colon L \to L^\perp$, as mentioned in the above point.

- Assume again that V is equipped with a hermitian product $\langle\cdot,\cdot\rangle$ and fix a chart (U_f,φ_f) for $\mathrm{P}V$. Given $H \in T_L(\mathrm{P}V)$, with $L \in U_f$. If $t \mapsto L(t)$ is a curve in U_f with $L(0) = L$ and $L'(0) = H$, writing $L(t) = \mathbb{C}x(t)$ for a curve $t \mapsto x(t)$ with $x(0) = x$, we compute $\mathrm{d}(\varphi_f)_L\colon T_L(\mathrm{P}V) \to T_{\varphi_f(L)}f^{-1}(1) \cong \ker f$ as

$$\mathrm{d}(\varphi_f)_L(H) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\frac{x(t)}{f(x(t))} = \frac{f(x)Hx - f(Hx)x}{f(x)^2} = \frac{Hx}{f(x)} - f\left(\frac{Hx}{f(x)}\right)\frac{x}{f(x)}$$

  with the quotient rule. It we denote the above by $v \in \ker f$ and project onto $(\mathbb{C}x)^\perp$ (using that H takes values in $L^\perp$), we obtain

$$\frac{Hx}{f(x)} = v - \frac{\langle v,x\rangle}{\langle x,x\rangle}x.$$

  This is how one expresses the isomorphism $H \mapsto v$ in coordinates.

Example 10. Given $T \in \mathrm{GL}(V)$, we may consider $\mathrm{P}T \in \mathrm{PGL}(V)$. Let's compute the derivative $\mathrm{d}(\mathrm{P}T)_L\colon T_L(\mathrm{P}V) \to T_{T[L]}(\mathrm{P}V)$. Given $H \in T_L(\mathrm{P}V) = \mathrm{Hom}(L, V/L)$, we want to define $\mathrm{d}(\mathrm{P}T)_L(H) \in T_{T[L]}(\mathrm{P}V) = \mathrm{Hom}(T[L], V/T[L])$. For this, note that T passes to the quotient as an isomorphism $\widetilde{T}\colon V/L \to V/T[L]$. So we'll have that

$$\mathrm{d}(\mathrm{P}T)_L(H) = \widetilde{T}\circ H\circ\big(T^{-1}|_{T[L]}\big).$$

To wit, using the preceding remark we may proceed as follows: take a smooth curve $L\colon (-\epsilon,\epsilon) \to \mathrm{P}V$ such that $L(0) = L$ and $L'(0) = H$. Take $y \in T[L]$ and consider a curve $y\colon(-\epsilon,\epsilon) \to V$ such that $y(0) = y$ and $y(t) \in T[L(t)]$ for all t. Then the inverted curve $t \mapsto T^{-1}(y(t))$ is a curve in V for which $T^{-1}(y(t)) \in L(t)$ for all t, and thus we have that $H(T^{-1}(y)) = T^{-1}(y'(0)) + L$. This means that $\widetilde{T}(H(T^{-1}(y))) = y'(0) + T[L]$. But the expression on the right is precisely $\mathrm{d}(\mathrm{P}V)_L(H)(y)$, by the very same principle.

Example 11 (Projections)**.** Assume that V is equipped with a hermitian[7] inner product $\langle\cdot,\cdot\rangle$. Fix a unit vector $u\in\Sigma$ and define $F\colon \mathrm{P}V\to V$ by $F(L)=\mathrm{pr}_L u$. We want to compute the derivative $\mathrm{d}F_L\colon T_L(\mathrm{P}V)\to V$. For this, note that $\langle\cdot,\cdot\rangle$ gives an identification $V/L\cong L^\perp$, so we may regard $H\in T_L(\mathrm{P}V)$ as a linear map $H\colon L\to L^\perp$, which may be thought of a "canonical lift" $\widetilde{H}$ which just happens to take values in $L^\perp$ (in other words, among all lifts of H there is only one taking values in $L^\perp$, and we fix this one once and for all). Take a unit vector $x\in V$ such that $L=\mathbb{C}x$, so that $\mathrm{pr}_L u=\langle u,x\rangle x$. Differentiate with respect to the variable x to get

$$\begin{aligned}\mathrm{d}F_L(H)&=\langle u,Hx\rangle x+\langle u,x\rangle Hx=\langle u,Hx\rangle x+H(\langle u,x\rangle x)\\&=\langle \mathrm{pr}_{L^\perp}u,Hx\rangle\, x+H\,\mathrm{pr}_L u\\&=\langle H^*\mathrm{pr}_{L^\perp}u,x\rangle x+H\,\mathrm{pr}_L u\\&=H^*\mathrm{pr}_{L^\perp}u+H\,\mathrm{pr}_L u,\end{aligned}$$

since H is linear and $H^*\colon L^\perp\to L$ denotes the adjoint of H. As a consequence, consider another function $G\colon \mathrm{P}V\to\mathbb{R}$ given by $G(L)=\|\mathrm{pr}_L(u)\|^2$. Since $G(L)=\|F(L)\|^2$, the chain rule gives that

$$\begin{aligned}\mathrm{d}G_L(H)&=2\,\mathrm{Re}\langle \mathrm{d}F_L(H),F(L)\rangle\\&=2\,\mathrm{Re}\,\langle H^*\mathrm{pr}_{L^\perp}u+H\,\mathrm{pr}_L u,\mathrm{pr}_L u\rangle\\&=2\,\mathrm{Re}\,\langle H^*\mathrm{pr}_{L^\perp}u,\mathrm{pr}_L u\rangle\\&=2\,\mathrm{Re}\langle H\,\mathrm{pr}_L u,\mathrm{pr}_{L^\perp}u\rangle\\&=2\,\mathrm{Re}\langle H\,\mathrm{pr}_L u,u\rangle.\end{aligned}$$

This should give us a basic grasp on tangent vectors. A vector field $\boldsymbol{X}\in\mathfrak{X}(\mathrm{P}V)$, in turn, is a map that takes $L\in\mathrm{P}V$ to a linear map $\boldsymbol{X}_L\colon L\to V/L$. In other words, it is a collection of linear maps with varying domain and codomain. But note that the union of these maps, seen as a map $V\to V$, is not linear (unless $\boldsymbol{X}=\mathbf{0}$), but just homogeneous of degree 1. Here's one important class of examples:

Example 12 (Linear fields)**.** Let $S\colon V\to V$ be any endomorphism. It can be regarded as a vector field in the manifold $V\setminus\{0\}$. We claim that it is projectable onto $\mathrm{P}V$. To wit, a vector field is projectable under a surjective submersion if and only if its flow is projectable. The flow of the linear field S is, not surprisingly, $\Phi_{t,S}(x)=\exp(tS)x$. Since $\exp(tS)$ is non-singular (as $\det\exp(tS)=\mathrm{e}^{t\,\mathrm{tr}\,(S)}\neq 0$), we have that

$$\Pi\circ\Phi_{t,S}(x)=\Pi(\exp(tS)x)=\mathbb{C}\exp(tS)x=(\mathrm{P}\exp(tS))(\mathbb{C}x)=(\mathrm{P}\exp(tS)\circ\Pi)(x)$$

for every $x\in V\setminus\{0\}$. So the flow of S is projectable.

Now, one important bundle built from $\mathrm{P}V$:

Definition 13. The *tautological line bundle* over $\mathrm{P}V$ is given by

$$\mathscr{L}\doteq\{(L,x)\mid L\in\mathrm{P}V\text{ and }x\in L\}\subseteq\mathrm{P}V\times V.$$

The bundle projection $\pi\colon\mathscr{L}\to\mathrm{P}V$ is given by $\pi(L,x)=L$.

[7]We'll take it to be linear in the first entry and anti-linear in the second entry.

Remark. The name "tautological" is because the fiber over the point $L \in \mathrm{P}V$ is the vector space L itself.

The word "bundle" has to be justified, and we'll already illustrate one application:

Proposition 14. *The tautological line bundle $\pi\colon \mathscr{L} \to \mathrm{P}V$ is a holomorphic line bundle and moreover we have that $T(\mathrm{P}V) \cong \mathrm{Hom}(\mathscr{L}, (\mathrm{P}V \times V)/\mathscr{L})$.*

Proof: We'll construct a VB-atlas for $\mathscr{L}$ from the manifold-atlas $\{(U_f, \varphi_f)\}_{f \in V^* \setminus \{0\}}$ for $\mathrm{P}V$. Note that for $L \in U_f$, $\dim L = 1$ implies that $f|_L\colon L \to \mathbb{C}$ is an isomorphism. With this, we define a trivialization $\Phi_f\colon \pi^{-1}[U_f] \to U_f \times \mathbb{C}$ by $\Phi_f(L, x) = (L, f(x))$. The inverse is $\Phi_f^{-1}\colon U_f \times \mathbb{C} \to \pi^{-1}[U_f]$ given by $\Phi_f^{-1}(L, \lambda) = (L, \lambda x/f(x))$, where $x \in L \setminus \{0\}$ is any vector (the normalization with $f(x)$ in the denominator makes the end result independent of the choice of x, as expected). And just like we had for the transitions $\varphi_f \circ \varphi_g^{-1}$, in this case we have that transition map between trivializations, $\Phi_f \circ \Phi_g^{-1}\colon (U_f \cap U_g) \times \mathbb{C} \to (U_f \cap U_g) \times \mathbb{C}$, is given by

$$(\Phi_f \circ \Phi_g^{-1})(L, \lambda) = \left(L, \frac{\lambda f(x)}{g(x)}\right),$$

and this is manifestly holomorphic. This shows that $\mathscr{L} \to \mathrm{P}V$ is a holomorphic line bundle. Lastly, we just write that

$$T(\mathrm{P}V) = \bigsqcup_{L \in \mathrm{P}V} T_L(\mathrm{P}V) \cong \bigsqcup_{L \in \mathrm{P}V} \mathrm{Hom}(L, V/L) = \mathrm{Hom}(\mathscr{L}, (\mathrm{P}V \times V)/\mathscr{L}),$$

as wanted. □

Remark. Local trivializations for a vector bundle are equivalent to local frames. For line bundles, a local frame is just a non-vanishing local section. In the above case, each $f \in V^* \setminus \{0\}$ gives rise to a local section $e_f \in \Gamma_{U_f}(\mathscr{L})$ by $e_f(L) = (L, x/f(x))$, where $x \in L \setminus \{0\}$ is any vector. And, as expected, if $g \in V^* \setminus \{0\}$ is another functional, we have the relation $e_g(L) = (f(x)/g(x))e_f(L)$, where again the choice of $x \in L \setminus \{0\}$ does not matter.

One eventually useful characterization of sections of $\mathscr{L}$ is given in the next:

Proposition 15. $\Gamma(\mathscr{L}) \cong \{\mu\colon V \setminus \{0\} \to \mathbb{C} \mid \mu$ *is smooth and homogeneous of degree* $-1\}$.

Proof: The construction is as follows: start with a section $\psi \in \Gamma(\mathscr{L})$. This is a map $\psi\colon \mathrm{P}V \to \mathrm{P}V \times V$ of the form $\psi(L) = (L, \widetilde{\psi}(L))$, where $\widetilde{\psi}\colon \mathrm{P}V \to V$ in addition satisfies $\widetilde{\psi}(L) \in L$. Then we may define $\widetilde{\widetilde{\psi}}\colon V \setminus \{0\} \to V$ by $\widetilde{\widetilde{\psi}}(x) = \widetilde{\psi}(\mathbb{C}x)$, which is homogeneous of degree 0, and now satisfies $\widetilde{\widetilde{\psi}}(x) \in \mathbb{C}x$. This means that there is $\mu_\psi(x) \in \mathbb{C}$ such that $\widetilde{\widetilde{\psi}}(x) = \mu_\psi(x)x$. This defines a function $\mu_\psi\colon V \setminus \{0\} \to \mathbb{C}$ which is homogeneous of degree -1 (because the degree of homogeneity is multiplicative), and has the same regularity as the original ψ. The map $\psi \mapsto \mu_\psi$ is a bijection. □

Example 16. If $E \to M$ is a vector subbundle of the trivial bundle $M \times V$, where V carries some inner product/hermitian product, there is a natural connection ∇ on E, obtained by projecting D. That is, $(\nabla_{\boldsymbol{X}}\psi)_x = \mathrm{pr}_{E_x}(\mathrm{D}_{\boldsymbol{X}}\psi)_x$, for all $x \in M$, $\boldsymbol{X} \in \mathfrak{X}(M)$ and $\psi \in \Gamma(E)$. Let's see what happens for $\mathscr{L} \subseteq \mathrm{P}V \times V$. Given $\mu \in \Gamma(\mathscr{L})$, we have that μ represents the function induced by $x \mapsto \mu(x)x$. So if $\boldsymbol{X} \in \mathfrak{X}(\mathrm{P}V)$, we have that

$$(\mathrm{D}_{\boldsymbol{X}}\mu)_x = \mathrm{d}\mu_x(\boldsymbol{X}x)x + \mu(x)\boldsymbol{X}x,$$

and projecting this onto $\mathbb{C}x$ we obtain $(\nabla_{\boldsymbol{X}}\mu)_x = \mathrm{d}\mu_x(\boldsymbol{X}x)$, since $\boldsymbol{X}x \in (\mathbb{C}x)^\perp$ and $x \mapsto \mathrm{d}\mu_x(\boldsymbol{X}x)$ is also homogeneous of degree -1 (as μ is). Do not be fooled, as this connection ∇ is not flat. In fact it admits no parallel sections: if $\nabla\mu = 0$, then $\mathrm{d}\mu_x|_{(\mathbb{C}x)^\perp} = 0$ for all $x \in V \setminus \{0\}$. This would imply that the integrability of a certain horizontal distribution, which is a contradiction (we will see in Section 3 the integrability tensor A of the submersion $\Sigma \to \mathrm{P}V$ is non-trivial).

Example 17. The dual $\mathscr{L}^* \to \mathrm{P}V$ of the tautological line bundle may also appear in some constructions. A small factory of sections come from linear functionals. Namely, if $f \in V^*$, we define $\psi_f \in \Gamma(\mathscr{L}^*)$ by $\psi_f(L) = (L, f|_L)$.

Proposition 18. *The tautological line bundle $\mathscr{L} \to \mathrm{P}V$ has no non-trivial holomorphic sections, and thus is not isomorphic to the trivial line bundle.*

Proof: Assume that $\psi \in \Gamma(\mathscr{L})$ is holomorphic. Take a basis $(e^1, \ldots, e^n)$ for V^*, and consider the maps $e^j \circ \psi \colon \mathrm{P}V \to \mathbb{C}$. They're holomorphic and bounded (since $\mathrm{P}V$ is compact), hence constant due to Liouville's Theorem. This means that ψ itself is constant. But $\psi(L) \in L$ for all $L \in \mathrm{P}V$ and the intersection of all lines in V is $\{0\}$, so $\psi = 0$. □

The above result also indicates that it might be hard for $\mathrm{P}V$ to be parallelizable as well. But it is "almost" stable, and from this we'll be able to conclude what we want:

Proposition 19. *There is a bundle isomorphism $T(\mathrm{P}V) \oplus (\mathrm{P}V \times \mathbb{C}) \cong \mathscr{L}^{\oplus n}$.*

Proof: We directly compute

$$\begin{aligned} T(\mathrm{P}V) \oplus (\mathrm{P}V \times \mathbb{C}) &\cong \mathrm{Hom}(\mathscr{L}, (\mathrm{P}V \times V)/\mathscr{L}) \oplus \mathrm{Hom}(\mathscr{L}, \mathscr{L}) \\ &= \mathrm{Hom}(\mathscr{L}, ((\mathrm{P}V \times V)/\mathscr{L}) \oplus \mathscr{L}) \\ &= \mathrm{Hom}(\mathscr{L}, \mathrm{P}V \times V) \\ &\cong \mathscr{L}^{\oplus n}, \end{aligned}$$

as wanted. The last isomorphism is not canonical, in general. □

Corollary 20. *$\mathrm{P}V$ is not a parallelizable manifold.*

Remark. Indeed, the previous isomorphism implies that the (total) Stiefel-Whitney class $\mathsf{w}(T(\mathrm{P}V)) = (1+a)^n$ is non-zero.

As a last application for this first section, we show that the tensor power $\mathscr{L}^{\otimes n}$ also plays a role in describing the geometry of $\mathrm{P}V$.

Proposition 21. *There is a holomorphic bundle isomorphism* $(T^*(\mathbb{P}V))^{\wedge(n-1)} \cong \mathscr{L}^{\otimes n}$.

Proof: Fix a volume form $\Omega\colon V^{\wedge n} \to \mathbb{C}$ for V. Given any vectors $v_1,\ldots,v_n \in L$ and linear maps $\widetilde{H}_2,\ldots,\widetilde{H}_n\colon L \to V$, consider the expression $\Omega(v_1,\widetilde{H}_2v_2,\ldots,\widetilde{H}_nv_n)$. For any $2 \le j \le n$, if we replace $\widetilde{H}_j$ with another lift $\widetilde{H}_j' = \widetilde{H}_j + B$, where $B\colon L \to L$, then Ω being multilinear and skew, $\dim L = 1$ and both v_1 and Bv_j being in L ensures that the extra term $\Omega(v_1,\widetilde{H}_2v_2,\ldots,Bv_j,\ldots,\widetilde{H}_nv_n)$ vanishes, so that we may pass the dependence on the $\widetilde{H}_j$'s to the quotient and consider the expression $\Omega(v_1,H_2v_2,\ldots,H_nv_n)$ as a function of $v_1,\ldots,v_n \in L$ and $H_2,\ldots,H_n \in T_L(\mathbb{P}V)$ instead. Since $\dim L = 1$, by moving complex scalars around, we see that the dependence on the v_j's is multilinear (and in this case, symmetric) and the dependence on the H_j's is multilinear and skew. With this in place, the desired isomorphism is characterized by

$$\mathscr{L}^{\otimes n} \ni \bigotimes_{i=1}^n v_i \mapsto \left(\bigwedge_{i=2}^n H_i \mapsto \Omega(v_1,H_2v_2,\ldots,H_nv_n)\right) \in (T^*(\mathbb{P}V))^{\wedge(n-1)}.$$

□

The tautological bundle $\mathscr{L}$ also has a natural connection with very nice properties. But we will leave this for later, when we introduce the Fubini-Study metric on $\mathbb{P}V$. We'll conclude this section exhibiting an embedding of $\mathbb{P}V$ into an Euclidean space.

Theorem 22. *Assume that V is equipped with a hermitian inner product $\langle\cdot,\cdot\rangle$, and consider the vector space $\mathfrak{gl}_0^{\rm sym}(V) \doteq \{A \in \mathfrak{gl}(V) \mid A^* = A \text{ and } \operatorname{tr}(A) = 0\}$ equipped with the inner product (also denoted by $\langle\cdot,\cdot\rangle$) given by $\langle A,B\rangle \doteq \operatorname{tr}(AB)$. The map $\Phi\colon \mathbb{P}V \to \mathfrak{gl}_0^{\rm sym}(V)$ given by $\Phi(L) = (n-1)\mathrm{Id}_L \oplus (-\mathrm{Id}_{L^\perp})$ is an embedding and the image $\Phi[\mathbb{P}V]$ is contained inside the sphere in $\mathfrak{gl}_0^{\rm sym}(V)$ of radius $\sqrt{n(n-1)}$.*

Proof: Another way to see that this function Φ is indeed smooth is by writing it as

$$\Phi(L) = (n-1)\mathrm{pr}_L - \mathrm{pr}_{L^\perp} = n\,\mathrm{pr}_L - \mathrm{Id}_V,$$

since $\mathrm{pr}_{L^\perp} = \mathrm{Id}_V - \mathrm{pr}_L$, so this follows from Example 11 (p. 9). We may also write Φ as $\Phi(L) = n\,\mathrm{pr}_{L,0}$, where $\mathrm{pr}_{L,0}$ denotes the traceless part[8] of pr_L (and the factor n can be though as to yield a nicer radius in the sphere mentioned in the original statement). Now, Φ is injective, since if $L' \in \mathbb{P}V$ is another line, then

$$n\,\mathrm{pr}_{L,0} = n\,\mathrm{pr}_{L',0} \implies \mathrm{pr}_{L,0} = \mathrm{pr}_{L',0} \implies \mathrm{pr}_L = \mathrm{pr}_{L'} \implies L = L',$$

as it should be. We also know that $\mathrm{d}\Phi_L(H) = n\left(\langle\cdot,Hx\rangle x + \langle\cdot,x\rangle Hx\right)$ where $x \in L\cap\Sigma$ is any unit spanning vector for L, and $H\colon L \to L^\perp$ is any linear map. This formula implies that $\mathrm{d}\Phi_L$ is injective, because if $\mathrm{d}\Phi_L(H) = 0$, then $L \cap L^\perp = \{0\}$ gives that $\langle\cdot,Hx\rangle = 0$ or $Hx = 0$, and in either case we obtain $H = 0$. Thus Φ is an immersion. Since $\mathbb{P}V$ is compact, Φ is an embedding, as wanted. Lastly, we note that

$$\Phi(L)^2 = (n\,\mathrm{pr}_L - \mathrm{Id}_V)\circ(n\,\mathrm{pr}_L - \mathrm{Id}_V) = n^2\,\mathrm{pr}_L - 2n\,\mathrm{pr}_L + \mathrm{Id}_V,$$

so that $\langle\Phi(L),\Phi(L)\rangle = n^2 - 2n + n = n^2 - n$, and hence $\|\Phi(L)\| = \sqrt{n(n-1)}$. □

[8] In general, if $A \in \mathfrak{gl}(V)$ is any given endomorphism, we may define the traceless part of A to be $A_0 = A - (\operatorname{tr}(A)/n)\mathrm{Id}_V$. Moreover, if $\mathfrak{gl}_0(V) \subseteq \mathfrak{gl}(V)$ denotes the ideal of traceless endomorphisms, we have that the map $\mathfrak{gl}(V) \ni A \mapsto (A_0, \operatorname{tr}(A)) \in \mathfrak{gl}_0(V) \oplus \mathbb{C}$ is a Lie algebra isomorphism.

2 Riemannian submersions

2.1 Definition and horizontal lifts

To move on to the next big step and define a canonical metric in PV, from a hermitian inner product $\langle\cdot,\cdot\rangle$ on V, the main tool to be used will be the concept of *Riemannian submersion*, so we will review it here. Fix smooth manifolds M and B and a smooth surjective submersion $\pi\colon M \to B$. Assume that M has a pseudo-Riemannian metric g^M. We start with the:

Definition 23. The *vertical space* of M at $x \in M$ is $\mathrm{Ver}_x(M) \doteq \ker \mathrm{d}\pi_x$, and the *horizontal space* at x is $\mathrm{Hor}_x(M) \doteq \mathrm{Ver}_x(M)^\perp$. The *fiber* over $b \in B$ is $M_b \doteq \pi^{-1}(b)$.

Naturally, we would like to write $T_xM = \mathrm{Hor}_x(M) \oplus \mathrm{Ver}_x(M)$ for all $x \in M$, but this need not be the case for an arbitrary pseudo-Riemannian metric, as all the fibers M_b may be[9] degenerate submanifolds of M. There is no way around this issue (which does not exist in the Riemannian case), and so this will need to be an assumption. This being the case, every $\boldsymbol{w} \in T_xM$ may be written as $\boldsymbol{w} = \mathsf{h}\boldsymbol{w} + \mathsf{v}\boldsymbol{w}$, where

$$\mathsf{h}\colon T_xM \to \mathrm{Hor}_x(M) \quad \text{and} \quad \mathsf{v}\colon T_xM \to \mathrm{Ver}_x(M)$$

are the associated projections. In other words, $\mathsf{h}\boldsymbol{w} \in \mathrm{Hor}_x(M)$ and $\mathsf{v}\boldsymbol{w} \in \mathrm{Ver}_x(M)$.

Lemma 24. *Let $b \in B$ and $\boldsymbol{v} \in T_bB$. For every $x \in M_b$ there is a unique $\boldsymbol{v}^{\mathsf{h}}_x \in \mathrm{Hor}_x(M)$ such that $\mathrm{d}\pi_x(\boldsymbol{v}^{\mathsf{h}}_x) = \boldsymbol{v}$. We call $\boldsymbol{v}^{\mathsf{h}}_x$ the* horizontal lift *of $\boldsymbol{v}$ to x. Moreover:*

(i) if $\boldsymbol{w} \in T_xM$, then the expected relation $\mathrm{d}\pi_x(\boldsymbol{w})^{\mathsf{h}}_x = \mathsf{h}\boldsymbol{w}$ holds.

(ii) $T_bB \ni \boldsymbol{v} \mapsto \boldsymbol{v}^{\mathsf{h}}_x \in \mathrm{Hor}_x(M)$ is a linear isomorphism.

Proof: Lifts of $\boldsymbol{v}$ always exist since π is a surjective submersion (to see this, consider coordinates around x and b for which π appears as a projection). But any two of those lifts differ by an element of $\mathrm{Ver}_x(M)$, so we choose $\boldsymbol{v}^{\mathsf{h}}_x$ to be the horizontal one (there is a unique one with zero vertical component). For the second part:

(i) just note that $\mathsf{h}\boldsymbol{w}$ is horizontal and satisfies $\mathrm{d}\pi_x(\mathsf{h}\boldsymbol{w}) = \mathrm{d}\pi_x(\boldsymbol{w})$.

(ii) the inverse isomorphism is the restriction of $\mathrm{d}\pi_x$ to $\mathrm{Hor}_x(M)$, by item (i).

$\square$

Remark. If $\psi\colon B \to M$ is a section of π (or just a local section), given $\boldsymbol{v} \in T_bB$, we have that $\boldsymbol{v}^{\mathsf{h}}_{\psi(b)} = \mathsf{h}\,\mathrm{d}\psi_b(\boldsymbol{v})$.

This construction can be done on the level of vector fields. If $\boldsymbol{X} \in \mathfrak{X}(B)$, there is a unique $\boldsymbol{X}^{\mathsf{h}} \in \mathfrak{X}(M)$ such that $\boldsymbol{X}^{\mathsf{h}}_x \in \mathrm{Hor}_x(M)$ and $\mathrm{d}\pi_x(\boldsymbol{X}^{\mathsf{h}}_x) = \boldsymbol{X}_{\pi(x)}$ for all $x \in M$.[10] One useful property of this lift operation to register now regards Lie brackets:

[9] Here's one Lorentzian example of this situation: $\pi\colon (\mathbb{R}^2, \mathrm{d}x^2 - \mathrm{d}y^2) \to \mathbb{R}$ given by $\pi(x,y) = y - x$. Then for every $b \in \mathbb{R}$, $(\mathbb{R}^2)_b$ is the affine light ray with positive slope 1 and y-intercept b, and the extreme equality case $\mathrm{Hor}_{(x,y)}(\mathbb{R}^2) = \mathrm{Ver}_{(x,y)}(\mathbb{R}^2) = \mathbb{R}(\partial_x|_{(x,y)} + \partial_y|_{(x,y)})$ happens.

[10] Of course we're just writing $\boldsymbol{X}^{\mathsf{h}}_x$ when $(\boldsymbol{X}^{\mathsf{h}})_x$ would be more precise, but no confusion should arise.

Lemma 25. *Let $X, Y \in \mathfrak{X}(B)$. Then $[X,Y]^{\mathsf{h}} = \mathsf{h}[X^{\mathsf{h}}, Y^{\mathsf{h}}]$.*

Proof: Clearly $\mathsf{h}[X^{\mathsf{h}}, Y^{\mathsf{h}}]$ is horizontal and satisfies

$$\mathrm{d}\pi(\mathsf{h}[X^{\mathsf{h}}, Y^{\mathsf{h}}]) = \mathrm{d}\pi([X^{\mathsf{h}}, Y^{\mathsf{h}}]) = [\mathrm{d}\pi(X^{\mathsf{h}}), \mathrm{d}\pi(Y^{\mathsf{h}})] \circ \pi = [X,Y] \circ \pi,$$

as wanted. □

For all of the above, only a metric in M was used. Now we'll equip B with a pseudo-Riemannian metric g^B as well.

Definition 26. The map $\pi\colon M \to B$ is a *pseudo-Riemannian submersion* if:

(i) it is a smooth surjective submersion;

(ii) all the fibers M_b are non-degenerate submanifolds of M, and;

(iii) for any $x \in M$ and $w_1, w_2 \in \mathrm{Hor}_x(M)$, the relation

$$g^M_x(w_1, w_2) = g^B_{\pi(x)}(\mathrm{d}\pi_x(w_1), \mathrm{d}\pi_x(w_2))$$

holds.

Remark. This definition is the natural weakening of an isometry. The restrictions of $\mathrm{d}\pi_x$ to subspaces of T_xM complementary to $\mathrm{Ver}_x(M)$ are isomorphisms, but in the presence of g^M there is a canonical choice of such a complement, namely, $\mathrm{Hor}_x(M)$. Condition (i) says that the fibers of π are non-empty and makes it possible to discuss lifts. Condition (ii) says that $\mathrm{Hor}_x(M)$ is a legitimate complement to $\mathrm{Ver}_x(M)$. And condition (iii) is requiring $\mathrm{d}\pi_x$ to be a linear isometry between $\mathrm{Hor}_x(M)$ and $T_{\pi(x)}B$, losing control of what happens on $\mathrm{Ver}_x(M)$.

Example 27 (Warped products)**.** If (B, g^B) and (F, g^F) are pseudo-Riemmanian manifolds and $\phi\colon B \to \mathbb{R}$ is a positive smooth function (the *warping function*), then their *warped product* $M \doteq B \times_\phi F = (B \times F, g^B \oplus \phi^2 g^F)$ equipped with the projection onto the first factor, $\pi\colon B \times_\phi F \to B$, is a pseudo-Riemannian submersion, since:

- the fibers $(\{b\} \times F, \phi(b)^2 g^F)$ are non-degenerate, as they're homothetic to (F, g^F) via the second projection $\{b\} \times F \to F$, and the latter is pseudo-Riemannian.
- for any $(b, x) \in B \times_\phi F$ we have

  $$\mathrm{Ver}_{(b,x)}(B \times_\phi F) = 0 \oplus T_xF \quad \text{and} \quad \mathrm{Hor}_{(b,x)}(B \times_\phi F) = T_bB \oplus 0,$$

  and the induced metric on the horizontal spaces is just g^B, i.e., the restriction $B \times \{x\} \to B$ of π is an isometry.

When $\phi = 1$ we have a usual pseudo-Riemannian product. And in general, since we have that $g^B \oplus \phi^2 g^F = \phi^2(\phi^{-2} g^B \oplus g^F)$ holds, warped products may be studied by combining usual products and conformal relations.

Example 28. A generalization of the construction of a warped product is done as follows: if (B, g^B) is pseudo-Riemannian and $(g^{F,b})_{b\in B}$ is a family of pseudo-Riemannian metrics on F, depending smoothly on the parameter b (in particular, all of the $g^{b,F}$'s have the same index). Then $(b,x) \mapsto g^B_b \oplus g^{b,F}_x$ is a pseudo-Riemannian metric on $B \times F$, and again the projection $\pi\colon B\times F \to B$ is a pseudo-Riemannian submersion, for the same reasons as above. If all of the $g^{b,F}$ are in the same homothetic class, say, $g^{b,F} = \phi(b)^2 g^F$ for some fixed pseudo-Riemannian metric g^F on F, we obtain a necessarily smooth (and without loss of generality, positive) function $\phi\colon B \to \mathbb{R}$ – then this construction yields $B \times_\phi F$.

An immediate consequence of this definition is that for $\boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{X}(B)$, the relation $g^M(\boldsymbol{X}^{\mathsf{h}}, \boldsymbol{Y}^{\mathsf{h}}) = g^B(\boldsymbol{X}, \boldsymbol{Y}) \circ \pi$ holds. Let ∇^M and ∇^B denote the Levi-Civita connections of (M, g^M) and (B, g^B), respectively. We'll see that lifting is compatible with the connections, in the sense of the following:

Proposition 29. *Let $\boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{X}(B)$. Then $(\nabla^B_{\boldsymbol{X}}\boldsymbol{Y})^{\mathsf{h}} = \mathsf{h}\nabla^M_{\boldsymbol{X}^{\mathsf{h}}}\boldsymbol{Y}^{\mathsf{h}}$. In particular, horizontal geodesics in M project onto geodesics in B.*

Proof: We start to argue as follows: since both $(\nabla^B_{\boldsymbol{X}}\boldsymbol{Y})^{\mathsf{h}}$ and $\mathsf{h}\nabla^M_{\boldsymbol{X}^{\mathsf{h}}}\boldsymbol{Y}^{\mathsf{h}}$ are horizontal vectors and g^M is non-degenerate, it suffices to show that these fields have the same product with all horizontal fields. But all horizontal fields are lifts of fields in B. Thus, it suffices to show that

$$g^M(\nabla^M_{\boldsymbol{X}^{\mathsf{h}}}\boldsymbol{Y}^{\mathsf{h}}, \boldsymbol{Z}^{\mathsf{h}}) = g^B(\nabla^B_{\boldsymbol{X}}\boldsymbol{Y}, \boldsymbol{Z}) \circ \pi$$

for all $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \mathfrak{X}(B)$. But this follows from the Koszul formula[11] in view of the relations

$$\boldsymbol{X}^{\mathsf{h}} g^M(\boldsymbol{Y}^{\mathsf{h}}, \boldsymbol{Z}^{\mathsf{h}}) = (\boldsymbol{X} g^B(\boldsymbol{Y}, \boldsymbol{Z})) \circ \pi \quad \text{and} \quad g^M(\boldsymbol{X}^{\mathsf{h}}, [\boldsymbol{Y}^{\mathsf{h}}, \boldsymbol{Z}^{\mathsf{h}}]) = g^B(\boldsymbol{X}, [\boldsymbol{Y}, \boldsymbol{Z}]) \circ \pi.$$

For the second part, fix a horizontal geodesic $\gamma^M\colon I \to M$, and consider the projected curve $\gamma^B \doteq \pi \circ \gamma^M\colon I \to B$. Note that $(\gamma^B)'(t) = \mathrm{d}\pi_{\gamma(t)}((\gamma^M)'(t))$, so that γ^M being horizontal implies that $(\gamma^B)'(t)^{\mathsf{h}}_{\gamma(t)} = (\gamma^M)'(t)$. Now we have that

$$\left(\frac{D(\gamma^B)'}{\mathrm{d}t}(t)\right)^{\mathsf{h}}_{\gamma^M(t)} = \mathsf{h}\frac{D(\gamma^M)'}{\mathrm{d}t}(t) = \boldsymbol{0} \implies \frac{D(\gamma^B)'}{\mathrm{d}t}(t) = \boldsymbol{0},$$

as wanted. □

Remark.

- This was to be expected: π behaves like an isometry when restricted to the horizontal spaces, so it should "preserve" horizontal components of geometric objects depending on the metric. More on this soon.
- The argument above has a simple corollary: if a horizontal curve in M was lifted from a geodesic in B, its covariant acceleration field is vertical. We will see in Corollary 33 ahead that in fact it vanishes.

[11] Given by $2\langle \nabla_{\boldsymbol{X}}\boldsymbol{Y}, \boldsymbol{Z}\rangle = (\mathscr{L}_{\boldsymbol{Y}} g)(\boldsymbol{X}, \boldsymbol{Z}) + \mathrm{d}(\boldsymbol{Y}_\flat)(\boldsymbol{X}, \boldsymbol{Z})$, where $\boldsymbol{Y}_\flat = \langle \boldsymbol{Y}, \cdot\rangle$.

One more basic property which is at our reach right now is given in the:

Proposition 30. *Given a smooth function $f\colon B \to \mathbb{R}$, the horizontal lift of the gradient of f is given by* $\operatorname{grad}_B(f)^{\mathsf{h}} = \mathsf{h}\operatorname{grad}_M(f\circ\pi)$.

Proof: It suffices to show that $\mathrm{d}\pi(\operatorname{grad}_M(f\circ\pi)) = \operatorname{grad}_B(f)$, and for that we'll use non-degeneracy of g^B, by checking that both fields give the same result when hit against a field $\boldsymbol{Y} \in \mathfrak{X}(B)$. This is a direct computation, done as follows:

$$\begin{aligned} g^B(\mathrm{d}\pi(\operatorname{grad}_M(f\circ\pi)),\boldsymbol{Y})\circ\pi &= g^M(\mathrm{d}\pi(\operatorname{grad}_M(f\circ\pi))^{\mathsf{h}},\boldsymbol{Y}^{\mathsf{h}}) = g^M(\mathsf{h}\operatorname{grad}_M(f\circ\pi),\boldsymbol{Y}^{\mathsf{h}}) \\ &= g^M(\operatorname{grad}_M(f\circ\pi),\boldsymbol{Y}^{\mathsf{h}}) = \mathrm{d}(f\circ\pi)(\boldsymbol{Y}^{\mathsf{h}}) \\ &= \mathrm{d}f(\mathrm{d}\pi(\boldsymbol{Y}^{\mathsf{h}})) = \mathrm{d}f(\boldsymbol{Y}\circ\pi) \\ &= g^B(\operatorname{grad}_B(f),\boldsymbol{Y})\circ\pi \end{aligned}$$

Since π is surjective, it may be cancelled on the right and we are done.

□

The geometry of the whole data encoded in the pseudo-Riemannian submersion $\pi\colon M\to B$ can be controlled, to some extent, by the geometry of the fibers M_b.

2.2 A digression on second fundamental forms

It pays off to recall the general construction: if $E\to M$ is a vector bundle equipped with a Koszul connection ∇, and we have a decomposition in Whitney sum of subbundles, $E = E^+\oplus E^-$, then ∇ projects onto two connections $\nabla^\pm$ on $E^\pm \to M$. One may form the van der Waerden sum $\nabla^+\oplus\nabla^-$ on E, but this does not need to equal the original ∇. The error term is a tensor $\alpha\colon \mathfrak{X}(M)\times\Gamma(E)\to\Gamma(E)$, called the *second fundamental form* of the decomposition, and the relation $\nabla = (\nabla^+\oplus\nabla^-)+\alpha$ holds. In particular, for $\boldsymbol{X}\in\mathfrak{X}(M)$ and $\psi^\pm\in\Gamma(E^\pm)$, we get the Gauss and Weingarten equations

$$\begin{cases} \nabla_{\boldsymbol{X}}(\psi^+) = \nabla^+_{\boldsymbol{X}}(\psi^+) + \alpha(\boldsymbol{X},\psi^+) \\ \nabla_{\boldsymbol{X}}(\psi^-) = \nabla^-_{\boldsymbol{X}}(\psi^-) + \alpha(\boldsymbol{X},\psi^-) \end{cases}$$

We may restrict α to obtain two maps $\alpha^\pm\colon \mathfrak{X}(M)\times\Gamma(E^\mp)\to\Gamma(E^\pm)$. The usual Palatini-like identity[12] implies that $R^\nabla = (R^+\oplus R^-) + \mathrm{d}^\nabla\alpha + [\alpha,\alpha]/2$ or, in more detail:

$$R^\nabla(\boldsymbol{X},\boldsymbol{Y})(\psi^\pm) = R^\pm(\boldsymbol{X},\boldsymbol{Y})(\psi^\pm) + (\mathrm{d}^\nabla\alpha)(\boldsymbol{X},\boldsymbol{Y})(\psi^\pm) + \alpha_{\boldsymbol{X}}\alpha_{\boldsymbol{Y}}(\psi^\pm) - \alpha_{\boldsymbol{Y}}\alpha_{\boldsymbol{X}}(\psi^\pm),$$

where $\boldsymbol{X},\boldsymbol{Y}\in\mathfrak{X}(M)$, $\psi^\pm\in\Gamma(E^\pm)$, and we abbreviate $\alpha_{\boldsymbol{X}} = \alpha(\boldsymbol{X},\cdot)$. Noting that d^∇ commutes with the projections $(\_)^\pm$, projecting the above we obtain the Gauss and Codazzi equations:

$$\begin{cases} R^\pm(\boldsymbol{X},\boldsymbol{Y})(\psi^\pm) = \big(R^\nabla(\boldsymbol{X},\boldsymbol{Y})(\psi^\pm)\big)^\pm - \alpha^\pm_{\boldsymbol{X}}\alpha^\mp_{\boldsymbol{Y}}(\psi^\pm) + \alpha^\pm_{\boldsymbol{Y}}\alpha^\mp_{\boldsymbol{X}}(\psi^\pm) \\ \big(R^\nabla(\boldsymbol{X},\boldsymbol{Y})(\psi^\pm)\big)^\mp = (\mathrm{d}^\nabla\alpha^\mp)(\boldsymbol{X},\boldsymbol{Y})(\psi^\pm) \end{cases}$$

[12] If ∇ is a connection in a vector bundle $E\to M$ and $A\colon\mathfrak{X}(M)\times\Gamma(E)\to\Gamma(E)$ is a tensor, then

$$R^{\nabla+A}(\boldsymbol{X},\boldsymbol{Y})\psi = R^\nabla(\boldsymbol{X},\boldsymbol{Y})\psi + (\mathrm{d}^\nabla A)(\boldsymbol{X},\boldsymbol{Y})\psi + A_{\boldsymbol{X}}A_{\boldsymbol{Y}}\psi - A_{\boldsymbol{Y}}A_{\boldsymbol{X}}\psi,$$

where $\mathrm{d}^\nabla\colon\Omega^1(M,\operatorname{End}(E))\to\Omega^2(M,\operatorname{End}(E))$ is the de Rham-like operator induced by ∇.

In the pseudo-Riemannian case, i.e., when $N \subseteq M$ is a non-degenerate submanifold, $E = TM|_N$, $E^+ = TN$ and $E^- = TN^\perp$, we have that $\alpha^+(\boldsymbol{X}, \cdot)$ and $\alpha^-(\boldsymbol{X}, \cdot)$ are negative adjoints, we write $\alpha^+ = \alpha$ (when convenient), $\alpha^-(\boldsymbol{X}, \boldsymbol{\xi}) = -A_{\boldsymbol{\xi}}(\boldsymbol{X})$ and $\nabla^- = \nabla^\perp$, so that these equations become

$$\begin{cases} \nabla^M_{\boldsymbol{X}}\boldsymbol{Y} = \nabla^N_{\boldsymbol{X}}\boldsymbol{Y} + \alpha(\boldsymbol{X}, \boldsymbol{Y}) \\ \nabla^M_{\boldsymbol{X}}\boldsymbol{\xi} = -A_{\boldsymbol{\xi}}(\boldsymbol{X}) + \nabla^\perp_{\boldsymbol{X}}\boldsymbol{\xi}, \end{cases}$$

for $\boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{X}(N)$ and $\boldsymbol{\xi} \in \mathfrak{X}^\perp(N)$, and also

$$\begin{cases} R^N(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{W}) = R^M(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{W}) + \langle \alpha(\boldsymbol{Y}, \boldsymbol{Z}), \alpha(\boldsymbol{X}, \boldsymbol{W})\rangle - \langle \alpha(\boldsymbol{X}, \boldsymbol{Z}), \alpha(\boldsymbol{Y}, \boldsymbol{W})\rangle \\ R^\perp(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{\xi}, \boldsymbol{\eta}) = R^M(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{\xi}, \boldsymbol{\eta}) + \langle [A_{\boldsymbol{\xi}}, A_{\boldsymbol{\eta}}](\boldsymbol{X}), \boldsymbol{Y}\rangle \\ (R^M(\boldsymbol{X}, \boldsymbol{Y})\boldsymbol{Z})^\perp = (\nabla_{\boldsymbol{X}}\alpha)(\boldsymbol{Y}, \boldsymbol{Z}) - (\nabla_{\boldsymbol{Y}}\alpha)(\boldsymbol{X}, \boldsymbol{Z}) \\ (R^M(\boldsymbol{X}, \boldsymbol{Y})\boldsymbol{\xi})^\top = A_{\nabla^\perp_{\boldsymbol{X}}\boldsymbol{\xi}}(\boldsymbol{Y}) - A_{\nabla^\perp_{\boldsymbol{Y}}\boldsymbol{\xi}}(\boldsymbol{X}) + (\nabla_{\boldsymbol{Y}}A_{\boldsymbol{\xi}})(\boldsymbol{X}) - (\nabla_{\boldsymbol{X}}A_{\boldsymbol{\xi}})(\boldsymbol{Y}), \end{cases}$$

for $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{W} \in \mathfrak{X}(N)$ and $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathfrak{X}^\perp(N)$. Each $A_{\boldsymbol{\xi}} \colon \mathfrak{X}(N) \to \mathfrak{X}(N)$ is called a *Weingarten operator*, and they're self-adjoint since α is symmetric (in view of both ∇^M and ∇^N being torsion-free).

2.3 Integrability tensors

So, for each $b \in B$, let ∇^b be the Levi-Civita connection of M_b equipped with the metric induced from g^M. Each M_b has a second fundamental form α^b. And this will lead us to the so-called *integrability tensors* of π. For this, note that for $x \in M_b$, we have that $T_x(M_b) = \mathrm{Ver}_x(M)$ and $T_x(M_b)^\perp = \mathrm{Hor}_x(M)$. So we want to look at the second fundamental form of the decomposition of $TM|_{M_b}$ provided by π, for each $b \in B$. Given $\boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{X}(M)$ and writing $\nabla^{b,\perp}$ for the normal connection of M_b, along each of them we may write

$$\nabla^M_{\mathsf{v}\boldsymbol{X}}(\mathsf{v}\boldsymbol{Y}) = \nabla^b_{\mathsf{v}\boldsymbol{X}}(\mathsf{v}\boldsymbol{Y}) + \alpha(\mathsf{v}\boldsymbol{X}, \mathsf{v}\boldsymbol{Y}) \quad \text{and} \quad \nabla^M_{\mathsf{v}\boldsymbol{X}}(\mathsf{h}\boldsymbol{Y}) = \alpha(\mathsf{v}\boldsymbol{X}, \mathsf{h}\boldsymbol{Y}) + \nabla^{b,\perp}_{\mathsf{v}\boldsymbol{X}}(\mathsf{h}\boldsymbol{Y}).$$

Applying h to the first relation and v to the second one (as to eliminate ∇^b and $\nabla^{b,\perp}$) and adding them, we obtain $\alpha(\mathsf{v}\boldsymbol{X}, \boldsymbol{Y}) = \mathsf{h}\nabla^M_{\mathsf{v}\boldsymbol{X}}(\mathsf{v}\boldsymbol{Y}) + \mathsf{v}\nabla^M_{\mathsf{v}\boldsymbol{X}}(\mathsf{h}\boldsymbol{Y})$. To obtain something controlling the geometry of the horizontal distribution instead, we dualize this formula (switching v and h everywhere). This leads us to the:

Definition 31. The *integrability tensors* of a pseudo-Riemannian submersion $\pi \colon M \to B$ are $\mathsf{T}, \mathsf{A} \colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by

$$\mathsf{T}_{\boldsymbol{X}}\boldsymbol{Y} = \mathsf{h}\nabla^M_{\mathsf{v}\boldsymbol{X}}(\mathsf{v}\boldsymbol{Y}) + \mathsf{v}\nabla^M_{\mathsf{v}\boldsymbol{X}}(\mathsf{h}\boldsymbol{Y}) \quad \text{and} \quad \mathsf{A}_{\boldsymbol{X}}\boldsymbol{Y} = \mathsf{v}\nabla^M_{\mathsf{h}\boldsymbol{X}}(\mathsf{h}\boldsymbol{Y}) + \mathsf{h}\nabla^M_{\mathsf{h}\boldsymbol{X}}(\mathsf{v}\boldsymbol{Y}).$$

Remark. For T, $\mathsf{v}\boldsymbol{X}$ always appears below, while for A, $\mathsf{h}\boldsymbol{X}$ always appears below. The projections for the $\boldsymbol{Y}$ argument are always "mixed".

So T encodes the second fundamental forms of the fibers M_b, while A is expected to measure the failure of integrability of the distribution $\mathrm{Hor}(M)$. Let's register some basic properties of T and A.

Proposition 32. *Given $\boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{X}(M)$, the following hold:*

(i) T *and* A *are indeed* $\mathscr{C}^\infty(M)$*-bilinear.*

(ii) $\mathsf{T}_{\boldsymbol{X}}$ *and* $\mathsf{A}_{\boldsymbol{X}}$ *are both skew-adjoint and reverse vertical and horizontal spaces.*

(iii) $\mathsf{T}_{\mathsf{v}\boldsymbol{X}} = \mathsf{T}_{\boldsymbol{X}}$ *and* $\mathsf{A}_{\mathsf{h}\boldsymbol{X}} = \mathsf{A}_{\boldsymbol{X}}$*. In particular,* $\mathsf{T}_{\mathsf{h}\boldsymbol{X}} = \mathsf{A}_{\mathsf{v}\boldsymbol{X}} = 0$.

(iv) $\mathsf{T}_{\mathsf{v}\boldsymbol{X}}(\mathsf{v}\boldsymbol{Y}) = \mathsf{T}_{\mathsf{v}\boldsymbol{Y}}(\mathsf{v}\boldsymbol{X})$ *and* $\mathsf{A}_{\mathsf{h}\boldsymbol{X}}(\mathsf{h}\boldsymbol{Y}) = -\mathsf{A}_{\mathsf{h}\boldsymbol{Y}}(\mathsf{h}\boldsymbol{X})$.

(v) $\mathsf{A}_{\mathsf{h}\boldsymbol{X}}(\mathsf{h}\boldsymbol{Y}) = (1/2)\mathsf{v}[\mathsf{h}\boldsymbol{X}, \mathsf{h}\boldsymbol{Y}]$.

(vi) the Gauss and Weingarten equations may be written in terms of T *and* A *as:*

$$\begin{cases} \nabla^M_{\mathsf{v}\boldsymbol{X}}(\mathsf{v}\boldsymbol{Y}) = \mathsf{T}_{\mathsf{v}\boldsymbol{X}}(\mathsf{v}\boldsymbol{Y}) + \nabla^b_{\mathsf{v}\boldsymbol{X}}(\mathsf{v}\boldsymbol{Y}) \\ \nabla^M_{\mathsf{v}\boldsymbol{X}}(\mathsf{h}\boldsymbol{Y}) = \nabla^{b,\perp}_{\mathsf{v}\boldsymbol{X}}(\mathsf{h}\boldsymbol{Y}) + \mathsf{T}_{\mathsf{v}\boldsymbol{X}}(\mathsf{h}\boldsymbol{Y}) \\ \nabla^M_{\mathsf{h}\boldsymbol{X}}(\mathsf{v}\boldsymbol{Y}) = \mathsf{A}_{\mathsf{h}\boldsymbol{X}}(\mathsf{v}\boldsymbol{Y}) + \mathsf{v}\nabla^M_{\mathsf{h}\boldsymbol{X}}(\mathsf{v}\boldsymbol{Y}) \\ \nabla^M_{\mathsf{h}\boldsymbol{X}}(\mathsf{h}\boldsymbol{Y}) = \mathsf{h}\nabla^M_{\mathsf{h}\boldsymbol{X}}(\mathsf{h}\boldsymbol{Y}) + \mathsf{A}_{\mathsf{h}\boldsymbol{X}}(\mathsf{h}\boldsymbol{Y}) \end{cases}$$

Horizontal terms are written first and vertical terms second.

Proof: In this proof, at least, we'll write $g^M = \langle \cdot, \cdot \rangle$, as the metric g^B will not appear and no confusion should arrive.

(i) Let's just show that T is $\mathscr{C}^\infty(M)$-linear in the $\boldsymbol{Y}$ variable, with everything else being clear or similar to this. If $f \in \mathscr{C}^\infty(M)$, we directly compute:

$$\begin{aligned} \mathsf{T}_{\boldsymbol{X}}(f\boldsymbol{Y}) &= \mathsf{h}\nabla^M_{\mathsf{v}\boldsymbol{X}}(\mathsf{v}(f\boldsymbol{Y})) + \mathsf{v}\nabla^M_{\mathsf{v}\boldsymbol{X}}(\mathsf{h}(f\boldsymbol{Y})) \\ &= \mathsf{h}\nabla^M_{\mathsf{v}\boldsymbol{X}}(f\mathsf{v}\boldsymbol{Y}) + \mathsf{v}\nabla^M_{\mathsf{v}\boldsymbol{X}}(f\mathsf{h}\boldsymbol{Y}) \\ &= \mathsf{h}\left(\mathrm{d}f(\mathsf{v}\boldsymbol{X})\,\mathsf{v}\boldsymbol{Y} + f\nabla^M_{\mathsf{v}\boldsymbol{X}}(\mathsf{v}\boldsymbol{Y})\right) + \mathsf{v}\left(\mathrm{d}f(\mathsf{v}\boldsymbol{X})\,\mathsf{h}\boldsymbol{Y} + f\nabla^M_{\mathsf{v}\boldsymbol{X}}(\mathsf{h}\boldsymbol{Y})\right) \\ &= \mathbf{0} + f\,\mathsf{h}\nabla^M_{\mathsf{v}\boldsymbol{X}}(\mathsf{v}\boldsymbol{Y}) + \mathbf{0} + f\,\mathsf{v}\nabla^M_{\mathsf{v}\boldsymbol{X}}(\mathsf{h}\boldsymbol{Y}) \\ &= f\,\mathsf{T}_{\boldsymbol{X}}\boldsymbol{Y}, \end{aligned}$$

since h and v are $\mathscr{C}^\infty(M)$-linear themselves.

(ii) Take $\boldsymbol{Y}, \boldsymbol{Z} \in \mathfrak{X}(M)$. We compute the symmetrizer $\langle \mathsf{T}_{\boldsymbol{X}}\boldsymbol{Y}, \boldsymbol{Z}\rangle + \langle \boldsymbol{Y}, \mathsf{T}_{\boldsymbol{X}}\boldsymbol{Z}\rangle$ as:

$$\begin{aligned} &\langle \mathsf{h}\nabla^M_{\mathsf{v}\boldsymbol{X}}(\mathsf{v}\boldsymbol{Y}) + \mathsf{v}\nabla^M_{\mathsf{v}\boldsymbol{X}}(\mathsf{h}\boldsymbol{Y}), \boldsymbol{Z}\rangle + \langle \boldsymbol{Y}, \mathsf{h}\nabla^M_{\mathsf{v}\boldsymbol{X}}(\mathsf{v}\boldsymbol{Z}) + \mathsf{v}\nabla^M_{\mathsf{v}\boldsymbol{X}}(\mathsf{h}\boldsymbol{Z})\rangle = \\ &\quad= \langle \mathsf{h}\nabla^M_{\mathsf{v}\boldsymbol{X}}(\mathsf{v}\boldsymbol{Y}), \mathsf{h}\boldsymbol{Z}\rangle + \langle \mathsf{v}\nabla^M_{\mathsf{v}\boldsymbol{X}}(\mathsf{h}\boldsymbol{Y}), \mathsf{v}\boldsymbol{Z}\rangle + \langle \mathsf{h}\boldsymbol{Y}, \mathsf{h}\nabla^M_{\mathsf{v}\boldsymbol{X}}(\mathsf{v}\boldsymbol{Z})\rangle + \langle \mathsf{v}\boldsymbol{Y}, \mathsf{v}\nabla^M_{\mathsf{v}\boldsymbol{X}}(\mathsf{h}\boldsymbol{Z})\rangle \\ &\quad= \langle \nabla^M_{\mathsf{v}\boldsymbol{X}}(\mathsf{v}\boldsymbol{Y}), \mathsf{h}\boldsymbol{Z}\rangle + \langle \nabla^M_{\mathsf{v}\boldsymbol{X}}(\mathsf{h}\boldsymbol{Y}), \mathsf{v}\boldsymbol{Z}\rangle + \langle \mathsf{h}\boldsymbol{Y}, \nabla^M_{\mathsf{v}\boldsymbol{X}}(\mathsf{v}\boldsymbol{Z})\rangle + \langle \mathsf{v}\boldsymbol{Y}, \nabla^M_{\mathsf{v}\boldsymbol{X}}(\mathsf{h}\boldsymbol{Z})\rangle \\ &\quad= (\mathsf{v}\boldsymbol{X})(\langle \mathsf{v}\boldsymbol{Y}, \mathsf{h}\boldsymbol{Z}\rangle) + (\mathsf{v}\boldsymbol{X})(\langle \mathsf{h}\boldsymbol{Y}, \mathsf{v}\boldsymbol{Z}\rangle) \\ &\quad= (\mathsf{v}\boldsymbol{X})(\langle \mathsf{v}\boldsymbol{Y}, \mathsf{h}\boldsymbol{Z}\rangle + \langle \mathsf{h}\boldsymbol{Y}, \mathsf{v}\boldsymbol{Z}\rangle) = 0, \end{aligned}$$

as ∇^M parallelizes $\langle \cdot, \cdot \rangle$. And similarly for A, we have that

$$\langle \mathsf{A}_{\boldsymbol{X}}\boldsymbol{Y}, \boldsymbol{Z}\rangle + \langle \boldsymbol{Y}, A_{\boldsymbol{X}}\boldsymbol{Z}\rangle = (\mathsf{h}\boldsymbol{X})(\langle \mathsf{v}\boldsymbol{Y}, \mathsf{h}\boldsymbol{Z}\rangle + \langle \mathsf{h}\boldsymbol{Y}, \mathsf{v}\boldsymbol{Z}\rangle) = 0.$$

Reversing horizontal and vertical spaces follows directly from the four simple relations $\mathsf{T}_{\boldsymbol{X}}(\mathsf{h}\boldsymbol{Y}) = \mathsf{v}\nabla^M_{\mathsf{v}\boldsymbol{X}}(\mathsf{h}\boldsymbol{Y})$, $\mathsf{T}_{\boldsymbol{X}}(\mathsf{v}\boldsymbol{Y}) = \mathsf{h}\nabla^M_{\mathsf{v}\boldsymbol{X}}(\mathsf{v}\boldsymbol{Y})$, $\mathsf{A}_{\boldsymbol{X}}(\mathsf{h}\boldsymbol{Y}) = \mathsf{v}\nabla^M_{\mathsf{h}\boldsymbol{X}}(\mathsf{h}\boldsymbol{Y})$ and $\mathsf{A}_{\boldsymbol{X}}(\mathsf{v}\boldsymbol{Y}) = \mathsf{h}\nabla^M_{\mathsf{h}\boldsymbol{X}}(\mathsf{v}\boldsymbol{Y})$.

(iii) Obvious.

(iv) The relations mentioned at the end of item (ii), together with item (iii) and the fact that ∇^M is torsion-free imply that $\mathsf{T}_{\mathsf{v}\boldsymbol{Y}}(\mathsf{v}\boldsymbol{X}) = \mathsf{h}[\mathsf{v}\boldsymbol{Y}, \mathsf{v}\boldsymbol{X}] + \mathsf{T}_{\mathsf{v}\boldsymbol{X}}(\mathsf{v}\boldsymbol{Y})$. But the Lie bracket of a vertical field with any field is vertical as well, so $\mathsf{h}[\mathsf{v}\boldsymbol{Y}, \mathsf{v}\boldsymbol{Y}] = \mathbf{0}$ and the conclusion follows for T. As for A, if suffices to show that $\mathsf{A}_{\mathsf{h}\boldsymbol{X}}(\mathsf{h}\boldsymbol{X}) = \mathbf{0}$ for all $\boldsymbol{X}$. Since $\mathsf{A}_{\mathsf{h}\boldsymbol{X}}(\mathsf{h}\boldsymbol{X})$ is always vertical, this can be achieved by showing that $\langle \mathsf{A}_{\mathsf{h}\boldsymbol{X}}(\mathsf{h}\boldsymbol{X}), \mathsf{v}\boldsymbol{Y}\rangle = 0$ for all $\boldsymbol{Y}$. As A is a tensor, we may as well assume that $\mathsf{h}\boldsymbol{X}$ is projectable, so that $2\langle \nabla^M_{\mathsf{v}\boldsymbol{Y}}(\mathsf{h}\boldsymbol{X}), \mathsf{h}\boldsymbol{X}\rangle = (\mathsf{v}\boldsymbol{Y})(\langle \mathsf{h}\boldsymbol{X}, \mathsf{h}\boldsymbol{X}\rangle) = 0$. With this in place, we recall again that $[\mathsf{h}\boldsymbol{X}, \mathsf{v}\boldsymbol{Y}]$ is vertical because $\mathsf{v}\boldsymbol{Y}$ is, thus

$$\begin{aligned} 0 = \langle [\mathsf{h}\boldsymbol{X}, \mathsf{v}\boldsymbol{Y}], \mathsf{h}\boldsymbol{X}\rangle &= \langle \nabla^M_{\mathsf{h}\boldsymbol{X}}(\mathsf{v}\boldsymbol{Y}) - \nabla^M_{\mathsf{v}\boldsymbol{Y}}(\mathsf{h}\boldsymbol{X}), \mathsf{h}\boldsymbol{X}\rangle \\ &= \langle \nabla^M_{\mathsf{h}\boldsymbol{X}}(\mathsf{v}\boldsymbol{Y}), \mathsf{h}\boldsymbol{X}\rangle = -\langle \mathsf{v}\boldsymbol{Y}, \nabla^M_{\mathsf{h}\boldsymbol{X}}(\mathsf{h}\boldsymbol{X})\rangle \\ &= -\langle \mathsf{v}\boldsymbol{Y}, \mathsf{A}_{\mathsf{h}\boldsymbol{X}}(\mathsf{h}\boldsymbol{X})\rangle, \end{aligned}$$

as wanted.

(v) Since $\mathsf{vh}\boldsymbol{Y} = 0$, we have that

$$\mathsf{A}_{\mathsf{h}\boldsymbol{X}}(\mathsf{h}\boldsymbol{Y}) = \mathsf{v}\nabla^M_{\mathsf{h}\boldsymbol{X}}(\mathsf{h}\boldsymbol{Y}) \stackrel{(*)}{=} \mathsf{v}\left([\mathsf{h}\boldsymbol{X}, \mathsf{h}\boldsymbol{Y}] + \nabla^M_{\mathsf{h}\boldsymbol{Y}}(\mathsf{h}\boldsymbol{X})\right) = \mathsf{v}[\mathsf{h}\boldsymbol{X}, \mathsf{h}\boldsymbol{Y}] + \mathsf{A}_{\mathsf{h}\boldsymbol{Y}}(\mathsf{h}\boldsymbol{X}),$$

where in $(*)$ we use that ∇^M is torsion-free. The conclusion now follows from the previous item.

(vi) This also follows from the relations mentioned in item (ii) above.

□

As a first consequence, we improve Proposition 29 (p. 15):

Corollary 33. *Let $\boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{X}(B)$. Then*

$$\nabla^M_{\boldsymbol{X}^{\mathsf{h}}}\boldsymbol{Y}^{\mathsf{h}} = (\nabla^B_{\boldsymbol{X}}\boldsymbol{Y})^{\mathsf{h}} + \frac{1}{2}\mathsf{v}[\boldsymbol{X}^{\mathsf{h}}, \boldsymbol{Y}^{\mathsf{h}}].$$

In particular, if $\gamma^B \colon I \to B$ is a curve and $\gamma^M \colon I \to M$ is a horizontal lift of γ^B (i.e., we have that $\pi \circ \gamma^M = \gamma^B$), then γ^B is a geodesic if and only if γ^M is.

Next, more interpretations for T and A:

Corollary 34. *The fibers M_b are totally geodesic if and only if $\mathsf{T} = 0$, and the horizontal distribution is integrable if and only if $\mathsf{A} = 0$ (i.e., up to a factor of 2, A is the Levi symbol of the horizontal distribution of π).*

Example 35. If $\mathsf{A} = 0$, then M is locally isometric to a generalized product as in Example 28 (p. 15). If $\mathsf{A} = \mathsf{T} = 0$, then M is locally isometric to a usual pseudo-Riemannian product.

With T and A in hands, we can get the analogous result of Proposition 29 (p. 15) for the curvature R^B:

Proposition 36. *Let $X, Y, Z \in \mathfrak{X}(B)$. Then*

$$(R^B(X,Y)Z)^{\mathsf{h}} = \mathsf{h}(R^M(X^{\mathsf{h}},Y^{\mathsf{h}})Z^{\mathsf{h}}) - [\mathsf{A}_{X^{\mathsf{h}}},\mathsf{A}_{Y^{\mathsf{h}}}](Z^{\mathsf{h}}) + 2\nabla^{b,\perp}_{\mathsf{A}_{X^{\mathsf{h}}}Y^{\mathsf{h}}}Z^{\mathsf{h}}.$$

In particular, if $\mathsf{A}=0$ we have $(R^B(X,Y)Z)^{\mathsf{h}} = \mathsf{h}(R^M(X^{\mathsf{h}},Y^{\mathsf{h}})Z^{\mathsf{h}})$, as expected.

Proof: We'll repeatedly use item (vi) of Proposition 32 (p. 18). We have that

$$\begin{aligned}(\nabla^B_X\nabla^B_Y Z)^{\mathsf{h}} &= \mathsf{h}\nabla^M_{X^{\mathsf{h}}}((\nabla^B_Y Z)^{\mathsf{h}}) = \mathsf{h}\nabla^M_{X^{\mathsf{h}}}(\mathsf{h}\nabla^M_{Y^{\mathsf{h}}}Z^{\mathsf{h}}) \\ &= \mathsf{h}\nabla^M_{X^{\mathsf{h}}}\nabla^M_{Y^{\mathsf{h}}}Z^{\mathsf{h}} - \mathsf{h}\nabla^M_{X^{\mathsf{h}}}\mathsf{A}_{Y^{\mathsf{h}}}Z^{\mathsf{h}} = \mathsf{h}\nabla^M_{X^{\mathsf{h}}}\nabla^M_{Y^{\mathsf{h}}}Z^{\mathsf{h}} - \mathsf{A}_{X^{\mathsf{h}}}\mathsf{A}_{Y^{\mathsf{h}}}Z^{\mathsf{h}}\end{aligned}$$

and

$$\begin{aligned}(\nabla^B_{[X,Y]}Z)^{\mathsf{h}} &= \mathsf{h}\nabla^M_{[X,Y]^{\mathsf{h}}}Z^{\mathsf{h}} = \mathsf{h}\nabla^M_{\mathsf{h}[X^{\mathsf{h}},Y^{\mathsf{h}}]}Z^{\mathsf{h}} \\ &= \mathsf{h}\nabla^M_{[X^{\mathsf{h}},Y^{\mathsf{h}}]}Z^{\mathsf{h}} - \mathsf{h}\nabla^M_{\mathsf{v}[X^{\mathsf{h}},Y^{\mathsf{h}}]}Z^{\mathsf{h}} = \mathsf{h}\nabla^M_{[X^{\mathsf{h}},Y^{\mathsf{h}}]}Z^{\mathsf{h}} - 2\nabla^{b,\perp}_{\mathsf{A}_{X^{\mathsf{h}}}Y^{\mathsf{h}}}Z^{\mathsf{h}},\end{aligned}$$

so putting everything together in the definition of curvature[13] we obtain that

$$(R^B(X,Y)Z)^{\mathsf{h}} = \mathsf{h}(R^M(X^{\mathsf{h}},Y^{\mathsf{h}})Z^{\mathsf{h}}) - [\mathsf{A}_{X^{\mathsf{h}}},\mathsf{A}_{Y^{\mathsf{h}}}](Z^{\mathsf{h}}) + 2\nabla^{b,\perp}_{\mathsf{A}_{X^{\mathsf{h}}}Y^{\mathsf{h}}}Z^{\mathsf{h}},$$

as required. □

This opens the path to the:

Corollary 37. *Let $X, Y, Z, W \in \mathfrak{X}(B)$. Then:*

(i) $$\begin{aligned}R^B(X,Y,Z,W)\circ\pi = R^M(X^{\mathsf{h}},Y^{\mathsf{h}},Z^{\mathsf{h}},W^{\mathsf{h}}) &+ g^M(\mathsf{A}_{Y^{\mathsf{h}}}Z^{\mathsf{h}},\mathsf{A}_{X^{\mathsf{h}}}W^{\mathsf{h}}) \\ &- g^M(\mathsf{A}_{X^{\mathsf{h}}}Z^{\mathsf{h}},\mathsf{A}_{Y^{\mathsf{h}}}W^{\mathsf{h}}) - 2g^M(\mathsf{A}_{X^{\mathsf{h}}}Y^{\mathsf{h}},\mathsf{A}_{Z^{\mathsf{h}}}W^{\mathsf{h}})\end{aligned}$$

(ii) $$K^B(X,Y)\circ\pi = K^M(X^{\mathsf{h}},Y^{\mathsf{h}}) + \frac{3g^M(\mathsf{A}_{X^{\mathsf{h}}}Y^{\mathsf{h}},\mathsf{A}_{X^{\mathsf{h}}}Y^{\mathsf{h}})}{g^B(X\wedge Y, X\wedge Y)\circ\pi}$$

Proof:

(i) We immediately have that

$$R^B(X,Y,Z,W)\circ\pi = R^M(X^{\mathsf{h}},Y^{\mathsf{h}},Z^{\mathsf{h}},W^{\mathsf{h}}) - g^M([\mathsf{A}_{X^{\mathsf{h}}},\mathsf{A}_{Y^{\mathsf{h}}}](Z^{\mathsf{h}}),W^{\mathsf{h}}) + 2g^M(\nabla^M_{\mathsf{A}_{X^{\mathsf{h}}}Y^{\mathsf{h}}}Z^{\mathsf{h}},W^{\mathsf{h}}).$$

Now we have to deal with the last two terms. First we have that:

$$\begin{aligned}-g^M([\mathsf{A}_{X^{\mathsf{h}}},\mathsf{A}_{Y^{\mathsf{h}}}](Z^{\mathsf{h}}),W^{\mathsf{h}}) &= -g^M(\mathsf{A}_{X^{\mathsf{h}}}\mathsf{A}_{Y^{\mathsf{h}}}Z^{\mathsf{h}},W^{\mathsf{h}}) + g^M(\mathsf{A}_{Y^{\mathsf{h}}}\mathsf{A}_{X^{\mathsf{h}}}Z^{\mathsf{h}},W^{\mathsf{h}}) \\ &= g^M(\mathsf{A}_{Y^{\mathsf{h}}}Z^{\mathsf{h}},\mathsf{A}_{X^{\mathsf{h}}}W^{\mathsf{h}}) - g^M(\mathsf{A}_{X^{\mathsf{h}}}Z^{\mathsf{h}},\mathsf{A}_{Y^{\mathsf{h}}}W^{\mathsf{h}}),\end{aligned}$$

and secondly that

$$\begin{aligned}g^M(\nabla^M_{\mathsf{A}_{X^{\mathsf{h}}}Y^{\mathsf{h}}}Z^{\mathsf{h}},W^{\mathsf{h}}) &= g^M(\nabla^M_{Z^{\mathsf{h}}}\mathsf{A}_{X^{\mathsf{h}}}Y^{\mathsf{h}},W^{\mathsf{h}}) + g^M([\mathsf{A}_{X^{\mathsf{h}}}Y^{\mathsf{h}},Z^{\mathsf{h}}],W^{\mathsf{h}}) \\ &= -g^M(\mathsf{A}_{X^{\mathsf{h}}}Y^{\mathsf{h}},\nabla^M_{Z^{\mathsf{h}}}W^{\mathsf{h}}) + 0 \\ &= -g^M(\mathsf{A}_{X^{\mathsf{h}}}Y^{\mathsf{h}},\mathsf{A}_{Z^{\mathsf{h}}}W^{\mathsf{h}}),\end{aligned}$$

since $[\mathsf{A}_{X^{\mathsf{h}}}Y^{\mathsf{h}},Z^{\mathsf{h}}]$ is vertical (as $\mathsf{A}_{X^{\mathsf{h}}}Y^{\mathsf{h}}$ is).

[13] We use the convention $R(X,Y) = [\nabla_X,\nabla_Y] - \nabla_{[X,Y]}$.

(ii) Make $(\boldsymbol{Z}, \boldsymbol{W}) = (\boldsymbol{Y}, \boldsymbol{X})$ in (i).

$\square$

Remark. Using item (v) of Proposition 32 (p. 18), one may rewrite the expression for $K^B(\boldsymbol{X}, \boldsymbol{Y})$ in terms of Lie brackets also. And in the Riemannian case, it follows that π is curvature-increasing over horizontal planes, i.e., $K^B(\boldsymbol{X}, \boldsymbol{Y}) \circ \pi \geq K^M(\boldsymbol{X}^{\mathsf{h}}, \boldsymbol{Y}^{\mathsf{h}})$ holds.

Now that horizontal planes are taken care of, the next natural step would be to move to vertical planes, i.e., planes tangent to the fibers. But the Gauss equation seen in the previous section takes care of that without proof:

Proposition 38. *Let $\mathsf{v}\boldsymbol{X}, \mathsf{v}\boldsymbol{Y} \in \mathfrak{X}(M)$ be vertical fields. Then*

$$K^b(\mathsf{v}\boldsymbol{X}, \mathsf{v}\boldsymbol{Y}) = K^M(\mathsf{v}\boldsymbol{X}, \mathsf{v}\boldsymbol{Y}) + \frac{g^M(\mathsf{T}_{\mathsf{v}\boldsymbol{X}}(\mathsf{v}\boldsymbol{X}), \mathsf{T}_{\mathsf{v}\boldsymbol{Y}}(\mathsf{v}\boldsymbol{Y})) - g^M(\mathsf{T}_{\mathsf{v}\boldsymbol{X}}(\mathsf{v}\boldsymbol{Y}), \mathsf{T}_{\mathsf{v}\boldsymbol{X}}(\mathsf{v}\boldsymbol{Y}))}{g^M(\mathsf{v}\boldsymbol{X} \wedge \mathsf{v}\boldsymbol{Y}, \mathsf{v}\boldsymbol{X} \wedge \mathsf{v}\boldsymbol{Y})},$$

where K^b denotes the sectional curvatures of the fibers M_b.

It is also possible to establish a formula (in terms of covariant derivatives of T and A, akin to the Codazzi-Mainardi equation) for the sectional curvature of a mixed plane, i.e., spanned by a horizontal vector and a vertical vector. We will not pursue this here further, but more details may be consulted in [8], for instance.

Under suitable conditions, a metric on B arises naturally from the metric on M. A general mechanism is that surjective submersions allow us to transfer whatever structure we're interested in the total space, to the base space, provided some extra symmetry assumption is satisfied. We'll start illustrating with the linear case (where the extra assumption is not needed).

Lemma 39.

(a) *Let Z be a real vector space and $F\colon V \to Z$ be $\mathbb{R}$-linear and surjective. Consider the complex space V as a pair (V, J_V), where $J_V\colon V \to V$ is $\mathbb{R}$-linear and satisfies $J_V^2 = -\mathrm{Id}_V$. There is at most one complex structure J_Z on Z which is compatible with F (i.e., such that $F \circ J_V = J_Z \circ F$). And there is exactly one if and only if* $\ker F$ *is a complex subspace of V (i.e., invariant under J_V).*

(b) *Let Z be a complex vector space and $F\colon V \to Z$ be $\mathbb{C}$-linear and surjective. If V is equipped with a hermitian product $\langle \cdot, \cdot \rangle_V$, there is a unique hermitian product $\langle \cdot, \cdot \rangle_Z$ for which the restriction $F|_{(\ker F)^\perp}\colon (\ker F)^\perp \to Z$ is a unitary map.*

Proof:

(a) First, assume that J_Z' and J_Z'' are two complex structures on Z which are compatible with F. Then $J_Z' \circ F = F \circ J_V = J_Z'' \circ F$, and since F is surjective it may be cancelled on the right, giving $J_Z' = J_Z''$, as wanted. This shows that there is at most one complex structure on Z compatible with F. Moving on, assume that $\ker F$ is a complex subspace of V. Let's define $J_Z\colon Z \to Z$ as follows: given $w \in Z$, there is $v \in V$ such that $F(v) = w$. Then set $J_Z(w) \doteq F(J_V(v))$. So:

- J_Z is well-defined: if $v, v' \in V$ are such that $F(v) = w$ and $F(v') = w$, then $F(v - v') = 0$, and $v - v' \in \ker F$. Since $\ker F$ is a complex subspace of V, we have that $J_V(v - v') \in \ker F$, so that $F(J_V(v - v')) = 0$. From this it follows that $F(J_v(v)) = F(J_V(v'))$, as wanted. And J_Z is compatible with F by construction.
- $J_Z^{-1} = -\mathrm{Id}_Z$: since we have that $F(J_V(v)) = J_Z(w)$, where $F(v) = w$, showing that $J_Z^2(w) = -w$ is showing that $F(J_V(J_V(v))) = -w$. But this is clear.

Conversely, if J_Z compatible with F exists, take $v \in \ker F$. Then we may directly compute $F(J_v(v)) = J_Z(F(v)) = J_Z(0) = 0$ and so $J_V(v) \in \ker F$ as well, so that $\ker F$ is a complex subspace of V.

(b) Note that $V = \ker F \oplus (\ker F)^\perp$, so that $V/\ker F \cong (\ker F)^\perp = Z$ says that for given $w \in Z$, among all the elements $v \in V$ for which $F(v) = w$, there is only one of them with vanishing $(\ker F)$-component. Call it $w^\uparrow$. Then we define the product on Z by $\langle w_1, w_2 \rangle_Z \doteq \langle w_1^\uparrow, w_2^\uparrow \rangle_V$. The above discussion says that $\langle \cdot, \cdot \rangle_Z$ is well-defined. Since $(\lambda w)^\uparrow = \lambda w^\uparrow$, $\langle \cdot, \cdot \rangle_Z$ is easily seen to be hermitian as well. The restriction $F|_{(\ker F)^\perp} \colon (\ker F)^\perp \to Z$ becomes unitary by construction and the uniqueness of $\langle \cdot, \cdot \rangle_Z$ in manifest from the definition.

□

Remark. Of course item (b) above also holds if V and W are real spaces and $\langle \cdot, \cdot \rangle_V$ is an inner product – then $\langle \cdot, \cdot \rangle$ will be an inner product as well.

From this, we get the non-linear version of the above:

Proposition 40. *Let M and B be smooth manifolds and $\pi \colon M \to B$ be a surjective submersion. Then:*

(a) If J^M is an almost complex structure on M, for every $x \in M$ we have that $\ker \mathrm{d}F_x$ is a complex subspace of T_xM, and there is a Lie group G acting transitively on the fibers of π by holomorphic transformations, there is a unique almost complex structure J^B on B that turns π into a holomorphic submersion. Moreover, J^B is integrable if J^M is.

(b) If g^M is a pseudo-Riemannian metric on M for which the fibers of π are non-degenerate submanifolds of M, and there is a Lie group G acting transitively on the fibers of π by isometries of (M, g^M), there is a unique pseudo-Riemannian metric g^B on B for which π becomes a pseudo-Riemannian submersion.

Proof: Both items follow from their respective linear versions, since the assumption of a suitable transitive action on the fibers says that the structure to be defined in a point $b \in B$ does not depend on the choice of particular $x \in M_b$. In item (a) to show that J^B is integrable if J^M is one can argue, for instance, that the Nijenhuis tensor of M projects onto the Nijenhuis tensor of B and then invoke the Newlander-Nirenberg theorem[14]. □

We will put this to action in the next section.

[14] Of course there are more elementary proofs, using the Newlander-Nirenberg sledgehammer is just the lazy man's way to get this done with quickly.

3 The Fubini-Study metric on $\mathbb{P}V$

3.1 Construction

Assume from here on that V is equipped with a hermitian inner product $\langle\cdot,\cdot\rangle$. Then $\langle\cdot,\cdot\rangle_{\mathbb{R}} = \mathrm{Re}\langle\cdot,\cdot\rangle$ is a real inner product, and in particular it defines a Riemannian metric on V (regarded as a real manifold of dimension $\dim_{\mathbb{R}} V = 2n$). This is precisely the procedure that gives the usual inner product in $\mathbb{R}^{2n}$ from the usual hermitian product in $\mathbb{C}^n$. In particular, this induces a Riemannian metric g^Σ on the unit sphere $\Sigma = \{x \in V \mid \langle x,x\rangle = 1\}$. Note that $\dim_{\mathbb{R}}\Sigma = 2n-1$ and that $T_x\Sigma = \{v \in V \mid \langle x,v\rangle_{\mathbb{R}} = 0\}$, as expected. Let's list a few properties:

- Multiplication by i is an unitary map, so we have that $\langle \mathrm{i}x,\mathrm{i}y\rangle_{\mathbb{R}} = \langle x,y\rangle_{\mathbb{R}}$, and substituting $y \mapsto -\mathrm{i}y$ we also conclude that $\langle \mathrm{i}x,y\rangle_{\mathbb{R}} = -\langle x,\mathrm{i}y\rangle_{\mathbb{R}}$ for all $x,y\in V$.
- The imaginary part $\mathrm{Im}\langle\cdot,\cdot\rangle$, in turn, is a real-valued skew-symmetric bilinear form due to hermitian symmetry of $\langle\cdot,\cdot\rangle$. More than that, it is in fact symplectic, since multiplication by i is an isomorphism and the relation $\mathrm{Im}\langle x,y\rangle = \langle x,\mathrm{i}y\rangle_{\mathbb{R}}$ holds for all $x,y\in V$, due to $\mathrm{Im}(z) = \mathrm{Re}(-\mathrm{i}z)$ for every $z\in\mathbb{C}$.
- Summarizing the above points, $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{\mathbb{R}} - \mathrm{i}\langle \mathrm{i}\cdot,\cdot\rangle_{\mathbb{R}}$. So $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ alone determines $\langle\cdot,\cdot\rangle$

The restriction $\Pi\colon \Sigma \to \mathbb{P}V$ is again a surjective submersion, so that we may apply what was done in the previous section. However, note that Σ is *not*, in general, a complex manifold itself, and $v \in T_x\Sigma$ does *not* imply that $\mathrm{i}v \in T_x\Sigma$ as well. First, note that the fibers $\Pi^{-1}[L] = L\cap\Sigma$ are circles – this may be easily seen by writing it explictly as $\Pi^{-1}[\mathbb{C}x] = \{\mathrm{e}^{\mathrm{i}t}x \mid t\in\mathbb{R}\}$. Next, we have the:

Lemma 41. *Let $x\in\Sigma$. Then $\mathrm{Ver}_x(\Sigma) = \mathbb{R}\mathrm{i}x$ and $\mathrm{Hor}_x(\Sigma) = (\mathbb{C}x)^\perp$.*

Proof: Since $\mathrm{d}\Pi_x\colon T_x\Sigma \to T_{\mathbb{C}x}(\mathbb{P}V)$ is surjective, the rank-nullity theorem says

$$\dim_{\mathbb{R}} T_x\Sigma = \dim_{\mathbb{R}}\ker \mathrm{d}\Pi_x + \dim_{\mathbb{R}}\,\mathrm{Im}\,\mathrm{d}\Pi_x$$

and so $\dim_{\mathbb{R}}\ker\mathrm{d}\Pi_x = 2n-1-(2n-2) = 1$. With this in place, it suffices to show that $\mathbb{R}\mathrm{i}x \subseteq \ker\mathrm{d}\Pi_x$. Indeed $\mathbb{R}\mathrm{i}x\subseteq T_x\Sigma$, as $\langle x,\mathrm{i}x\rangle_{\mathbb{R}} = 0$, and we also have that

$$\mathrm{d}\Pi_x(\mathrm{i}x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\Pi(\mathrm{e}^{\mathrm{i}t}x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\mathbb{C}x = 0.$$

This settles that $\mathrm{Ver}_x(\Sigma) = \mathbb{R}\mathrm{i}x$. Now, if $x\in T_x\Sigma$ is in the g^Σ-orthogonal complement of $\mathrm{i}\mathbb{R}x$, then $\langle x,v\rangle_{\mathbb{R}} = 0$ (because v is tangent) and also $g^\Sigma_x(\mathrm{i}x,v) = 0$ (because v is horizontal), which together imply that $\langle x,v\rangle = 0$. Since $\mathrm{Hor}_x(\Sigma)\subseteq(\mathbb{C}x)^\perp$ and we have the equality $\dim_{\mathbb{R}}\mathrm{Hor}_x(\Sigma) = \dim_{\mathbb{R}}(\mathbb{C}x)^\perp = 2n-2$ between dimensions, it follows that $\mathrm{Hor}_x(\Sigma) = (\mathbb{C}x)^\perp$. □

Corollary 42. *Let $w,w'\in T_x\Sigma$. Then:*

(i) $\mathsf{v}w = \mathrm{i}g^\Sigma_x(w,\mathrm{i}x)x$ *and* $\mathsf{h}w = w - \mathrm{i}g^\Sigma_x(w,\mathrm{i}x)x$.

(ii) $g^\Sigma_x(\mathsf{v}w, \mathsf{v}w') = g^\Sigma_x(w, \mathrm{i}x) g^\Sigma_x(w', \mathrm{i}x)$.

(iii) $g^\Sigma_x(\mathsf{h}w, \mathsf{h}w') = g^\Sigma_x(w, w') - g^\Sigma_x(w, \mathrm{i}x) g^\Sigma_x(w', \mathrm{i}x)$.

(iv) $\mathrm{i}(\mathsf{h}w)$ *is tangent to* Σ *and horizontal, while* $\mathrm{i}(\mathsf{v}w)$ *is normal to* Σ.

Remark. It is meaningless to look at $\mathsf{v}(\mathrm{i}w)$ and $\mathsf{h}(\mathrm{i}w)$ instead because, as mentioned before, w tangent does not imply $\mathrm{i}w$ tangent. In fact, it follows from (iv) that $\mathrm{i}w$ is tangent if and only if w is horizontal.

Proof: Only (iv) deserves some comment. We have that $\mathrm{i}(\mathsf{h}w)$ is tangent to Σ because $g^\Sigma_x(\mathrm{i}(\mathsf{h}w), x) = -g^\Sigma_x(\mathsf{h}w, \mathrm{i}x) = 0$. And it is also horizontal because its vertical part vanishes, due to $\mathsf{v}(\mathrm{i}(\mathsf{h}w)) = \mathrm{i}g^\Sigma_x(\mathrm{i}(\mathsf{h}w), \mathrm{i}x)x = \mathrm{i}\langle \mathsf{h}w, x\rangle_{\mathbb{R}} x = 0$. Lastly, $\mathrm{i}(\mathsf{v}w)$ is normal to Σ because $g^\Sigma_x(\mathrm{i}(\mathsf{v}w), x) = -g^\Sigma_x(\mathsf{v}w, \mathrm{i}x) = 0$. $\square$

Corollary 43. *Let* $H \in T_L(\mathrm{P}V)$ *and* $x \in L \cap \Sigma$. *Then* $H^{\mathsf{h}}_x = Hx$.

Proof: Just note that H takes values in $(\mathbb{C}x)^\perp = \mathrm{Hor}_x(\Sigma)$ and $\mathrm{d}\Pi_x(Hx) = H$ under the isomorphism $T_L(\mathrm{P}V) = \mathrm{Hom}(L, L^\perp)$. $\square$

Moving on, we need to transfer the geometry of Σ to $\mathrm{P}V$ via π, and for that we need a suitable group action on Σ.

Proposition 44. *The natural action* $\mathrm{U}(1) \circlearrowright V \setminus \{0\}$ *given by scalar multiplication is fiber-preserving and consists of holomorphic isometries. It also restricts to an action* $\mathrm{U}(1) \circlearrowright \Sigma$ *with is in addition transitive on each fiber.*

Remark. We write $\mathrm{U}(1)$ instead of $\mathbb{S}^1$ above because this will more easily hint at the generalization to be done in Section 4.

Proof: Since $\mathrm{U}(1)$ consists of unit complex numbers and multiplications by scalar are $\mathbb{C}$-linear, the action is fiber-preserving and consists of holomorphic transformation. In particular, $\mathrm{U}(1) \subseteq \mathrm{O}(V, \mathrm{Re}\langle\cdot,\cdot\rangle) = \mathrm{Iso}(\Sigma, g^\Sigma)$, so we also have isometries. And given two points $x, y \in \Sigma$ in the same fiber, there is $\lambda \in \mathrm{U}(1)$ such that $y = \lambda x$ (namely, $\lambda = \langle y, x\rangle$). $\square$

Definition 45. The unique Riemannian metric g^{FS} on $\mathrm{P}V$ that turns $\Pi\colon \Sigma \to \mathrm{P}V$ into a Riemannian submersion is called the *Fubini-Study metric* on $\mathrm{P}V$. The metric g^{FS} is in fact hermitian (i.e., compatible with the natural complex structure of $\mathrm{P}V$).

Remark.

- In practice, this means that given $L \in \mathrm{P}V$ and $H_1, H_2 \in T_L(\mathrm{P}V)$, we have that $g^{\mathrm{FS}}_L(H_1, H_2) = g^\Sigma_x(H_1 x, H_2 x)$, where $x \in L$ is any unit vector.
- We'll denote the Levi-Civita connection of $(\mathrm{P}V, g^{\mathrm{FS}})$ by ∇^{FS}.

3.2 Properties

To move on, we need to understand ∇^{FS}. For that, we'll need to understand the Levi-Civita connection ∇^{Σ} of (Σ, g^{Σ}). We have that $\boldsymbol{N}\colon \Sigma \to V$ given by $\boldsymbol{N}(x) = x$ is a unit real-normal field along Σ, since for any tangent vector $v \in T_x\Sigma$, we have that $\langle \boldsymbol{N}(x), v\rangle_{\mathbb{R}} = \langle x, v\rangle_{\mathbb{R}} = 0$. Now, vector fields on V are just smooth functions $V \to V$, while a vector field $\boldsymbol{X} \in \mathfrak{X}(\Sigma)$ is just any smooth function $\boldsymbol{X}\colon \Sigma \to V$ such that $\langle \boldsymbol{X}_x, x\rangle_{\mathbb{R}} = 0$ for all $x \in \Sigma$. In terms of $\boldsymbol{N}$, Corollary 42 (p. 23) reads:

Corollary 46. *Let $\boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{X}(\Sigma)$. Then:*

(i) $\mathsf{v}\boldsymbol{X} = \mathrm{i}g^{\Sigma}(\boldsymbol{X}, \mathrm{i}\boldsymbol{N})\boldsymbol{N}$ *and* $\mathsf{h}\boldsymbol{X} = \boldsymbol{X} - \mathrm{i}g^{\Sigma}(\boldsymbol{X}, \mathrm{i}\boldsymbol{N})\boldsymbol{N}$.

(ii) $g^{\Sigma}(\mathsf{v}\boldsymbol{X}, \mathsf{v}\boldsymbol{Y}) = g^{\Sigma}(\boldsymbol{X}, \mathrm{i}\boldsymbol{N})g^{\Sigma}(\boldsymbol{Y}, \mathrm{i}\boldsymbol{N})$.

(iii) $g^{\Sigma}(\mathsf{h}\boldsymbol{X}, \mathsf{h}\boldsymbol{Y}) = g^{\Sigma}(\boldsymbol{X}, \boldsymbol{Y}) - g^{\Sigma}(\boldsymbol{X}, \mathrm{i}\boldsymbol{N})g^{\Sigma}(\boldsymbol{Y}, \mathrm{i}\boldsymbol{N})$.

(iv) $\mathrm{i}(\mathsf{h}\boldsymbol{X})$ *is tangent to Σ and horizontal, while* $\mathrm{i}(\mathsf{v}\boldsymbol{X})$ *is normal to Σ.*

The Levi-Civita connection of $(V, \langle\cdot,\cdot\rangle_{\mathbb{R}})$ is the standard flat connection D, which is given by $(\mathrm{D}_{\boldsymbol{X}}\boldsymbol{Y})_x = D\boldsymbol{Y}(x)(\boldsymbol{X})$, where $D\boldsymbol{Y}(x)$ is the total derivative of $\boldsymbol{Y}$. Thus, ∇^{Σ} is obtained from projecting D onto $T\Sigma$ and we may write $\nabla^{\Sigma}_{\boldsymbol{X}}\boldsymbol{Y} = \mathrm{D}_{\boldsymbol{X}}\boldsymbol{Y} - \langle \mathrm{D}_{\boldsymbol{X}}\boldsymbol{Y}, \boldsymbol{N}\rangle_{\mathbb{R}}\boldsymbol{N}$. However, $\langle \boldsymbol{Y}, \boldsymbol{N}\rangle_{\mathbb{R}} = 0$, so differentiating that and evaluating at $\boldsymbol{X}$ we get the crucial relation $\langle \mathrm{D}_{\boldsymbol{X}}\boldsymbol{Y}, \boldsymbol{N}\rangle_{\mathbb{R}} + g^{\Sigma}(\boldsymbol{Y}, \boldsymbol{X}) = 0$, since the total derivative $D\boldsymbol{N}(x)$ is the identity map. We have proven item (i) in:

Lemma 47. *For $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \mathfrak{X}(\Sigma)$, we have that:*

(i) $\nabla^{\Sigma}_{\boldsymbol{X}}\boldsymbol{Y} = \mathrm{D}_{\boldsymbol{X}}\boldsymbol{Y} + g^{\Sigma}(\boldsymbol{X}, \boldsymbol{Y})\boldsymbol{N}$.

(ii) $R^{\Sigma}(\boldsymbol{X}, \boldsymbol{Y})\boldsymbol{Z} = g^{\Sigma}(\boldsymbol{Y}, \boldsymbol{Z})\boldsymbol{X} - g^{\Sigma}(\boldsymbol{X}, \boldsymbol{Z})\boldsymbol{Y}$.

(iii) $K^{\Sigma} = 1$ *is a constant.*

Proof:

(i) Done above.

(ii) From (i), the second fundamental form of Σ relative to V is $\alpha(\boldsymbol{X}, \boldsymbol{Y}) = g^{\Sigma}(\boldsymbol{X}, \boldsymbol{Y})\boldsymbol{N}$. Since $R^{\mathrm{D}} = 0$, for any additional $\boldsymbol{W} \in \mathfrak{X}(\Sigma)$ we have that

$$\begin{aligned} R^{\Sigma}(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{W}) &= \langle g^{\Sigma}(\boldsymbol{Y}, \boldsymbol{Z})\boldsymbol{N}, g^{\Sigma}(\boldsymbol{X}, \boldsymbol{W})\boldsymbol{N}\rangle_{\mathbb{R}} - \langle g^{\Sigma}(\boldsymbol{X}, \boldsymbol{Z})\boldsymbol{N}, g^{\Sigma}(\boldsymbol{Y}, \boldsymbol{W})\boldsymbol{N}\rangle_{\mathbb{R}} \\ &= g^{\Sigma}(\boldsymbol{Y}, \boldsymbol{Z})g^{\Sigma}(\boldsymbol{X}, \boldsymbol{W}) - g^{\Sigma}(\boldsymbol{X}, \boldsymbol{Z})g^{\Sigma}(\boldsymbol{Y}, \boldsymbol{W}) \\ &= g^{\Sigma}(g^{\Sigma}(\boldsymbol{Y}, \boldsymbol{Z})\boldsymbol{X} - g^{\Sigma}(\boldsymbol{X}, \boldsymbol{Z})\boldsymbol{Y}, \boldsymbol{W}). \end{aligned}$$

Since g^{Σ} is non-degenerate, we may now drop $\boldsymbol{W}$.

(iii) If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are orthonormal fields, then

$$K^{\Sigma}(\boldsymbol{X}, \boldsymbol{Y}) = \frac{R^{\Sigma}(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Y}, \boldsymbol{X})}{g^{\Sigma}(\boldsymbol{X}, \boldsymbol{X})g^{\Sigma}(\boldsymbol{Y}, \boldsymbol{Y}) - g^{\Sigma}(\boldsymbol{X}, \boldsymbol{Y})^2} = \frac{g^{\Sigma}(1\cdot\boldsymbol{X} - 0\cdot\boldsymbol{Y}, \boldsymbol{X})}{1\cdot 1 - 0^2} = 1,$$

as required.

□

With this in place, let's compute T and A. We will see that the vector field $\mathrm{i}\boldsymbol{N}$ will play an important role here. So we will summarize some of its properties:

Proposition 48.

(i) $\nabla^\Sigma_{\mathrm{i}\boldsymbol{N}}(\mathrm{i}\boldsymbol{N}) = \boldsymbol{0}$.

(ii) For any $\boldsymbol{X} \in \mathfrak{X}(\Sigma)$, *we have that* $\nabla^\Sigma_{\boldsymbol{X}}(\mathrm{i}\boldsymbol{N}) = \mathrm{i}(\mathsf{h}\boldsymbol{X})$.

(iii) The flow of $\mathrm{i}\boldsymbol{N}$ *is* $\Phi_{\mathrm{i}\boldsymbol{N}}\colon \mathbb{R}\times\Sigma \to \Sigma$ *given by* $\Phi_{t,\mathrm{i}\boldsymbol{N}}(x) = \mathrm{e}^{\mathrm{i}t}x$.

Proof:

(i) Since $\mathrm{i}\boldsymbol{N}$ is the restriction of a $\mathbb{R}$-linear map, we have

$$\nabla^\Sigma_{\mathrm{i}\boldsymbol{N}}(\mathrm{i}\boldsymbol{N}) = \mathrm{D}_{\mathrm{i}\boldsymbol{N}}(\mathrm{i}\boldsymbol{N}) + g^\Sigma(\mathrm{i}\boldsymbol{N},\mathrm{i}\boldsymbol{N})\boldsymbol{N} = \mathrm{i}(\mathrm{i}\boldsymbol{N}) + \boldsymbol{N} = \boldsymbol{0}.$$

(ii) It is another direct computation:

$$\nabla^\Sigma_{\boldsymbol{X}}(\mathrm{i}\boldsymbol{N}) \overset{(i)}{=} \nabla^\Sigma_{\mathsf{h}\boldsymbol{X}}(\mathrm{i}\boldsymbol{N}) = \mathrm{D}_{\mathsf{h}\boldsymbol{X}}(\mathrm{i}\boldsymbol{N}) + g^\Sigma(\mathsf{h}\boldsymbol{X},\mathrm{i}\boldsymbol{N})\boldsymbol{N} = \mathrm{i}(\mathsf{h}\boldsymbol{X}) + 0 = \mathrm{i}(\mathsf{h}\boldsymbol{X}).$$

This is horizontal due to item (iv) in Corollary 46 (p. 25).

(iii) Straightforward.

□

Now the next result readily follows:

Theorem 49. *Given* $\boldsymbol{X},\boldsymbol{Y} \in \mathfrak{X}(\Sigma)$, *we have that* $\mathsf{T} = 0$ *and*

$$\mathsf{A}_{\boldsymbol{X}}\boldsymbol{Y} = \mathrm{i}g^\Sigma(\boldsymbol{Y},\mathrm{i}\boldsymbol{N})\,\mathsf{h}\boldsymbol{X} + \mathrm{i}g^\Sigma(\boldsymbol{X},\mathrm{i}(\mathsf{h}\boldsymbol{Y}))\boldsymbol{N}.$$

Proof: A fiber $\Pi^{-1}[\mathbb{C}x]$ can be parametrized by $\gamma\colon \mathbb{R}\to\Sigma$ given by $\gamma(t) = \mathrm{e}^{\mathrm{i}t}x$. We have that $\gamma'(t) = \mathrm{i}\mathrm{e}^{\mathrm{i}t}x$ is a unit vector and $\gamma''(t) = -\mathrm{e}^{\mathrm{i}t}x = -\boldsymbol{N}(\gamma(t))$ is normal to Σ, so γ is a geodesic and $\Pi^{-1}[\mathbb{C}x]$ is totally geodesic (of course, one can also argue that $\mathrm{i}\boldsymbol{N}$ has constant length, so that $\nabla^\Sigma_{\mathrm{i}\boldsymbol{N}}(\mathrm{i}\boldsymbol{N}) = \boldsymbol{0}$ means that its integral curves are geodesics). Hence $\mathsf{T} = 0$. Another way to see this is by directly computing T: since $\nabla^\Sigma_{\mathrm{i}\boldsymbol{N}}(\mathrm{i}\boldsymbol{N}) = \boldsymbol{0}$, it follows that $\mathsf{T}_{\mathrm{i}\boldsymbol{N}}(\mathrm{i}\boldsymbol{N}) = \boldsymbol{0}$ as well. Next, we use that for $\boldsymbol{Y}\in\mathfrak{X}(\Sigma)$, the relation

$$\mathsf{T}_{\mathrm{i}\boldsymbol{N}}(\mathsf{h}\boldsymbol{Y}) = \mathsf{v}\nabla^\Sigma_{\mathrm{i}\boldsymbol{N}}(\mathsf{h}\boldsymbol{Y}) = \mathrm{i}g^\Sigma(\nabla^\Sigma_{\mathrm{i}\boldsymbol{N}}(\mathsf{h}\boldsymbol{Y}),\mathrm{i}\boldsymbol{N})\boldsymbol{N} = -\mathrm{i}g^\Sigma(\mathsf{h}\boldsymbol{Y},\nabla^\Sigma_{\mathrm{i}\boldsymbol{N}}(\mathrm{i}\boldsymbol{N}))\boldsymbol{N} = \boldsymbol{0}$$

holds due to the above. Then items (i) and (iii) from Proposition 32 (p. 18) together imply that $\mathsf{T} = 0$. As for A, we proceed directly in two steps: first we compute a vertical part as $\mathsf{A}_{\boldsymbol{X}}(\mathrm{i}\boldsymbol{N}) = \mathsf{h}\nabla^\Sigma_{\mathsf{h}\boldsymbol{X}}(\mathrm{i}\boldsymbol{N}) = \mathsf{h}(\mathrm{i}(\mathsf{h}\boldsymbol{X})) = \mathrm{i}(\mathsf{h}\boldsymbol{X})$, and then a horizontal part as

$$\begin{aligned}\mathsf{A}_{\boldsymbol{X}}(\mathsf{h}\boldsymbol{Y}) &= \mathsf{v}\nabla^\Sigma_{\mathsf{h}\boldsymbol{X}}(\mathsf{h}\boldsymbol{Y}) = \mathrm{i}g^\Sigma(\nabla^\Sigma_{\mathsf{h}\boldsymbol{X}}(\mathsf{h}\boldsymbol{Y}),\mathrm{i}\boldsymbol{N})\boldsymbol{N} \\ &= -\mathrm{i}g^\Sigma(\mathsf{h}\boldsymbol{Y},\nabla^\Sigma_{\mathsf{h}\boldsymbol{X}}(\mathrm{i}\boldsymbol{N}))\boldsymbol{N} = -\mathrm{i}g^\Sigma(\mathsf{h}\boldsymbol{Y},\mathrm{i}(\mathsf{h}\boldsymbol{X}))\boldsymbol{N} \\ &= \mathrm{i}g^\Sigma(\mathrm{i}(\mathsf{h}\boldsymbol{Y}),\mathsf{h}\boldsymbol{X})\boldsymbol{N} = \mathrm{i}g^\Sigma(\boldsymbol{X},\mathrm{i}(\mathsf{h}\boldsymbol{Y}))\boldsymbol{N}.\end{aligned}$$

Putting everything together using that A is a tensor, the result follows. □

Now we obtain the sectional curvature $K^{\rm FS}$ of $(\mathrm{P}V, g^{\rm FS})$:

Theorem 50. *Let $L \in \mathrm{P}V$ and $H_1, H_2 \in T_L(\mathrm{P}V)$ be linearly independent vectors. Then the relation*

$$K^{\rm FS}(H_1,H_2)_L = 1 + \frac{3g_L^{\rm FS}(\mathrm{i}H_1,H_2)^2}{\|H_1\wedge H_2\|^2}$$

holds. In particular, the minimum value 1 is achieved when $\{H_1,H_2\}$ is orthonormal and $\mathrm{i}H_2$ is also orthogonal to H_1, while the maximum value 4 is achieved for holomorphic curvatures, *i.e., given a unit vector $H \in T_L(\mathrm{P}V)$ we have that $K^{\rm FS}(H,\mathrm{i}H)_L = 4$.*

Corollary 51. *$(\mathrm{P}V, g^{\rm FS})$ is an Einstein manifold with $\mathrm{Ric}^{\rm FS} = 2ng^{\rm FS}$, where we recall that $n = \dim_{\mathbb{C}}(\mathrm{P}V)$. It follows that the scalar curvature is $\mathrm{s}^{\rm FS} = 2n(2n-2)$.*

Proof: Let $L \in \mathrm{P}V$ and $H \in T_L(\mathrm{P}V)$ be a unit vector. Extend the single vector H to an orthonormal basis $(H, E_2, \ldots, E_{2n-2})$ of $T_L(\mathrm{P}V)$ (since $\dim_{\mathbb{R}}(\mathrm{P}V) = 2n-2$) and compute:

$$\begin{aligned}\mathrm{Ric}_L^{\rm FS}(H,H) &= \sum_{j=2}^{2n-2} K^{\rm FS}(H,E_j) = \sum_{j=2}^{2n-2}(1+3g_L^{\rm FS}(\mathrm{i}H,E_j)^2) \\ &= 2n-3+3\|\mathrm{i}H\|^2 = 2n-3+3 \\ &= 2n.\end{aligned}$$

Renormalizing and polarizing, it follows that $\mathrm{Ric}^{\rm FS} = 2ng^{\rm FS}$. ☐

Remark. Adjusting the dimension for $(\mathbb{C}\mathrm{P}^n, g^{\rm FS})$, we obtain $\mathrm{Ric}^{\rm FS} = 2(n+1)g^{\rm FS}$. The above computations show that while $\mathbb{C}\mathrm{P}^1$ is *diffeomorphic* to a sphere $\mathbb{S}^2$, it is actually *isometric* only to a sphere $\mathbb{S}^2(1/2)$ of radius $r = 1/2$ because, in this case, the holomorphic curvature is actually the constant Gaussian curvature of the surface (and $1/r^2 = 4$ with $r > 0$ implies $r = 1/2$, as claimed).

To discuss more properties of $(\mathrm{P}V, g^{\rm FS})$, we turn to its isometries:

Proposition 52.

(i) $\mathrm{PU}(V, \langle\cdot,\cdot\rangle) \subseteq \mathrm{Iso}(\mathrm{P}V, g^{\rm FS})$.

(ii) Every transformation in $\mathfrak{u}(V,\langle\cdot,\cdot\rangle)$, regarded as a vector field[15] *on Σ, projects to a Killing field on $(\mathrm{P}V, g^{\rm FS})$.*

Proof:

(i) Let $T \in \mathrm{U}(V,\langle\cdot,\cdot\rangle)$. Since for any line $L \in \mathrm{P}V$ we have that $V/L = L^\perp$ and $T[L]^\perp = T[L^\perp]$, we have that $V/T[L] \cong T[L^\perp]$. So, the formula for the derivative $\mathrm{d}(\mathrm{P}T)_L \colon T_L(\mathrm{P}V) \to T_{T[L]}(\mathrm{P}V)$ seen in Example 10 (p. 8) indeed becomes just

[15]Since $\mathfrak{u}(V,\langle\cdot,\cdot\rangle) = \{B \in \mathrm{End}(V) \mid \langle Bx,y\rangle + \langle x,By\rangle = 0 \text{ for all } x,y \in V\}$, hermitian symmetry of $\langle\cdot,\cdot\rangle$ gives that for all $B \in \mathfrak{u}(V,\langle\cdot,\cdot\rangle)$ and all $x \in \Sigma$ we have $g^V(Bx,x) = 0$, which means that $Bx \in T_x\Sigma$.

$\mathrm{d}(\mathrm{P}V)_L(H) = T \circ H \circ T^{-1}|_{T[L]}$. So, given $H_1, H_2 \in T_L(\mathrm{P}V)$, we choose $x \in L \cap \Sigma$ and compute

$$\begin{aligned} g^{\mathrm{FS}}_{T[L]}(\mathrm{d}(\mathrm{P}T)_L(H_1), \mathrm{d}(\mathrm{P}T)_L(H_2)) &= g^{\Sigma}_{Tx}(\mathrm{d}(\mathrm{P}T)_L(H_1)Tx, \mathrm{d}(\mathrm{P}T)_L(H_2)Tx) \\ &= g^{\Sigma}_{Tx}(TH_1T^{-1}Tx, TH_2T^{-1}Tx) \\ &= g^{\Sigma}_{Tx}(TH_1x, TH_2x) \\ &= g^{\Sigma}_{x}(H_1x, H_2x) \\ &= g^{\mathrm{FS}}_{L}(H_1, H_2), \end{aligned}$$

as wanted.

(ii) This follows from Example 12 (p. 9) together with item (i) above.

$\square$

With this in place, we can establish that $(\mathrm{P}V, g^{\mathrm{FS}})$ is "homogeneous", in the "weak" following sense:

Proposition 53. *Let $L_1, L_2 \in \mathrm{P}V$ and consider a linear isometry $A\colon T_{L_1}(\mathrm{P}V) \to T_{L_2}(\mathrm{P}V)$. There is an isometry $F \in \mathrm{PU}(V, \langle\cdot,\cdot\rangle)$ such that $F(L_1) = L_2$ and $\mathrm{d}F_{L_1} = A$.*

Proof: Pick points $x_1, x_2 \in \Sigma$ such that $L_j = \mathbb{C}x_j$ (for $j = 1, 2$), and recall that we have $\mathrm{Hor}_{x_j}(\Sigma) = (\mathbb{C}x_j)^{\perp} \cong T_{L_j}(\mathrm{P}V)$ under $\mathrm{d}\pi_{x_j}$ (again for $j = 1, 2$). So there is a (unique) linear isometry $\widetilde{A}$ such that the diagram

$$\begin{tikzcd} (\mathbb{C}x_1)^{\perp} \arrow[r, dashed, "\widetilde{A}"] \arrow[d, "\mathrm{d}\pi_{x_1}"', "\cong"] & (\mathbb{C}x_2)^{\perp} \arrow[d, "\cong"', "\mathrm{d}\pi_{x_2}"] \\ T_{L_1}(\mathrm{P}V) \arrow[r, "A"] & T_{L_2}(\mathrm{P}V) \end{tikzcd}$$

commutes. Then $T\colon V \to V$ defined by $Tx_1 = x_2$ and $T|_{(\mathbb{C}x_1)^{\perp}} = \widetilde{A}$ is also a linear isometry, and $F \doteq \mathrm{P}T$ is an isometry with $F(L_1) = L_2$ and $\mathrm{d}F_{L_1} = A$, as wanted. $\square$

This has lots of interesting consequences:

Corollary 54. $\mathrm{PU}(V, \langle\cdot,\cdot\rangle) = \mathrm{Iso}(\mathrm{P}V, g^{\mathrm{FS}})$.

Proof: In connected pseudo-Riemannian manifolds, isometries are determined by its 1-jet at any point. By the previous result, the 1-jet of any isometry of $(\mathrm{P}V, g^{\mathrm{FS}})$ can be realized by an element of $\mathrm{PU}(V, \langle\cdot,\cdot\rangle)$, so the original isometry must be in $\mathrm{PU}(V, \langle\cdot,\cdot\rangle)$ as well. $\square$

Corollary 55. $(\mathrm{P}V, g^{\mathrm{FS}})$ *is a locally symmetric Kähler-Einstein manifold.*

Proof: We have already seen that $(\mathrm{P}V, g^{\mathrm{FS}})$ is an Einstein manifold. Proving that if $J\colon T(\mathrm{P}V) \to T(\mathrm{P}V)$ is the complex structure of $\mathrm{P}V$ then $\nabla^{\mathrm{FS}}J = 0$ is a pointwise statement: fix $L \in \mathrm{P}V$ and take an isometry $F\colon \mathrm{P}V \to \mathrm{P}V$ such that $F(L) = L$ and $\mathrm{d}F_L = -\mathrm{Id}_{T_L(\mathrm{P}V)}$. Then the rank of the $(1,2)$-tensor $\nabla^{\mathrm{FS}}J$ being 3 (since J has type $(1,1)$) implies that $(F^*(\nabla^{\mathrm{FS}}J))_L = (-1)^3(\nabla^{\mathrm{FS}}J)_L = -(\nabla^{\mathrm{FS}}J)_L$, while F being an isometry that fixes L also means that $(F^*(\nabla^{\mathrm{FS}}J))_L = (\nabla^{\mathrm{FS}}J)_L$. So $(\nabla^{\mathrm{FS}}J)_L = 0$ and L being arbitrary implies that $\nabla^{\mathrm{FS}}J = 0$, and thus $(\mathrm{P}V, g^{\mathrm{FS}})$ is Kähler. To see that $(\mathrm{P}V, g^{\mathrm{FS}})$ is locally symmetric, we repeat the same argument using that the rank of $\nabla^{\mathrm{FS}}R^{\mathrm{FS}}$ is 5, which is odd (here R^{FS} stands for the curvature tensor), so that $\nabla^{\mathrm{FS}}R^{\mathrm{FS}} = 0$. $\square$

Corollary 56. *For every subspace $W \subseteq V$ the inclusion $\mathrm{P}W \subseteq \mathrm{P}V$ is an isometric embedding, $\mathrm{P}W$ is a totally geodesic submanifold of $(\mathrm{P}V, g^{\mathrm{FS}})$, and every connected totally geodesic complex submanifold equals $\mathrm{P}W$ for some subspace $W \subseteq V$.*

Proof: The products in V restrict to products in W, and the unit sphere of W is the intersection of W with the unit sphere of V. This means that the metric induced in $\mathrm{P}W$ by the Fubini-Study metric of $\mathrm{P}V$ is in fact the Fubini-Study metric of $\mathrm{P}W$. Now consider the Householder reflection $T = \mathrm{Id}_V - 2\,\mathrm{pr}_{W^\perp} \in \mathrm{U}(V, \langle\cdot,\cdot\rangle)$. Note that we have $T|_W = \mathrm{Id}_W$. Then $\mathrm{P}T \in \mathrm{Iso}(\mathrm{P}V, g^{\mathrm{FS}})$ has $\mathrm{P}W$ as the set of its fixed points. So, $\mathrm{P}W$ being the fixed point set of an isometry of the ambient space, is totally geodesic. Conversely, if $N \subseteq \mathrm{P}V$ is a connected totally geodesic complex submanifold, N is completely determined by a point $L \in N$ and a tangent space T_LN. Thus, if $\dim N = k$, we choose a basis $H_1, \ldots, H_k$ for T_LN and take $W = L \oplus \bigoplus_{i=1}^k H_i[L]$. Then $L \in \mathrm{P}W$ and $H_i \in T_L(\mathrm{P}W)$ for all $i = 1, \ldots, k$ (since $H_i[L] \subseteq L^\perp$ and the definition of W means that $H_i[L]$ also lies in the orthogonal complement of L relative to W). Since we have $\dim \mathrm{P}W = \dim N$, this means that $T_L(\mathrm{P}W) = T_LN$. Thus $\mathrm{P}W = N$ as required. $\square$

Remark. If V were a real space, this classifies all the totally geodesic submanifolds of $\mathrm{P}V$. It is possible to modify the above argument to classify all totally real totally geodesic submanifolds of $\mathrm{P}V$ as fixed points of antiholomorphic isometries (induced by anti-linear reflections).

Corollary 57. *Any element of $\mathfrak{u}(V, \langle\cdot,\cdot\rangle)$, regarded as a vector field on Σ, projects onto a Killing field on $\mathrm{P}V$.*

Proof: From Example 12 (p. 9) we have that the flow of the projected field consists of isometries. $\square$

Remark. Assume here that $\langle\cdot,\cdot\rangle$ is complex-Lorentzian, i.e., it is $\mathbb{C}$-linear in the first entry, has hermitian symmetry, and $\langle\cdot,\cdot\rangle$-orthonormal basis for V has one timelike vector and $n-1$ spacelike vectors, where a non-zero $x \in V$ is timelike if $\langle x, x\rangle < 0$, spacelike if $\langle x, x\rangle > 0$ and lightlike if $\langle x, x\rangle = 0$. We say that $L \in \mathrm{P}V$ is timelike if $\langle\cdot,\cdot\rangle$ restricted to L is negative-definite. The *complex hyperbolic space* associated to V is

$$\mathrm{H}V \doteq \{L \in \mathrm{P}V \mid L \text{ is timelike.}\}.$$

It is an open submanifold of $\mathrm{P}V$ due to continuity of $\langle\cdot,\cdot\rangle$. Note that $\mathrm{H}V$ is the image under Π of the hyperboloid $\Sigma = \{x \in V \mid \langle x, x\rangle = -1\}$. Then $\langle\cdot,\cdot\rangle_{\mathbb{R}} = \mathrm{Re}\langle\cdot,\cdot\rangle$ is a real pseudo-Euclidean scalar product on V with index 2 (i.e., the highest dimension of a subspace on which $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ is negative-definite is 2), whose restriction defines a metric g^Σ on Σ. Again g^Σ survives in the quotient and we obtain a metric in $\mathrm{H}V$. It can be studied using the same strategy as presented above.

4 Calculus in Grassmannians

4.1 General features

Now we generalize projective spaces.

Definition 58. Let $1 \leq k \leq n$. The *Grassmannian of k-planes* of V is defined as

$$\mathrm{Gr}_k(V) \doteq \{W \subseteq V \mid W \text{ is a subspace of } V \text{ with } \dim W = k\}.$$

It will be useful to also consider the *Stiefel manifold of k-frames* of V, defined as

$$\mathrm{St}_k(V) \doteq \{\mathfrak{x} \in V^k \mid \mathfrak{x} \text{ is a } k\text{-tuple of linearly independent vectors}\}.$$

As a default notation, we will always use the same kernel letter[16] for a k-uple of vectors, as in $\mathfrak{x} = (x_1, \ldots, x_k)$, and this will be regarded as a row vector with vector entries when being acted on by the right with a matrix. Since linear independence is an open condition, $\mathrm{St}_k(V) \subseteq V^k$ is open, and hence a complex submanifold of V^k. We have a span-projection $\Pi\colon \mathrm{St}_k(V) \to \mathrm{Gr}_k(V)$ given by $\Pi(\mathfrak{x}) = \mathrm{span}(\mathfrak{x})$. And $\mathrm{Gr}_k(V)$ may also be described as a quotient, by saying that $\mathfrak{x} \sim \mathfrak{x}'$ if there is $A \in \mathrm{GL}(k,\mathbb{C})$ such that[17] $\mathfrak{x}' = \mathfrak{x}A$. So we may write that $\mathrm{Gr}_k(V) = \mathrm{St}_k(V)/_\sim$, and say that it is the orbit-space of the action $\mathrm{St}_k(V) \circlearrowleft \mathrm{GL}(k,\mathbb{C})$. And so, $\mathrm{Gr}_k(V)$ gains a quotient topology. For $k = 1$, note that this is precisely the projection $\Pi\colon V \setminus \{0\} \to \mathrm{P}V$ we have seen before. Let's see how much of the previous arguments we can repeat here:

Proposition 59. $\mathrm{Gr}_k(V)$ *is a compact Hausdorff space.*

Proof: Fix a hermitian inner product $\langle\cdot,\cdot\rangle$ on V. Every subspace of V admits an orthonormal basis, which means that the restriction of the projection Π to the Stiefel manifold of *orthonormal* k-frames, $\mathrm{St}_k(V,\langle\cdot,\cdot\rangle)$, is surjective. This way, compactness of $\mathrm{Gr}_k(V)$ follows from the compactness of $\mathrm{St}_k(V,\langle\cdot,\cdot\rangle)$, which in turn may be verified in several ways (for instance, express $\mathrm{St}_k(V,\langle\cdot,\cdot\rangle)$ as a quotient of $\mathrm{U}(V,\langle\cdot,\cdot\rangle)$ or argue that it is a closed and bounded subset of V^k). For Hausdorffness, we again verify (just like Proposition 2, p. 2) that the defining equivalence relation $\sim \subseteq \mathrm{St}_k(V) \times \mathrm{St}_k(V)$ is closed. Assume that $\mathfrak{x}, \mathfrak{x}' \in \mathrm{St}_k(V)$ and that we have sequences $(\mathfrak{x}_n)_{n\geq 0}$, $(\mathfrak{x}'_n)_{n\geq 0}$ in $\mathrm{St}_k(V)$ with $\mathfrak{x}_n \sim \mathfrak{x}'_n$ for all $n \geq 0$, with $\mathfrak{x}_n \to \mathfrak{x}$ and $\mathfrak{x}'_n \to \mathfrak{x}'$. Our goal is to show that $\mathfrak{x} \sim \mathfrak{x}'$. So, for every $n \geq 0$, write $\mathfrak{x}'_n = \mathfrak{x}_n A_n$, with $A_n \in \mathrm{GL}(k,\mathbb{C})$. Since we have that $\mathfrak{x}_n = [x_{n,1} \cdots x_{n,k}]$ is linearly independent, the matrix $G_n = (\langle x_{n,i}, x_{n,j}\rangle)_{i,j=1}^k$ is non-singular. And if $\mathfrak{x}'_n = [x'_{n,1} \cdots x'_{n,k}]$ and we set $B_n = (\langle x'_{n,i}, x_{n,j}\rangle)_{i,j=1}^k$, it follows

[16] The symbol $\mathfrak{x}$ is a Fraktur x.

[17] If $\mathfrak{x} = [x_1 \cdots x_k]$ and $\mathfrak{x}' = [x'_1 \cdots x'_k]$ are related via $x'_j = \sum_{i=1}^k a^i{}_j x_i$ with $A = (a^i{}_j)_{i,j=1}^k \in \mathrm{GL}(k,\mathbb{C})$, this is written as $\mathfrak{x}' = \mathfrak{x}A$, with A acting on the right.

by taking inner products that $B_n = A_n^\top G_n$. However, since $\mathfrak{x}_n \to \mathfrak{x}$ and $\mathfrak{x}'_n \to \mathfrak{x}'$ (and this convergence is taken entrywise), we know that $B_n \to B$ and $G_n \to G$, where $B \in \mathrm{Mat}(k,\mathbb{C})$ and $G \in \mathrm{GL}(k,\mathbb{C})$ (invertibility coming from $\mathfrak{x} \in \mathrm{St}_k(V)$). Making $n \to +\infty$ we conclude that $(A_n)_{n\geq 0}$ converges[18] to some matrix A, and it remains to show that this matrix is non-singular. But making $n \to +\infty$ in $\mathfrak{x}'_n = \mathfrak{x}_n A_n$ gives that $\mathfrak{x}' = \mathfrak{x}A$, and so $\mathfrak{x}, \mathfrak{x}' \in \mathrm{St}_k(V)$ implies that $A \in \mathrm{GL}(k,\mathbb{C})$, as wanted. $\square$

We keep moving on:

Proposition 60. $\mathrm{Gr}_k(V)$ *is a complex manifold (hence it is orientable), the quotient projection* $\Pi\colon \mathrm{St}_k(V) \to \mathrm{Gr}_k(V)$ *becomes a holomorphic submersion, and its dimension is given by* $\dim_{\mathbb{C}} \mathrm{Gr}_k(V) = k(n-k)$.

Remark. In particular, $\dim_{\mathbb{C}} \mathrm{Gr}_k(\mathbb{C}^{n+k}) = nk$.

Proof: Instead of considering linear functionals $f\colon V \to \mathbb{C}$, as we did before in Proposition 3 (p. 2) when constructing an atlas for PV, we'll adapt the strategy accordingly and now consider linear maps[19] $f\colon V \to \mathbb{C}^k$. Let $U_f = \{W \in \mathrm{Gr}_k(V) \mid f[W] = \mathbb{C}^k\}$, and note that $\Pi^{-1}[U_f] = \{\mathfrak{x} \in \mathrm{St}_k(V) \mid f\mathfrak{x} \in \mathrm{St}_k(\mathbb{C}^k)\}$ is open in $\mathrm{St}_k(V)$, as linear independence is an open condition, so that $U_f \subseteq \mathrm{Gr}_k(V)$ is open by definition of quotient topology (it is also saturated). Fix the standard basis $\mathfrak{e}$ of $\mathbb{C}^k$, and now define $\varphi_f\colon U_f \to \prod_{i=1}^k f^{-1}(e_i)$ by setting $\varphi_f(W) = \mathfrak{x}$, where $\mathfrak{x} \in W^{\times k} \cap \mathrm{St}_k(V)$ is such that $f\mathfrak{x} = \mathfrak{e}$. These vectors are uniquely characterized by W itself. Namely, we have that the restriction $f|_W\colon W \to \mathbb{C}^k$ is an isomorphism, and $x_i = (f|_W)^{-1}(e_i)$ for each $i = 1,\ldots,k$. Choosing any reference basis $\mathfrak{x}_0$ for W and noting that the entries of $[f|_W]^{-1}_{\mathfrak{e},\mathfrak{x}_0}$ (which are the components of the x_i's relative to the basis $\mathfrak{x}_0$) are rational (hence continuous) functions of the entries of $[f|_W]_{\mathfrak{x}_0,\mathfrak{e}}$ (which in turn depend smoothly on $\mathfrak{x}_0$ via the usual hermitian product in $\mathbb{C}^k$), we see that the function

$$\Pi^{-1}[U_f] \ni \mathfrak{w} \mapsto \left(\left(f|_{\Pi(\mathfrak{w})}\right)^{-1}(e_1),\ldots,\left(f|_{\Pi(\mathfrak{w})}\right)^{-1}(e_k)\right) \in \prod_{i=1}^k f^{-1}(e_i)$$

is continuous. By the characteristic property of the quotient topology in $\mathrm{Gr}_k(V)$, since U_f is saturated, φ_f is continuous. Also, φ_f is bijective for the simple reason we can directly exhibit the inverse as the span-map

$$\prod_{i=1}^k f^{-1}(e_i) \ni (u_1,\ldots,u_k) \mapsto \bigoplus_{i=1}^k \mathbb{C}u_i \in U_f.$$

It follows that φ_f is a homeomorphism and $\mathrm{Gr}_k(V)$ is a topological manifold. Let's proceed and look at the transition maps. If $f,g\colon V \to \mathbb{C}^k$ are surjective, then the composition $\varphi_f \circ \varphi_g^{-1}\colon \varphi_g[U_f \cap U_g] \to \varphi_f[U_f \cap U_g]$ is given by

$$\varphi_f \circ \varphi_g^{-1}(u_1,\ldots,u_k) = \left(\left(f|_{\bigoplus_{i=1}^k \mathbb{C}u_i}\right)^{-1}(e_1),\ldots,\left(f|_{\bigoplus_{i=1}^k \mathbb{C}u_i}\right)^{-1}(e_k)\right),$$

[18] Namely, to $A = (G^{-1})^\top B^\top$, but this is not important.

[19] Just as we have dealt only with non-zero linear functionals before, this time we'll only work with surjective f's. We'll also write $f\mathfrak{x}$ for the element $(f(x_1),\ldots,f(x_k)) \in (\mathbb{C}^k)^k$.

which is holomorphic (because again, from the point of view of matrices, we have rational functions). $\square$

That being understood, some properties and examples seen before for $\mathbb{P}V$ are directly adapted to the Grassmannian $\mathrm{Gr}_k(V)$.

Proposition 61. *For $W \in \mathrm{Gr}_k(V)$, we have an isomorphism $T_W\mathrm{Gr}_k(V) \cong \mathrm{Hom}(W, V/W)$.*

Proof: We have two group actions on $\mathrm{St}_k(V)$:

- $\mathrm{GL}(V) \circlearrowright \mathrm{St}_k(V)$ defined by $F \cdot [x_1 \cdots x_k] = [Fx_1 \cdots Fx_k]$. We will write the latter just as $F\mathfrak{x}$, as done above. In fact, this restricts from a bigger action $\mathfrak{gl}(V) \circlearrowright V^k$ given by the same formula.

- $\mathrm{St}_k(V) \circlearrowleft \mathrm{GL}(k, \mathbb{C})$ defined by $\mathfrak{x} \cdot A \doteq \mathfrak{x}A$, as mentioned before.

The key is to note that these actions are compatible, in the sense that $(F\mathfrak{x})A = F(\mathfrak{x}A)$. Thus, we'll show that the isomorphism takes a linear map $H\colon W \to V/W$ to $\mathrm{d}\Pi_{\mathfrak{x}}(\widetilde{H}\mathfrak{x})$, where $\widetilde{H}\colon W \to V$ is any linear lift of H and $\mathfrak{x}$ is any basis for W. We will only show that this is independent on the choices of $\mathfrak{x}$ and $\widetilde{H}$, and that $H \mapsto \mathrm{d}\Pi_{\mathfrak{w}}(\widetilde{H}\mathfrak{x})$ is an isomorphism follows just as in Proposition 9 (p. 7).

- It is independent of the choice of linear lift $\widetilde{H}$. For if $\widetilde{H}_1, \widetilde{H}_2\colon W \to V$ are two linear lifts of H, we have that $\widetilde{H}_2 = \widetilde{H}_1 + B$, where $B\colon W \to W$ is linear. This implies that $\mathrm{d}\Pi_{\mathfrak{x}}(\widetilde{H}_2\mathfrak{x}) = \mathrm{d}\Pi_{\mathfrak{x}}(\widetilde{H}_1\mathfrak{x}) + \mathrm{d}\Pi_{\mathfrak{x}}(B\mathfrak{x})$, but we have that
$$\mathrm{d}\Pi_{\mathfrak{x}}(B\mathfrak{x}) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Pi(\mathfrak{x} + tB\mathfrak{x}) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} W = 0,$$
since for t small enough we have that $\mathfrak{x} + tB\mathfrak{x}$ is also a basis for W (by continuity and since B takes values in W).

- It is independent on the choice of $\mathfrak{x}$. For if $\mathfrak{x}' \in \mathrm{St}_k(V)$ is another basis for W, then $\mathfrak{x}' = \mathfrak{x}A$ for some $A \in \mathrm{GL}(k, \mathbb{C})$. The multiplication map by this A, which we'll also denote by $A\colon V \to V$, is linear. So $\Pi \circ A = \Pi$. The chain rule implies that $\mathrm{d}\Pi_{\mathfrak{x}A} \circ A = \mathrm{d}\Pi_{\mathfrak{x}}$. Evaluating at $\widetilde{H}\mathfrak{x}$ gives that
$$\mathrm{d}\Pi_{\mathfrak{x}}(\widetilde{H}\mathfrak{x}) = \mathrm{d}\Pi_{\mathfrak{x}A}((\widetilde{H}\mathfrak{x})A) = \mathrm{d}\Pi_{\mathfrak{x}A}(\widetilde{H}(\mathfrak{x}A)) = \mathrm{d}\Pi_{\mathfrak{x}'}(\widetilde{H}\mathfrak{x}')$$
as wanted, since $\widetilde{H}$ is linear.

$\square$

The same remarks done for the discussion regarding $\mathbb{P}V$ also apply here. Let's see more examples.

Example 62 (A full duality)**.** Let's go back to what we started mentioning back in Example 5 (p. 5). Consider the map $\mathrm{Ann}\colon \mathrm{Gr}_k(V) \to \mathrm{Gr}_{n-k}(V^*)$ taking a subspace to its polar space, i.e., given by $\mathrm{Ann}(W) = W^0$. Let's compute the derivative
$$\mathrm{d}(\mathrm{Ann})_W\colon T_W\mathrm{Gr}_k(V) \to T_{W^0}\mathrm{Gr}_{n-k}(V^*)$$

by recalling[20] that $V^*/W^0 \cong W^*$. Given $H \in T_W\mathrm{Gr}_k(V)$ we want to find the linear map $\mathrm{d}(\mathrm{Ann})_W(H)\colon W^0 \to W^*$. Given $f \in W^0$, we know that $\mathrm{d}(\mathrm{Ann})_W(H)(f) = f'(0)|_W$, where $t \mapsto f(t)$ is a curve in V^* with $f(t) \in W(t)^0$ and $f(0) = 0$, where $t \mapsto W(t)$ is a curve in $\mathrm{Gr}_k(V)$ with $W(0) = W$ and $W'(0) = H$. We find its value at $w \in W$ by taking a curve $t \mapsto w(t)$ in V with $w(t) \in W(t)$ for all t and differentiating $f(t)w(t) = 0$ at $t = 0$ to obtain $f'(0)w = -f(w'(0))$. But $H(w) = w'(0) + W$, so that $f \in W^0$ implies that $f'(0)w = -f(H(w))$. We conclude that $\mathrm{d}(\mathrm{Ann})_W(H) = -H^*$. Since (minus[21]) the pull-back operation is involutive under the identification $V^{**} \cong V$, we conclude that $\mathrm{d}(\mathrm{Ann})_W$ is an isomorphism, so Ann is a local diffeomorphism. But Ann is bijective, so it is in fact a global diffeomorphism.

Example 63. The situation discussed in Example 10 (p. 8) is directly adapted: any automorphism $T \in \mathrm{GL}(V)$ induces a map $\mathrm{P}T\colon \mathrm{Gr}_k(V) \to \mathrm{Gr}_k(V)$ via direct images, i.e., $\mathrm{P}T(W) \doteq T[W]$. The derivative $\mathrm{d}(\mathrm{P}T)_W\colon T_W\mathrm{Gr}_k(V) \to T_{T[W]}\mathrm{Gr}_k(V)$ is given by

$$\mathrm{d}(\mathrm{P}T)_W(H) = \widetilde{T} \circ H \circ \big(T^{-1}|_{T[W]}\big),$$

where $\widetilde{T}\colon V/W \to V/T[W]$ is the isomorphism induced by T on the quotient level.

Example 64 (Projections again)**.** The computation done in Example 11 (p. 9) can be generalized without issues here. Assume that $\langle \cdot, \cdot \rangle$ is a hermitian inner product on V and fix a unit vector $u \in V$. Define $F\colon \mathrm{Gr}_k(V) \to V$ by $F(W) = \mathrm{pr}_W u$. Let's regard a tangent vector as a linear map $H\colon W \to W^\perp$ and compute the derivative $\mathrm{d}F_W\colon T_W\mathrm{Gr}_k(V) \to V$. So, fix an orthonormal basis $\mathfrak{x}$ for W, so that

$$F(W) = \sum_{i=1}^k \langle u, x_i \rangle x_i \implies \mathrm{d}F_W(H) = \sum_{i=1}^k \langle u, Hx_i \rangle x_i + \sum_{i=1}^k \langle u, x_i \rangle Hx_i.$$

Again, this boils down to $\mathrm{d}F_W(H) = H^*\mathrm{pr}_{W^\perp} u + H\,\mathrm{pr}_W u$. Formally, this implies that the derivative of $G\colon \mathrm{Gr}_k(V) \to \mathbb{R}$ given by $G(W) = \|\mathrm{pr}_W u\|^2$ is just

$$\mathrm{d}G_W(H) = 2\,\mathrm{Re}\langle H\,\mathrm{pr}_W u, u \rangle,$$

as before.

Example 65 (Intersection)**.** Fix $W_0 \in \mathrm{Gr}_r(V)$ and let

$$\mathrm{Gr}_k^{\pitchfork}(V; W_0) \doteq \{W \in \mathrm{Gr}_k(V) \mid W + W_0 = V\}.$$

Since transversality is an open condition, $\mathrm{Gr}_k^{\pitchfork}(V; W_0)$ is open in $\mathrm{Gr}_k(V)$. Consider the function "cut with W_0", $\mathrm{c}\colon \mathrm{Gr}_k^{\pitchfork}(V; W_0) \to \mathrm{Gr}_{n-k-r}(W_0)$, given by $\mathrm{c}(W) = W \cap W_0$. Let's compute the derivative $\mathrm{dc}_W\colon T_W\mathrm{Gr}_k(V) \to T_{W\cap W_0}\mathrm{Gr}_{n-k-r}(W_0)$. For a given $H\colon W \to V/W$, we want to make explicit $\mathrm{dc}_W(H)\colon W \cap W_0 \to W_0/(W \cap W_0)$. But note that the transversality assumption implies that $W_0/(W \cap W_0) \cong V/W$, so in fact we seek $\mathrm{dc}_W(H)\colon W \cap W_0 \to V/W$. Now what happens is clear: since c restricts W to W_0, its derivative will also be a restriction: $\mathrm{dc}_W(H) = H|_{W\cap W_0}$.

[20] The restriction map $V^* \to W^*$ is linear, surjective (by Hahn-Banach, if you'll allow me to overkill) and its kernel is W^0.

[21] And thankfully $(-1)^2 = 1$.

Example 66 (Product)**.** Fix another vector space Z with $\dim_{\mathbb{C}} Z = m$. Consider the product map $\mathrm{p}_Z\colon \mathrm{Gr}_k(V) \to \mathrm{Gr}_{k+m}(V \times Z)$ given by $\mathrm{p}_Z(W) = W \times Z$. Note that

$$\frac{V \times Z}{W \times Z} \cong \frac{V}{W}.$$

With this, let's compute the derivative $\mathrm{d}(\mathrm{p}_Z)_W\colon T_W\mathrm{Gr}_k(V) \to T_{W\times Z}\mathrm{Gr}_{k+m}(V \times Z)$. For $H \in T_W\mathrm{Gr}_k(V)$, we want to find $\mathrm{d}(\mathrm{p}_Z)_W(H)\colon W \times Z \to V/W$. It is to be expected that $\mathrm{d}(\mathrm{p}_Z)_W(H) = H \circ \mathrm{pr}_W$, where $\mathrm{pr}_W\colon W \times Z \to W$ is the projection. Indeed, given $(w, z) \in W \times Z$, consider a curve $t \mapsto (w(t), z(t))$ in $V \times Z$ with $(w(0), z(0)) = (w, z)$ and $(w(t), z(t)) \in W(t) \times Z$ for all t, where $t \mapsto W(t)$ is a curve in $\mathrm{Gr}_k(V)$ with $W(0) = 0$ and $W'(0) = H$. Noting that $Hw = w'(0) + W$, we have that

$$\begin{aligned}\mathrm{d}(\mathrm{p}_Z)_W(H)(w,z) &= (w'(0), z'(0)) + (W \times Z)\\ &= (w'(0) + W, z'(0) + Z)\\ &= (Hw, 0),\end{aligned}$$

as claimed.

Example 67 (Tensorization)**.** Fix again another complex vector space Z, with dimension $\dim_{\mathbb{C}} Z = m$. Consider the tensorization map $\mathrm{t}_Z\colon \mathrm{Gr}_k(V) \to \mathrm{Gr}_{km}(V \otimes Z)$ given by $\mathrm{t}_Z(W) = W \otimes Z$. Since vector spaces are flat modules, we have that

$$\frac{V \otimes Z}{W \otimes Z} \cong \frac{V}{W} \otimes Z,$$

which allows us to compute the derivative $\mathrm{d}(\mathrm{t}_Z)_W\colon T_W\mathrm{Gr}_k(V) \to T_{W\otimes Z}\mathrm{Gr}_{km}(V \otimes Z)$. Given $H \in T_W\mathrm{Gr}_k(V)$, we want to find $\mathrm{d}(\mathrm{t}_Z)_W(H)\colon W \otimes Z \to (V/W) \otimes Z$. The natural guess is just $\mathrm{d}(\mathrm{t}_Z)_W(H) = H \otimes \mathrm{Id}_Z$, and we argue that this is the case as follows: take $\Phi \in W \otimes Z$, and consider a curve $t \mapsto \Phi(t)$ in $V \otimes Z$ such that $\Phi(0) = \Phi$ and $\Phi(t) \in W(t) \otimes Z$ for all t, where $t \mapsto W(t)$ is a curve in $\mathrm{Gr}_k(V)$ with $W(0) = 0$ and $W'(0) = H$. By definition of tensor product, write

$$\Phi(t) = \sum_{i=1}^{r} \Phi_i^W(t) \otimes \Phi_i^Z(t),$$

where $r \geq 1$ and we have $\Phi_i^W(t) \in W(t)$ and $\Phi_i^Z(t) \in Z$ for all t. Moreover, note that $H\Phi_i^W(0) = (\Phi_i^W)'(0) + W$ for all $i = 1, \ldots, r$. With this in place, we use the product rule (since the $\otimes$ operation is bilinear) to get

$$\Phi'(0) = \sum_{i=1}^{r} (\Phi_i^W)'(0) \otimes \Phi_i^Z(0) + \sum_{i=1}^{r} \Phi_i^W(0) \otimes (\Phi_i^Z)'(0).$$

Projecting onto $(V/W)\otimes Z$ we obtain

$$\begin{aligned}
\mathrm{d}(\mathrm{t}_Z)_W(H)\Phi &= \Phi'(0) + \left(\frac{V}{W}\otimes Z\right) \\
&= \sum_{i=1}^{r} (\Phi_i^W)'(0)\otimes \Phi_i^Z(0) + \left(\frac{V}{W}\otimes Z\right) \\
&= \sum_{i=1}^{r} \big((\Phi_i^W)'(0) + W\big)\otimes \Phi_i^Z(0) \\
&= \sum_{i=1}^{r} H\Phi_i^W(0)\otimes \Phi_i^Z(0) = (H\otimes \mathrm{Id}_Z)(\Phi),
\end{aligned}$$

as wanted.

There is also a version of the tautological bundle for $\mathrm{Gr}_k(V)$.

Definition 68. The tautological bundle over $\mathrm{Gr}_k(V)$ is given by

$$\mathscr{L}_k \doteq \{(W,x) \mid W\in \mathrm{Gr}_k(V) \text{ and } x\in W\} \subseteq \mathrm{Gr}_k(V)\times V.$$

The bundle projection $\pi\colon \mathscr{L}_k \to \mathrm{Gr}_k(V)$ is given by $\pi(W,x) = W$.

Remark. With our previous notation, $\mathscr{L} = \mathscr{L}_1$.

Proposition 69. *The tautological bundle $\pi\colon \mathscr{L}_k\to \mathrm{Gr}_k(V)$ is a holomorphic vector bundle of rank k and moreover we have that $T\mathrm{Gr}_k(V)\cong \mathrm{Hom}(\mathscr{L}_k, (\mathrm{Gr}_k(V)\times V)/\mathscr{L}_k)$.*

Proof: Just like in Proposition 14 (p. 10), we'll use surjective linear maps $f\colon V\to \mathbb{C}^k$ and the associated domains $U_f\subseteq \mathrm{Gr}_k(V)$ to construct a VB-atlas for $\mathscr{L}_k$. The VB-charts will be $\Phi_f\colon \pi^{-1}[U_f]\to U_f\times \mathbb{C}^k$ given by $\Phi_f(W,x) = (W, f(x))$ with inverses $\Phi_f^{-1}\colon U_f\times\mathbb{C}^k\to \pi^{-1}[U_f]$ given by $\Phi_f^{-1}(W,u) = (W, (f|_W)^{-1}(u))$. The last assertion is now clear. $\square$

Remark. Mimicking Proposition 15 (p. 10), one obtains that

$$\Gamma(\mathscr{L}_k)\cong \{\mu\colon \mathrm{St}_k(V)\to\mathbb{C}^k \mid \mu(\mathfrak{x}A) = A^{-1}\mu(\mathfrak{x}), \text{ for all } \mathfrak{x}\in \mathrm{St}_k(V) \text{ and } A\in \mathrm{GL}(k,\mathbb{C})\}.$$

Example 70. Still on the topic of bundles, if $E\to M$ is a vector bundle, we can form the fiber bundle $\mathrm{Gr}_k(E)\to M$, where

$$\mathrm{Gr}_k(E) = \bigsqcup_{x\in M} \mathrm{Gr}_k(E_x),$$

and the projection takes $W\in \mathrm{Gr}_k(E)$ to the point $x\in M$ for which W is a subspace of E_x. Note that if M is compact, so is $\mathrm{Gr}_k(E)$. This allows us to express some other functions in a more elegant way. For instance, assume that (M,g) is a Riemannian manifold. Then:

- its *sectional curvature* is a certain smooth function $K\colon \mathrm{Gr}_2(TM)\to\mathbb{R}$.

- the *Ricci curvature* $\mathrm{ric}_x(\boldsymbol{v}) \doteq \mathrm{Ric}_x(\boldsymbol{v},\boldsymbol{v})/\|\boldsymbol{v}\|^2$, defined only for non-zero tangent vectors $\boldsymbol{v}$ in T_xM, in turn, may be seen as a smooth function $\mathrm{P}TM \to \mathbb{R}$, where $\mathrm{P}TM \doteq \mathrm{Gr}_1(TM)$ is the *projectivized tangent bundle* of M – and this function is a constant if and only if (M,g) is an Einstein manifold.

- if $M^k \subseteq \mathbb{R}^n$ is isometrically embedded in $\mathbb{R}^n$, we may consider the *Gauss map* $G\colon M \to \mathrm{Gr}_k(\mathbb{R}^n)$ given by $G(p) \doteq T_pM$. Its derivative
$$\mathrm{d}G_p\colon T_pM \to T_{G(p)}\mathrm{Gr}_k(\mathbb{R}^n) \cong \mathrm{Hom}(T_pM, T_p^\perp M)$$
may be seen as a bilinear map $\mathrm{d}G_p\colon T_pM \times T_pM \to T_p^\perp M$. This is, naturally, the second fundamental form II_p of M at p, relative to $\mathbb{R}^n$. By our usual principles, we have that $\mathrm{d}G_p(\boldsymbol{v},\boldsymbol{w}) = \boldsymbol{w}'(0)$, where $t \mapsto \gamma(t)$ is a curve in M with $\gamma(0) = p$ and $\gamma'(0) = \boldsymbol{v}$ and $t \mapsto \boldsymbol{w}(t)$ is a vector field along γ with $\boldsymbol{w}(0) = \boldsymbol{w}$. The general relation
$$\boldsymbol{w}'(t) = \frac{\mathrm{D}\boldsymbol{w}}{\mathrm{d}t}(t) + \mathrm{II}_{\gamma(t)}(\gamma'(t),\boldsymbol{w}(t))$$
holds, but we may assume that $t \mapsto \boldsymbol{w}(t)$ is parallel along γ, so that evaluating at $t = 0$ gives $\boldsymbol{w}'(0) = \mathrm{II}_p(\boldsymbol{v},\boldsymbol{w})$, as wanted. We have two consequences of this:
  - If M is connected and totally geodesic, then $\mathrm{II} = 0$ says that G is constant. Thus M is contained in the affine subspace $p + T_pM$, where $p \in M$ is any chosen point.
  - we have that if $\boldsymbol{\xi} \in T_p^\perp M$ is any normal vector, for all $\boldsymbol{w} \in T_pM$ we have that
$$\langle \mathrm{d}G_p(\boldsymbol{v})\boldsymbol{w},\boldsymbol{\xi}\rangle = \langle \mathrm{II}_p(\boldsymbol{v},\boldsymbol{w}),\boldsymbol{\xi}\rangle = \langle A_{\boldsymbol{\xi}}(\boldsymbol{v}),\boldsymbol{w}\rangle,$$
which says that the *Weingarten map* $A_{\boldsymbol{\xi}}$ is given by $A_{\boldsymbol{\xi}}(\boldsymbol{v}) = \mathrm{d}G_p(\boldsymbol{v})^*(\boldsymbol{\xi})$, where $\mathrm{d}G_p(\boldsymbol{v})^*\colon T_p^\perp M \to T_pM$ is the adjoint of $\mathrm{d}G_p(\boldsymbol{v})$.

- if we are still in the setting of the above item, the *mean curvature vector* $\boldsymbol{H}$ of M is just the metric trace $\boldsymbol{H} = \mathrm{tr}\circ \mathrm{d}G$ (divided by k, if you feel like it). If $k = 2$ (i.e., M is a surface), the *Gaussian curvature* is just the composition $K \circ G$.

- we'll see soon that $\mathrm{Gr}_k(\mathbb{R}^n)$ has a natural $\mathrm{SO}(n,\mathbb{R})$-invariant Riemannian metric. The volume form of the pull-back of this natural metric is a function multiple of the volume form of g, which for $k = n-1$ is the *Gauss-Kronecker curvature* of M, up to a sign that depends on the parity of k.

Example 71 (Grassmannians as metric spaces)**.** In $\mathfrak{gl}(V)$, one may define an inner product by $\langle T,S\rangle = \mathrm{tr}\,(T^\dagger S)$, there $T^\dagger$ stands for the adjoint of T. Then the function $d\colon \mathrm{Gr}_k(V)\times\mathrm{Gr}_k(V) \to \mathbb{R}_{\geq 0}$ defined by $d(W_1,W_2) = \|\mathrm{pr}_{W_1} - \mathrm{pr}_{W_2}\|$ is a distance function. It is clearly symmetric, the triangle inequality for d follows from the one for the norm $\|\cdot\|$ in $\mathfrak{gl}(V)$, and if $d(W_1,W_2) = 0$, then $\mathrm{pr}_{W_1} = \mathrm{pr}_{W_2}$ implies that $W_1 = W_2$, since projections are characterized by their images. Note that if we choose an orthonormal basis $(e_1,\ldots,e_n)$ for V, then
$$\langle \mathrm{pr}_{W_1},\mathrm{pr}_{W_2}\rangle = \sum_{i=1}^n \langle \mathrm{pr}_{W_1}(e_i),\mathrm{pr}_{W_2}(e_i)\rangle.$$

Moving on, we can adapt Theorem 22 (p. 12) for Grassmannians:

Theorem 72. *Assume that V is equipped with a hermitian inner product $\langle\cdot,\cdot\rangle$, and consider the vector space $\mathfrak{gl}_0^{\rm sym}(V) \doteq \{A \in \mathfrak{gl}(V) \mid A^* = A \text{ and } \operatorname{tr}(A) = 0\}$ equipped with the inner product (also denoted by $\langle\cdot,\cdot\rangle$) given by $\langle A, B\rangle \doteq \operatorname{tr}(AB)$. The map $\Phi\colon \mathrm{Gr}_k(V) \to \mathfrak{gl}_0^{\rm sym}(V)$ given by $\Phi(W) = (n-k)\mathrm{Id}_W \oplus (-k\,\mathrm{Id}_{W^\perp})$ is an embedding and the image $\Phi[\mathrm{Gr}_k(V)]$ is contained inside the sphere in $\mathfrak{gl}_0^{\rm sym}(V)$ of radius $\sqrt{nk(n-k)}$.*

Proof: First note that $\operatorname{tr}\Phi(W) = (n-k)k - k(n-k) = 0$, and also that by its definition we have $\Phi(W)$ self-adjoint, so that $\Phi(W) \in \mathfrak{gl}_0^{\rm sym}(V)$. Clearly Φ is smooth and injective (due to the second summand), and for every $W \in \mathrm{Gr}_k(V)$ we have that $\mathrm{d}\Phi_W$ is injective, for if $H\colon W \to W^\perp$ is linear, then $\mathrm{d}\Phi_W(H) = n(H^* \circ \mathrm{pr}_{W^\perp} + H \circ \mathrm{pr}_W) = 0$ and $W \cap W^\perp = \{0\}$ imply that $H \circ \mathrm{pr}_W = 0$, but pr_W is surjective, so $H = 0$. Thus Φ is an immersion and since $\mathrm{Gr}_k(V)$ is compact, it follows that Φ is an embedding. Now $\Phi(W)^2 = (n-k)^2\mathrm{Id}_W \oplus k^2\,\mathrm{Id}_{W^\perp}$ gives that

$$\langle \Phi(W), \Phi(W)\rangle = (n-k)^2k + k^2(n-k) = nk(n-k),$$

and we are done. $\square$

Remark. The radius is minimized for $k = 1$ and $k = n-1$, and it is maximized for $n/2$ when n is even, or $\lfloor n/2\rfloor$ and $\lceil n/2\rceil$ when n is odd.

4.2 Some embeddings

Let's start generalizing the Fubini-Study metric for Grassmannians, adapting what was done in Section 3. We'll try to keep the notation the same as back then. The hermitian product $\langle\cdot,\cdot\rangle$ in V induces a hermitian product in the cartesian power $V^{\times k}$, and by taking the real part this induces a Riemannian metric in the Stiefel manifold of orthonormal k-frames of V, $\Sigma_k \doteq \mathrm{St}_k(V, \langle\cdot,\cdot\rangle)$, which will play the role of Σ in the $k = 1$ case (namely, $\Sigma_1 = \Sigma$). Naturally, the goal will be to turn $\Pi\colon \Sigma_k \to \mathrm{Gr}_k(V)$ into a Riemannian submersion by using Proposition 40 (p. 22). So, we'll start by studying the geometry of Σ_k inside $V^{\times k}$. The big issue here is that Σ_k is not, in general, a sphere. So, consider the Gram map $G\colon V^{\times k} \times V^{\times k} \to \mathfrak{gl}(k,\mathbb{C})$ given by $G(\mathfrak{x},\mathfrak{y}) = (\langle x_i, y_j\rangle)_{i,j=1}^k$. It is as close to a hermitian inner product as it can get:

Proposition 73 (Properties of G)**.** *For the Gram map defined above, the following hold:*

(i) *G is $\mathbb{C}$-linear in the first entry, $G(\lambda\mathfrak{x}_1 + \mathfrak{x}_2, \mathfrak{y}) = \lambda G(\mathfrak{x}_1,\mathfrak{y}) + G(\mathfrak{x}_2,\mathfrak{y})$ for all k-uples $\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{y} \in V^{\times k}$ and $\lambda \in \mathbb{C}$.*

(ii) *G has hermitian symmetry, $G(\mathfrak{y},\mathfrak{x}) = G(\mathfrak{x},\mathfrak{y})^\dagger$ for all $\mathfrak{x},\mathfrak{y} \in V^{\times k}$.*

(iii) *For each $\mathfrak{x} \in V^{\times k}$, $G(\mathfrak{x},\mathfrak{x})$ is positive-semidefinite and it is zero only when $\mathfrak{x} = \mathbf{0}$. In particular, G is non-degenerate.*

(iv) *$G(T\mathfrak{x},\mathfrak{y}) = G(\mathfrak{x}, T^\dagger\mathfrak{y})$ for all $\mathfrak{x},\mathfrak{y} \in V^{\times k}$ and $T \in \mathfrak{gl}(V)$. In particular, $\mathrm{U}(V, \langle\cdot,\cdot\rangle)$ still acts as "isometries" for G.*

(v) $G(\mathfrak{x}A, \mathfrak{y}B) = A^\top G(\mathfrak{x}, \mathfrak{y})\overline{B}$ *for all* $\mathfrak{x}, \mathfrak{y} \in V^{\times k}$ *and* $A, B \in \mathfrak{gl}(k, \mathbb{C})$.

This map G will be extremely useful in what follows. The hermitian product on $V^{\times k}$ induced by $\langle\cdot,\cdot\rangle$, which we'll keep denoting by $\langle\cdot,\cdot\rangle$, turns out to be given by $\langle\mathfrak{x}, \mathfrak{y}\rangle = \operatorname{tr} G(\mathfrak{x}, \mathfrak{y})$. So we have a natural Riemannian metric on V, again to be denoted by $\langle\cdot,\cdot\rangle_{\mathbb{R}}$, given by $\langle\mathfrak{x}, \mathfrak{y}\rangle_{\mathbb{R}} = \operatorname{Re}\operatorname{tr} G(\mathfrak{x}, \mathfrak{y})$. Again, $\langle\cdot,\cdot\rangle$ is recovered from $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ via the relation $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{\mathbb{R}} - \mathrm{i}\langle \mathrm{i}\cdot,\cdot\rangle_{\mathbb{R}}$. The Riemannian metric on Σ_k will be denoted by g^{Σ_k}. In terms of G, we may write $\Sigma_k = \{\mathfrak{x} \in V^{\times k} \mid G(\mathfrak{x}, \mathfrak{x}) = \mathrm{Id}_k\}$. This allows us to give a rather simple proof that Σ_k is indeed a smooth manifold.

Proposition 74. *If* $\mathfrak{h}(k) = \{A \in \mathfrak{gl}(k, \mathbb{C}) \mid A = A^\dagger\}$ *is the (real) vector space of hermitian matrices, then the identity matrix* Id_k *is a regular value of the smooth function* $F\colon V^{\times k} \to \mathfrak{h}(k)$ *given by* $F(\mathfrak{x}) = G(\mathfrak{x}, \mathfrak{x})$. *Thus the Stiefel manifold of orthonormal k-frames* $\Sigma_k = F^{-1}[\mathrm{Id}_k]$ *is a real embedded submanifold of* $V^{\times k}$ *of dimension* $\dim \Sigma_k = 2nk - k^2$ *and the tangent spaces to* Σ_k *are given by* $T_{\mathfrak{x}}\Sigma_k = \{\mathfrak{v} \in V^{\times k} \mid G(\mathfrak{v}, \mathfrak{x}) \in \mathfrak{u}(k)\}$.

Proof: Smooothness of F follows G being $\mathbb{R}$-bilinear, hence smooth. Also, property (ii) of Proposition 73 (p. 37) says that F indeed takes values in $\mathfrak{h}(k)$. Thus, our goal is to show that if $\mathfrak{x}$ is an orthonormal k-frame for V, then the $\mathbb{R}$-linear derivative $DF(\mathfrak{x})\colon V^{\times k} \to \mathfrak{h}(k)$ given by $DF(\mathfrak{x})(\mathfrak{v}) = G(\mathfrak{v}, \mathfrak{x}) + G(\mathfrak{x}, \mathfrak{v})$ is surjective. And this is surprisingly simple: given $A \in \mathfrak{h}(k)$, we use property (v) of Proposition 73 together with $A = A^\dagger$ to compute

$$DF(\mathfrak{x})\left(\mathfrak{x}\frac{A^\top}{2}\right) = \frac{A}{2} + \frac{A^\dagger}{2} = A,$$

as required. So, Σ_k is a real embedded submanifold of $V^{\times k}$, and since $\dim_{\mathbb{R}} V = 2n$, we have that

$$\dim_{\mathbb{R}} \Sigma_k = \dim_{\mathbb{R}}(V^{\times k}) - \dim_{\mathbb{R}} \mathfrak{h}(k) = 2nk - \left(k + 2\frac{k(k-1)}{2}\right) = 2nk - k^2.$$

Lastly, property (ii) of Proposition 73 again says that $DF(\mathfrak{x})(\mathfrak{v}) = 0$ if and only if $G(\mathfrak{v}, \mathfrak{x}) \in \mathfrak{u}(k)$, so the description for the tangent spaces follows. □

Remark.

- This is consistent with $\Sigma_k \cong \mathrm{U}(n)/\mathrm{U}(k)$, as $n^2 - (n-k)^2 = 2nk - k^2$.
- Note that $\mathfrak{h}(k)$ is a real form of $\mathfrak{gl}(k, \mathbb{C})$, as $\mathfrak{gl}(k, \mathbb{C}) = \mathfrak{h}(k) \oplus \mathfrak{u}(k)$ (as real vector spaces, under $A = (A + A^\dagger)/2 + (A - A^\dagger)/2$) and $\mathfrak{u}(k) = \mathrm{i}\mathfrak{h}(k)$. Moreover, $\mathfrak{h}(k)$ and $\mathfrak{u}(k)$ are orthogonal relative to the real inner product

$$\mathfrak{gl}(k, \mathbb{C}) \times \mathfrak{gl}(k, \mathbb{C}) \ni (A, B) \mapsto \operatorname{Re}\operatorname{tr}(A^\dagger B) \in \mathbb{R},$$

for the following reason: if $A \in \mathfrak{h}(k)$ and $\mathrm{i}B \in \mathfrak{u}(k)$ (with $B \in \mathfrak{h}(k)$), we have that

$$\overline{\operatorname{tr}(AB)} = \operatorname{tr}(\overline{AB}) = \operatorname{tr}((\overline{AB})^\top) = \operatorname{tr}(B^\dagger A^\dagger) = \operatorname{tr}(BA) = \operatorname{tr}(AB),$$

which means that $\operatorname{tr}(AB) \in \mathbb{R}$, and thus

$$\operatorname{Re}\operatorname{tr}(A^\dagger(\mathrm{i}B)) = \operatorname{Re}(\mathrm{i}\operatorname{tr}(AB)) = -\operatorname{Im}\operatorname{tr}(AB) = 0,$$

as wanted. As far as the Lie algebra structure goes, we have the general formula $[A,B]^\dagger = -[A^\dagger, B^\dagger]$ for all $A, B \in \mathfrak{gl}(k,\mathbb{C})$, so that the three relations

$$[\mathfrak{h}(k),\mathfrak{h}(k)] \subseteq \mathfrak{u}(k), \quad [\mathfrak{u}(k),\mathfrak{u}(k)] \subseteq \mathfrak{u}(k) \quad \text{and} \quad [\mathfrak{h}(k),\mathfrak{u}(k)] \subseteq \mathfrak{h}(k)$$

follow.

- In particular, $\mathfrak{x}$ is normal to $T_{\mathfrak{x}}\Sigma_k$ itself, as matrices in $\mathfrak{u}(k)$ have imaginary trace.

Now, we focus again on $\Pi\colon \Sigma_k \to \mathrm{Gr}_k(V)$. Given $W \in \mathrm{Gr}_k(V)$, we have that $\Pi^{-1}[W] = W^{\times k} \cap \Sigma_k$ is just the collection of all orthonormal bases of W, so that each fiber is diffeomorphic to a $\mathrm{U}(k)$. As in the $k = 1$ case, we might expect the vertical spaces to be isomorphic to $\mathfrak{u}(k)$. And indeed, for $\mathfrak{x} \in \Sigma_k$ and $A \in \mathfrak{u}(k)$, we have that $G(\mathfrak{x}A, \mathfrak{x}) = A^\top \in \mathfrak{u}(k)$, so $\mathfrak{x}A \in T_{\mathfrak{x}}\Sigma_k$. With this in place, we move on:

Lemma 75. *Let $\mathfrak{x} \in \Sigma_k$. Then*

$$\mathrm{Ver}_{\mathfrak{x}}(\Sigma_k) = \{\mathfrak{x}A \in T_{\mathfrak{x}}\Sigma_k \mid A \in \mathfrak{u}(k)\} \quad \textit{and} \quad \mathrm{Hor}_{\mathfrak{x}}(\Sigma_k) = \{\mathfrak{v} \in T_{\mathfrak{x}}(\Sigma_k) \mid G(\mathfrak{v},\mathfrak{x}) = 0\}.$$

Proof: Let $\mathfrak{v} \in \mathrm{Ver}_{\mathfrak{x}}(\Sigma_k)$. Then write $\mathfrak{v} = \gamma'(0)$, where $t \mapsto \gamma(t)$ is a curve of orthonormal bases for W, i.e., a curve in $W^{\times k} \cap \Sigma_k$ with $\gamma(0) = \mathfrak{x}$. Then for each t there is a unitary matrix $U(t) \in \mathrm{U}(k)$ such that $\gamma(t) = \mathfrak{x}U(t)$, and $t \mapsto U(t)$ is a curve with $U(0) = \mathrm{Id}_k$. If we set $A \doteq U'(0) \in \mathfrak{u}(k)$, differentiating both sides at $t = 0$ gives that $\mathfrak{v} = \mathfrak{x}S$. So $\mathrm{Ver}_{\mathfrak{x}}(\Sigma_k) \subseteq \{\mathfrak{x}A \in T_{\mathfrak{x}}\Sigma_k \mid A \in \mathfrak{u}(k)\}$. For the opposite inclusion, start with $A \in \mathfrak{u}(k)$, and note that $\mathrm{e}^{tA} \in \mathrm{U}(k)$ for all t small enough. Then $\mathfrak{x}\mathrm{e}^{tA} \in W^{\times k} \cap \Sigma_k$ for all such small t, so that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathfrak{x}\mathrm{e}^{tA} = \mathfrak{x}A$$

establishes that $\mathrm{Ver}_{\mathfrak{x}}(\Sigma_k) = \{\mathfrak{x}S \mid S \in \mathfrak{u}(k)\}$ as wanted. As for the horizontal spaces, we proceed as follows: if $\mathfrak{x}A \in \mathrm{Ver}_{\mathfrak{x}}(\Sigma_k)$ and $\mathfrak{v} \in T_{\mathfrak{x}}\Sigma_k$ satisfies $G(\mathfrak{v},\mathfrak{x}) = 0$, we compute

$$g^{\Sigma_k}_{\mathfrak{x}}(\mathfrak{v},\mathfrak{x}A) = \operatorname{Re}\operatorname{tr} G(\mathfrak{v},\mathfrak{x}A) = \operatorname{Re}\operatorname{tr}\left(G(\mathfrak{v},\mathfrak{x})\overline{A}\right) = \operatorname{Re}\operatorname{tr} 0 = 0,$$

which shows that $\mathfrak{v} \in \mathrm{Hor}_{\mathfrak{x}}(\Sigma_k)$. So the desired equality follows from a dimension computation. On one hand, we have that

$$\dim_{\mathbb{R}} \mathrm{Hor}_{\mathfrak{x}}(\Sigma_k) = \dim_{\mathbb{R}} \Sigma_k - \dim_{\mathbb{R}} \mathrm{Ver}_{\mathfrak{x}}(\Sigma_k) = 2nk - k^2 - k^2 = 2nk - 2k^2,$$

and on the other hand the linear map $G(\cdot,\mathfrak{x})\colon T_{\mathfrak{x}}\Sigma_k \to \mathfrak{u}(k)$ is surjective (due to property (v) of Proposition 73, p. 37), so that

$$\dim_{\mathbb{R}} \ker G(\cdot,\mathfrak{x}) = \dim_{\mathbb{R}} \Sigma_k - \dim_{\mathbb{R}} \mathfrak{u}(k) = 2nk - k^2 - k^2 = 2nk - 2k^2.$$

□

Remark.

- Looking at the full (surjective) map $G(\cdot,\mathfrak{x})\colon V^{\times k} \to \mathfrak{gl}(k,\mathbb{C})$ gives the same result, since $\dim_{\mathbb{R}} \ker G(\cdot,\mathfrak{x}) = \dim_{\mathbb{R}}(V^{\times k}) - \dim_{\mathbb{R}} \mathfrak{gl}(k,\mathbb{C}) = 2nk - 2k^2$.
- Another argument for establishing the description of $\mathrm{Hor}_{\mathfrak{x}}(\Sigma_k)$ that avoids a dimension computation to prove the direct inclusion goes as follows: if we take $\mathfrak{v} \in \mathrm{Hor}_{\mathfrak{x}}(\Sigma_k)$, for all $A \in \mathfrak{u}(k)$ we have that $\mathrm{Re}\,\mathrm{tr}\,(G(\mathfrak{v},\mathfrak{x})A) = 0$ (since $A \in \mathfrak{u}(k)$ if and only if $\overline{A} \in \mathfrak{u}(k)$). This means that $G(\mathfrak{v},\mathfrak{x}) \in \mathfrak{h}(k)$. But since $\mathfrak{v}$ is in particular a tangent vector, we also have $G(\mathfrak{v},\mathfrak{x}) \in \mathfrak{u}(k)$. So $G(\mathfrak{v},\mathfrak{x}) = 0$.

Now, if $\mathfrak{w} \in T_{\mathfrak{x}}\Sigma_k$, write $\mathfrak{w} = \mathfrak{w} - \mathfrak{x}A + \mathfrak{x}A$, note that $G(\mathfrak{w} - \mathfrak{x}A,\mathfrak{x}) = G(\mathfrak{w},\mathfrak{x}) - A^\top$. We obtain the version of Corollary 42 (p. 23) in this setting:

Corollary 76. *Let $\mathfrak{w},\mathfrak{w}' \in T_{\mathfrak{x}}\Sigma_k$. Then:*

(i) $\mathrm{v}\mathfrak{w} = \mathfrak{x}G(\mathfrak{w},\mathfrak{x})^\top$ *and* $\mathrm{h}\mathfrak{w} = \mathfrak{w} - \mathfrak{x}G(\mathfrak{w},\mathfrak{x})^\top$.

(ii) $g_{\mathfrak{x}}^{\Sigma_k}(\mathrm{v}\mathfrak{w},\mathrm{v}\mathfrak{w}') = \mathrm{Re}\,\mathrm{tr}\,(G(\mathfrak{w},\mathfrak{x})G(\mathfrak{w}',\mathfrak{x})^\dagger)$.

(iii) $g_{\mathfrak{x}}^{\Sigma_k}(\mathrm{h}\mathfrak{w},\mathrm{h}\mathfrak{w}') = g_{\mathfrak{x}}^{\Sigma_k}(\mathfrak{w},\mathfrak{w}') - \mathrm{Re}\,\mathrm{tr}\,(G(\mathfrak{w},\mathfrak{x})G(\mathfrak{w}',\mathfrak{x})^\dagger)$.

(iv) for all $A \in \mathfrak{u}(k)$, $(\mathrm{h}\mathfrak{w})A$ are tangent to Σ_k and horizontal.

Remark. But $(\mathrm{v}\mathfrak{w})A$ might not be normal!

Proof: Let's start by just checking that

$$\begin{aligned} g_{\mathfrak{x}}^{\Sigma_k}(\mathfrak{w} - \mathfrak{x}G(\mathfrak{w},\mathfrak{x})^\top, \mathfrak{x}G(\mathfrak{w},\mathfrak{x})^\top) &= \mathrm{Re}\,\mathrm{tr}\,\big(G(\mathfrak{w} - \mathfrak{x}G(\mathfrak{w},\mathfrak{x})^\top, \mathfrak{x}G(\mathfrak{w},\mathfrak{x})^\top)\big) \\ &= \mathrm{Re}\,\mathrm{tr}\,\big(G(\mathfrak{w},\mathfrak{x})G(\mathfrak{w},\mathfrak{x})^\dagger - G(\mathfrak{w},\mathfrak{x})G(\mathfrak{w},\mathfrak{x})^\dagger\big) \\ &= 0, \end{aligned}$$

hence since the expression for $\mathrm{v}\mathfrak{w}$ is clearly correct, so is the one for $\mathrm{h}\mathfrak{w}$. Item (ii) is clear, and (iii) is a consequence of (ii). As for (iv), we just compute

$$G((\mathrm{h}\mathfrak{w})A,\mathfrak{x}) = A^\top G(\mathfrak{w} - \mathfrak{x}G(\mathfrak{w},\mathfrak{x})^\top,\mathfrak{x}) = A^\top(G(\mathfrak{w},\mathfrak{x}) - G(\mathfrak{w},\mathfrak{x})) = 0,$$

which establishes that $(\mathrm{h}\mathfrak{w})A$ is horizontal.

$\square$

Example 77 (Some vector fields). Since $\mathfrak{gl}(k,\mathbb{C}) = \mathfrak{h}(k) \oplus \mathfrak{u}(k)$, we can see how linear maps behave as vector fields when restricted to Σ_k. If $A \in \mathfrak{u}(k)$ and $B \in \mathfrak{h}(k)$, then $\mathfrak{x} \mapsto \mathfrak{x}A$ is a tangent vector field along Σ_k, while $\mathfrak{x} \mapsto \mathfrak{x}B$ is normal. Indeed, we have that $G(\mathfrak{x}A,\mathfrak{x}) = A^\top$ is in $\mathfrak{u}(k)$ as well, and for all tangent vectors $\mathfrak{v} \in T_{\mathfrak{x}}\Sigma_k$ we have that $\mathrm{Re}\,\mathrm{tr}\,G(\mathfrak{x}B,\mathfrak{v}) = \mathrm{Re}\,\mathrm{tr}\,(B^\top G(\mathfrak{x},\mathfrak{v})) = 0$, since $\mathfrak{h}(k) \perp \mathfrak{u}(k)$. And since each $\mathfrak{x} \in \Sigma_k$ is linearly independent, the map $B \mapsto \mathfrak{x}B$ is injective, so that by a dimension count it follows that $(T_{\mathfrak{x}}\Sigma_k)^\perp = \{\mathfrak{x}B \mid B \in \mathfrak{h}(k)\}$ (where this $\perp$ is relative to $\mathrm{Re}\,\mathrm{tr}\,G$). Moreover, a similar thing happens with operators: if $T \in \mathfrak{u}(V,\langle\cdot,\cdot\rangle)$, then T is a vector field tangent to Σ_k, since

$$G(T\mathfrak{x},\mathfrak{x})^\dagger = G(\mathfrak{x},T\mathfrak{x}) = -G(T\mathfrak{x},\mathfrak{x}) \implies G(T\mathfrak{x},\mathfrak{x}) \in \mathfrak{u}(k).$$

Corollary 78. *Let* $W \in \mathrm{Gr}_k(V)$ *and* $\mathfrak{x} \in \Pi^{-1}[W]$*. Then* $H^{\mathrm{h}}_{\mathfrak{x}} = H\mathfrak{x}$*.*

Proof: Since $H\colon W \to W^{\perp}$, we have that $G(H\mathfrak{x}, \mathfrak{x}) = 0$. $\square$

Proposition 79. *The natural action* $\mathrm{St}_k(V) \circlearrowleft \mathrm{U}(k)$ *given by change of basis is fiber-preserving and consists of holomorphic isometries. It also restricts to an action* $\Sigma_k \circlearrowleft \mathrm{U}(k)$ *with is in addition transitive on each fiber.*

Proof: For any $A \in \mathrm{U}(k)$ and $\mathfrak{x} \in \mathrm{St}_k(V)$, we have that $\mathfrak{x}$ and $\mathfrak{x}A$ are in the same fiber, and each orbit-map is holomorphic. In particular, we have the sequence of inclusions $\mathrm{U}(k) \subseteq \mathrm{O}(V^{\times k}, \mathrm{Re}\,\mathrm{tr}\, G) \subseteq \mathrm{Iso}(\Sigma_k, g^{\Sigma_k})$, since for all $A \in \mathrm{U}(k)$ we may use the cyclic invariance of the trace to obtain

$$\begin{aligned} \mathrm{Re}\,\mathrm{tr}\, G(\mathfrak{x}A, \mathfrak{y}B) &= \mathrm{Re}\,\mathrm{tr}\,(A^{\top} G(\mathfrak{x},\mathfrak{y})\overline{A}) = \mathrm{Re}\,\mathrm{tr}\,(A^{\dagger}\overline{G(\mathfrak{x},\mathfrak{y})}A) \\ &= \mathrm{Re}\,\mathrm{tr}\,(AA^{\dagger}\overline{G(\mathfrak{x},\mathfrak{y})}) = \mathrm{Re}\,\mathrm{tr}\,(\overline{G(\mathfrak{x},\mathfrak{y})}) \\ &= \mathrm{Re}\,\mathrm{tr}\, G(\mathfrak{x},\mathfrak{y}). \end{aligned}$$

Lastly, for the restricted action, two bases $\mathfrak{x}$ and $\mathfrak{y}$ in the same fiber are related via $\mathfrak{y} = \mathfrak{x}A$ for some (unique) $A \in \mathrm{GL}(k, \mathbb{C})$, and now $\mathfrak{x}, \mathfrak{y} \in \Sigma_k$ implies that in fact we have $A \in \mathrm{U}(k)$. $\square$

Definition 80. The unique Riemannian metric g^{G} on $\mathrm{Gr}_k(V)$ that turns the projection $\Pi\colon \Sigma_k \to \mathrm{Gr}_k(V)$ into a Riemannian submersion is called a *Grassmannian metric*. The metric g^{G} is in fact hermitian (i.e., compatible with the natural complex structure of $\mathrm{Gr}_k(V)$).

Remark.

- In practice, this means that given $W \in \mathrm{Gr}_k(V)$ and $H_1, H_2 \in T_W\mathrm{Gr}_k(V)$, we have that $g^{\mathrm{G}}_W(H_1, H_2) = g^{\Sigma_k}_{\mathfrak{x}}(H_1\mathfrak{x}, H_2\mathfrak{x})$, where $\mathfrak{x}$ is any orthonormal basis for W.
- We'll denote the Levi-Civita connection of $(\mathrm{Gr}_k(V), g^{\mathrm{G}})$ by ∇^{G}.

One could follow the same strategy as before to compute the connection and the curvature of $(\mathrm{Gr}_k(V), g^{\mathrm{G}})$ by studying the geometry of Σ_k and the integrability tensors T and A of the submersion $\Pi\colon \Sigma_k \to \mathrm{Gr}_k(V)$. The computations are not exactly pleasant, so we will not pursue them (see, e.g., [12] and [13]). To make our point, let's determine the Levi-Civita connection ∇^{Σ_k} of (Σ_k, g^{Σ_k}). It is described as the projection of $\mathrm{D}^{\times k}$ onto the tangent spaces of Σ_k. So we'll have to turn out attention to vector fields. A vector field $\mathfrak{X} \in \mathfrak{X}(V^{\times k})$ is just a map $\mathfrak{X}\colon V^{\times k} \to V^{\times k}$, so we may put $(\mathrm{D}^{\times k}_{\mathfrak{X}}\mathfrak{Y})_{\mathfrak{x}} = D\mathfrak{Y}(\mathfrak{x})(\mathfrak{X})$, where $D\mathfrak{Y}(\mathfrak{x})$ is the total derivative of $\mathfrak{Y}$. This is, of course, the standard flat connection of $V^{\times k}$. It parallelizes $\mathrm{Re}\,\mathrm{tr}\, G$ because this is already constant (when regarded as a $(0,2)$-tensor field on the manifold $V^{\times k}$). Note that we also have the canonical normal field $\mathfrak{N}\colon V^{\times k} \to V^{\times k}$ given by $\mathfrak{N}(\mathfrak{x}) = \mathfrak{x}$, and $D\mathfrak{N}(\mathfrak{x})$ is always the identity map. In particular, this implies (as before) that $\mathrm{Re}\,\mathrm{tr}\, G(\mathrm{D}^{\times k}_{\mathfrak{X}}\mathfrak{Y}, \mathfrak{N}) = -\mathrm{Re}\,\mathrm{tr}\, G(\mathfrak{Y}, \mathfrak{X})$, but this does *not* hold if we remove $\mathrm{Re}\,\mathrm{tr}$.

Lemma 81.

(i) *Let* $\mathfrak{v} \in V^{\times k}$. *Then the real orthogonal projection of* $\mathfrak{v}$ *onto* $T_{\mathfrak{x}}\Sigma_k$ *is* $\mathfrak{v} - \mathfrak{x}B$, *while the normal component is* $\mathfrak{x}B$, *where*

$$B = \left(\frac{G(\mathfrak{v},\mathfrak{x}) + G(\mathfrak{v},\mathfrak{x})^\dagger}{2}\right)^\top.$$

Note that if $\mathfrak{v}$ *is already tangent to* Σ_k, *then* $B \in \mathfrak{h}(k) \cap \mathfrak{u}(k) = \{0\}$.

(ii) *Given* $\mathfrak{X}, \mathfrak{Y} \in \mathfrak{X}(\Sigma_k)$, *we have that*

$$\nabla^{\Sigma_k}_{\mathfrak{X}}\mathfrak{Y} = \mathrm{D}^{\times k}_{\mathfrak{X}}\mathfrak{Y} - \mathfrak{N}\left(\frac{G(\mathrm{D}^{\times k}_{\mathfrak{X}}\mathfrak{Y}, \mathfrak{N}) + G(\mathrm{D}^{\times k}_{\mathfrak{X}}\mathfrak{Y}, \mathfrak{N})^\dagger}{2}\right)^\top$$

Proof:

(i) Write $\mathfrak{v} = \mathfrak{v} - \mathfrak{x}B + \mathfrak{x}B$. Our goal is to solve the system

$$\begin{cases} G(\mathfrak{v} - \mathfrak{x}B, \mathfrak{x}) \in \mathfrak{u}(k) \\ B \in \mathfrak{h}(k) \end{cases}$$

for B. Since $G(\mathfrak{v} - \mathfrak{x}B, \mathfrak{x})^\dagger = G(\mathfrak{v},\mathfrak{x})^\dagger - B^\top$, this being in $\mathfrak{u}(k)$ means that

$$G(\mathfrak{v},\mathfrak{x})^\dagger - \overline{B} = -G(\mathfrak{v},\mathfrak{x}) + B^\top \implies \overline{G(\mathfrak{v},\mathfrak{x})} - B^\dagger = -G(\mathfrak{v},\mathfrak{x})^\top + B,$$

so that

$$B = \frac{1}{2}\left(\overline{G(\mathfrak{v},\mathfrak{x})} + G(\mathfrak{v},\mathfrak{x})^\top\right) = \left(\frac{G(\mathfrak{v},\mathfrak{x}) + G(\mathfrak{v},\mathfrak{x})^\dagger}{2}\right)^\top.$$

(ii) Immediate from (i).

$\square$

From this, one can compute the curvature tensor of (Σ_k, g^{Σ_k}), and so on. In the case $k = 1$, we turned our attention to the field $\mathrm{i}\boldsymbol{N}$. Here, for $A \in \mathfrak{u}(k)$, one can easily see that the flow of A (regarded as a vector field on Σ_k is just $\Phi_{t,A}\colon \Sigma_k \to \Sigma_k$ given by $\Phi_{t,A}(\mathfrak{x}) = \mathfrak{x}\mathrm{e}^{tA}$. Since $\mathrm{e}^{tA} \in \mathrm{U}(k)$, we see that A is a Killing vector field. Moreover, if $W \in \mathrm{Gr}_k(W)$, the fiber $\Pi^{-1}[W]$ is precisely the fixed point set of the Householder reflection (used for the first time here in Corollary 56, p. 29) about W acting on Σ_k, restricted from $V^{\times k}$. Thus the fibers are totally geodesic and so $\mathsf{T} = 0$. The computation for A would follow from combining Lemma 81 (p. 41) above with Corollary 76 (p. 40).

However, aiming to move on, we can still obtain a great deal of information from symmetries. We'll begin by using Example 63 (p. 33) to mimic Proposition 52 (p. 27):

Proposition 82.

(i) $\mathrm{PU}(V, \langle\cdot,\cdot\rangle) \subseteq \mathrm{Iso}(\mathrm{Gr}_k(V), g^{\mathrm{G}})$.

(ii) Every transformation in $\mathfrak{u}(V,\langle\cdot,\cdot\rangle)$ and every matrix in $\mathfrak{u}(k)$, regarded as vector fields on Σ_k, project to Killing fields on $(\mathrm{Gr}_k(V), g^{\mathrm{G}})$.

Proof:

(i) We'll repeat the proof given before here just for the sake of completeness. Let $T\in \mathrm{U}(V,\langle\cdot,\cdot\rangle)$. Since for any subspace $W\in\mathrm{Gr}_k(V)$ we have that $V/W = W^\perp$ and $T[W]^\perp = T[W^\perp]$, we have that $V/T[W]\cong T[W^\perp]$. So, the formula for the derivative $\mathrm{d}(\mathrm{P}T)_W\colon T_W\mathrm{Gr}_k(V)\to T_{T[W]}\mathrm{Gr}_k(V)$ seen in Example 63 (p. 33) indeed becomes just $\mathrm{d}(\mathrm{P}V)_W(H) = T\circ H\circ T^{-1}|_{T[W]}$. So, given $H_1,H_2\in T_W\mathrm{Gr}_k(V)$, we choose $\mathfrak{x}\in W^{\times k}\cap\Sigma_k$ and compute

$$\begin{aligned}
g^{\mathrm{G}}_{T[W]}(\mathrm{d}(\mathrm{P}T)_W(H_1),\mathrm{d}(\mathrm{P}T)_W(H_2)) &= g^{\Sigma_k}_{T\mathfrak{x}}(\mathrm{d}(\mathrm{P}T)_W(H_1)T\mathfrak{x},\mathrm{d}(\mathrm{P}T)_W(H_2)T\mathfrak{x})\\
&= g^{\Sigma_k}_{T\mathfrak{x}}(TH_1T^{-1}T\mathfrak{x},TH_2T^{-1}T\mathfrak{x})\\
&= g^{\Sigma_k}_{T\mathfrak{x}}(TH_1\mathfrak{x},TH_2\mathfrak{x})\\
&= g^{\Sigma_k}_{\mathfrak{x}}(H_1\mathfrak{x},H_2\mathfrak{x})\\
&= g^{\mathrm{G}}_W(H_1,H_2),
\end{aligned}$$

as wanted.

(ii) This follows from the brief discussion preceding this proposition, from adapting Example 12 (p. 9) to $\mathrm{Gr}_k(V)$, and from Example 77 (p. 40).

□

Proposition 53 (p. 28), in turn, is not easily adapted. In fact, it is not even true that $\mathrm{Iso}(\mathrm{Gr}_k(V),g^{\mathrm{G}})$ equals $\mathrm{PU}(V,\langle\cdot,\cdot\rangle)$ anymore (acting diagonally, or even the product of k copies of it). Nevertheless, we still have enough symmetry to obtain good results:

Proposition 83. *Let $W\in\mathrm{Gr}_k(V)$ be a subspace. There is an isometry $F\in\mathrm{PU}(V,\langle\cdot,\cdot\rangle)$ such that $F(W)=W$ and $\mathrm{d}F_W = -\mathrm{Id}_{T_W\mathrm{Gr}_k(V)}$.*

Proof: Again, consider the Householder reflection $R\in\mathrm{U}(V,\langle\cdot,\cdot\rangle)$ about W, given by $R = 2\,\mathrm{pr}_W - \mathrm{Id}_V$. Then take $F=\mathrm{P}R$. So for $H\in T_W\mathrm{Gr}_k(V)$, using that R is an involution which fixes W pointwise, we compute

$$\begin{aligned}
\mathrm{d}F_W(H)w &= (2\,\mathrm{pr}_W - \mathrm{Id}_V)\circ H\circ(2\,\mathrm{pr}_W-\mathrm{Id}_V)w\\
&= (2\,\mathrm{pr}_W-\mathrm{Id}_V)\circ H(w)\\
&= -H(w)
\end{aligned}$$

for all $w\in W$, since H takes values in $W^\perp$. □

Example 84. Here is an isometry between two different Grassmannians: translate the duality given in Example 62 (p. 32) using $\langle\cdot,\cdot\rangle$ to consider the orthogonal-complement diffeomorphism $\mathrm{Ort}\colon\mathrm{Gr}_k(V)\to\mathrm{Gr}_{n-k}(V)$ given by $\mathrm{Ort}(W)\doteq W^\perp$. Let's compute the derivative $\mathrm{d}(\mathrm{Ort})_W\colon T_W\mathrm{Gr}_k(V)\to T_{W^\perp}\mathrm{Gr}_{n-k}(V)$. Given $H\in T_W\mathrm{Gr}_k(V)$ and $y\in W^\perp$, we need to determine $\mathrm{d}(\mathrm{Ort})_W(H)y = y'(0)$, where $t\mapsto y(t)$ is any curve in V with

$y(0) = y$ and $y(t) = W(t)^\perp$ for all t, where $t \mapsto W(t)$ is a curve in $\mathrm{Gr}_k(V)$ with $W(0) = W$ and $W'(0) = H$. If $x \in W$ is arbitrary, we may consider a curve $t \mapsto x(t)$ with $x(0) = x$ and $x(t) \in W(t)$ for all t. Differentiating $\langle y(t), x(t)\rangle = 0$ gives that:

$$\langle y'(0), x\rangle + \langle y, x'(0)\rangle = 0 \implies \langle \mathrm{d}(\mathrm{Ort})_W(H)y, x\rangle + \langle y, Hx\rangle = 0.$$

Since x and y are arbitrary, by definition we conclude that $\mathrm{d}(\mathrm{Ort})_W(H) = -H^*$, where $H^*\colon W^\perp \to W$ is the adjoint of H. So far this is not a surprise. Now we check that Ort is an isometry as follows: take orthonormal bases $\mathfrak{x}$ for W and $\mathfrak{y}$ for $W^\perp$, and compute

$$\begin{aligned}
g^{\mathrm{G}}_{W^\perp}(\mathrm{d}(\mathrm{Ort})_W(H), \mathrm{d}(\mathrm{Ort}_W(H)) &= g^{\mathrm{G}}_{W^\perp}(-H^*, -H^*) = g^{\mathrm{G}}_{W^\perp}(H^*, H^*) \\
&= g^{\Sigma_{n-k}}_{\mathfrak{y}}(H^*\mathfrak{y}, H^*\mathfrak{y}) = \operatorname{Re}\operatorname{tr} G(H^*\mathfrak{y}, H^*\mathfrak{y}) \\
&= \operatorname{Re}\operatorname{tr} G(HH^*\mathfrak{y}, \mathfrak{y}) = \operatorname{Re}\operatorname{tr}(HH^*) \\
&= \operatorname{Re}\operatorname{tr}(H^*H) = \operatorname{Re}\operatorname{tr} G(H^*H\mathfrak{x}, \mathfrak{x}) \\
&= \operatorname{Re}\operatorname{tr} G(H\mathfrak{x}, H\mathfrak{x}) = g^{\Sigma_k}_{\mathfrak{x}}(H\mathfrak{x}, H\mathfrak{x}) \\
&= g^{\mathrm{G}}_W(H, H)
\end{aligned}$$

for every $H \in T_W\mathrm{Gr}_k(V)$. By polarization, the conclusion follows. If $n = 2k \geq 4$, this is a map $\mathrm{Gr}_k(V) \to \mathrm{Gr}_k(V)$ which is not induced by any transformation in $\mathrm{U}(V, \langle\cdot,\cdot\rangle)$, see [5].

Remark. As a consequence, if $M^k \subseteq \mathbb{R}^n$ is an embedded submanifold and we consider the normal Gauss map $G^\perp\colon M \to \mathrm{Gr}_k(\mathbb{R}^n)$ given by $G^\perp(p) = (T_pM)^\perp$, then we have the relation $G^\perp = \mathrm{Ort} \circ G$ and the chain rule implies that $\mathrm{d}G^\perp_p(\boldsymbol{v})\boldsymbol{\xi} = -A_{\boldsymbol{\xi}}(\boldsymbol{v})$, for all $\boldsymbol{v} \in T_pM$ and $\boldsymbol{\xi} \in (T_pM)^\perp$.

In any case, the same arguments as in the previous section gives the:

Corollary 85. $(\mathrm{Gr}_k(V), g^{\mathrm{G}})$ *is a locally symmetric Kähler manifold.*

Corollary 86. *For every subspace $W \subseteq V$, the inclusion $\mathrm{Gr}_k(W) \subseteq \mathrm{Gr}_k(V)$ is an isometric embedding, and $\mathrm{Gr}_k(W)$ is a totally geodesic submanifold of $(\mathrm{Gr}_k(V), g^{\mathrm{G}})$.*

Since this time we did not explictly compute the Ricci tensor of $(\mathrm{Gr}_k(V), g^{\mathrm{G}})$, showing that this is an Einstein manifold will require another strategy. Also, the classification of all connected totally geodesic complex submanifolds of $\mathrm{Gr}_k(V)$ is non-trivial, as there are submanifolds of $\mathrm{Gr}_k(V)$ whose dimension is not divisible by k. In any case, there is some research done about that under additional assumptions (for example, in [11] it is proven that a totally geodesic submanifold N of $\mathrm{Gr}_k(V)$ whose points have trivial intersection as subspaces of V is isometric to a sphere or to a complex projective space).

In any case, the strategy will use the homogeneity of $\mathrm{Gr}_k(V)$. More precisely, note that given the stabilizer of a given $W \in \mathrm{Gr}_k(V)$ under the action $\mathrm{U}(V, \langle\cdot,\cdot\rangle) \circlearrowright \mathrm{Gr}_k(V)$ is the product $\mathrm{U}(W) \times \mathrm{U}(W^\perp)$, since W is T-invariant if and only if $W^\perp$ is T-invariant, provided $T \in \mathrm{U}(V, \langle\cdot,\cdot\rangle)$. The orbit-stabilizer theorem then gives

$$\mathrm{Gr}_k(V) \cong \frac{\mathrm{U}(V, \langle\cdot,\cdot\rangle)}{\mathrm{U}(W) \times \mathrm{U}(W^\perp)},$$

and in fact it is even true that $\mathrm{Gr}_k(V)$ is a principal $(\mathrm{U}(W) \times \mathrm{U}(W^\perp))$-bundle. Recall that if we have a Lie group action $G \circlearrowright M$ on a manifold, G also acts on the tangent bundle TM via derivatives. However, if $x \in M$ and $g \in \mathrm{Stab}(x)$, the derivative is a map $\mathrm{d}g_x \colon T_xM \to T_xM$. This means that by restricting, we obtain the so-called *isotropy representation* $\mathrm{Stab}(x) \to \mathrm{GL}(T_xM)$, which in particular cases (as we will see next) may help us understand the geometry of M.

Lemma 87. *The isotropy representation* $\mathrm{Stab}(W) \to \mathrm{U}(T_W\mathrm{Gr}_k(V), g^{\mathrm{G}}_W)$ *is irreducible.*

Proof: Every $T \in \mathrm{Stab}(W)$ may be written as $T = T_1 \oplus T_2$ with $T_1 \in \mathrm{U}(W)$ and $T_2 \in \mathrm{U}(W^\perp)$, and by Example 63 (p. 33) we have that $T \cdot H = T_2 \circ H \circ T_1^{-1}$, for every $H \in T_W\mathrm{Gr}_k(V)$. Let $\mathscr{W} \subseteq T_W\mathrm{Gr}_k(V)$ be a non-trivial invariant subspace. We proceed in three steps:

- First, we will show that the minimal rank of a non-zero transformation in $\mathscr{W}$ is 1. So, assume by contradiction that an arbitrary $H \in \mathscr{W} \setminus \{0\}$ has rank $r \geq 2$, and let $e_1, \ldots, e_r$ be an orthonormal basis for $(\ker H)^\perp$ (relative to W). Define $T_1 \colon W \to W$ by $T_1(e_1) = e_1$, $T_1(e_j) = -e_j$ for $j = 2, \ldots, r$, and acting as the identity on $\ker H$. Then we have a unitary map $T = T_1 \oplus \mathrm{Id}_{W^\perp}$, and due to invariance of $\mathscr{W}$, consider the transformation $H + T \cdot H \in \mathscr{W}$. The contradiction is obtained by noting that the rank of $H + T \cdot H$ is equal to 1.

- Second, this implies that $\mathscr{W}$, not being the zero subspace, indeed contains one element of rank 1, say, $\langle \cdot, x_0 \rangle \otimes y_0$, and we may assume that $x_0 \in W$ and $y_0 \in W^\perp$ (we have $y_0 \in W^\perp$ because the operator must take values in $W^\perp$, and $x_0 \in W$ because the component of x_0 in $W^\perp$ may be discarded). With transitivity of the actions of $\mathrm{U}(W)$ and $\mathrm{U}(W^\perp)$ on W and $W^\perp$, invariance of $\mathscr{W}$ and the fact that $\mathscr{W}$ is closed under multiplication by scalar, we conclude that $\langle \cdot, x \rangle \otimes y \in \mathscr{W}$ for *any* $x \in W$ and $y \in W^\perp$.

- Lastly, every map $H \colon W \to W^\perp$ is a sum of rank 1 maps as above, so since $\mathscr{W}$ is closed under sums, we conclude that $H \in \mathscr{W}$ and thus $\mathscr{W} = T_W\mathrm{Gr}_k(V)$.

$\square$

Corollary 88. $(\mathrm{Gr}_k(V), g^{\mathrm{G}})$ *is an Einstein manifold.*

This corollary is a general phenomenon regarding irreducible isotropy representations. So let's state and prove it in generality:

Theorem 89. *Let* (M, g) *be a Riemannian manifold,* G *be a Lie group, and* $G \circlearrowright M$ *be a transitive Lie group action given by isometries. If the isotropy representations are irreducible, then* (M, g) *is Einstein.*

Proof: Fix a reference point $x \in M$ and then consider the isotropy representation $\mathrm{Stab}(x) \to \mathrm{O}(T_xM, g_x)$. The Ricci tensor $\mathrm{Ric}_x \colon T_xM \times T_xM \to \mathbb{R}$ corresponds to a self-adjoint endomorphism of T_xM, which we'll also denote by Ric_x, under the relation $\mathrm{Ric}_x(\boldsymbol{v}, \boldsymbol{w}) = g_x(\mathrm{Ric}_x(\boldsymbol{v}), \boldsymbol{w})$. Since Ric_x is self-adjoint and g_x is positive-definite, there is an eigenvector for Ric_x with associated eigenvalue λ. The eigenspace associated to λ is a $\mathrm{Stab}(x)$-invariant subspace of T_xM, since G acts on M by isometries. Since the isotropy representation is irreducible, we necessarily have $\mathrm{Ric}_x = \lambda g_x$. But the action of G on M is also transitive, so $\mathrm{Ric} = \lambda g$ as tensor fields on M. $\square$

Remark. The conclusion still holds if (M, g) is pseudo-Riemannian, provided the Ricci endomorphism still has an eigenvalue.

Now, with the overall geometry of $\mathrm{Gr}_k(V)$ being more or less understood, we'll devote the rest of this text by discussing a couple (more) of examples of embeddings of Grassmannians. The first one morally tries to answer the question: why is the projective space $\mathrm{P}V$ the protagonist of the story instead of an arbitrary Grassmannian $\mathrm{Gr}_k(V)$? If we take a subspace $W \in \mathrm{Gr}_k(V)$ and fix a basis $\mathfrak{x}$ for W, we may consider the (non-zero) k-blade $x_1 \wedge \cdots \wedge x_k \in V^{\wedge k}$, so that for any other basis $\mathfrak{x}'$ for W, the k-blades $x_1' \wedge \cdots \wedge x_k'$ and $x_1 \wedge \cdots \wedge x_k$ are proportional (namely, the scalar multiple is the determinant of the matrix that changes one basis to the other). This means that given the subspace W, we may create the line $\mathbb{C}(x_1 \wedge \cdots \wedge x_k)$ inside $V^{\wedge k}$. And $V^{\wedge k}$ receives a hermitian product from V (which we'll keep denoting by $\langle \cdot, \cdot \rangle$), defined by

$$\langle x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_k \rangle = \det G(\mathfrak{x}, \mathfrak{y}) = \det \left((\langle x_i, y_j \rangle)_{i,j=1}^k \right).$$

And this is well-defined since the right side is multilinear and alternating in both $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$. We also obtain a Riemannian metric g^{V^k} on $V^{\wedge k}$ by

$$g^{V^k}(x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_k) = \det \operatorname{Re} G(\mathfrak{x}, \mathfrak{y})$$

and, in particular, $\mathrm{P}(V^{\wedge k})$ gains a Fubini-Study metric. With this in place, we may write the:

Theorem 90 (The Plücker Embedding). *The mapping $\Gamma\colon \mathrm{Gr}_k(V) \to \mathrm{P}(V^{\wedge k})$ given by $\Gamma(W) \doteq \mathbb{C}(x_1 \wedge \cdots \wedge x_k)$, where $\mathfrak{x}$ is any basis for W, is a holomorphic isometric embedding.*

Proof: We have just argued above that Γ is indeed well-defined. Also, Γ is smooth, because if $\beta\colon V^{\wedge k} \to \mathbb{C}$ is a linear functional (which we'll think of as a multilinear alternating map $V^{\times k} \to \mathbb{C}$), on the open set $U_\beta = \{L \in \mathrm{P}(V^{\wedge k}) \mid \beta[L] = \mathbb{C}\}$ we may consider the associated chart map $\varphi_\beta\colon U_\beta \to \beta^{-1}(1)$, and consider the map

$$\mathrm{St}_k(V) \ni \mathfrak{x} \mapsto \frac{x_1 \wedge \cdots \wedge x_k}{b(x_1, \ldots, x_k)} \in \beta^{-1}(1),$$

which is manifestly smooth. So smoothness of Γ follows from the arbitrariety of β and the fact that the projection $\mathrm{St}_k(V) \to \mathrm{Gr}_k(V)$ is a surjective submersion. Also, Γ is injective, for if $\Gamma(W) = \Gamma(W')$ and we choose bases $\mathfrak{x}$ and $\mathfrak{x}'$ for W and W', there are $\lambda \in \mathbb{C} \setminus \{0\}$ and $A \in \mathrm{GL}(k, \mathbb{C})$ such that

$$\mathfrak{x} = \begin{bmatrix} \lambda x_1' & \cdots & x_k \end{bmatrix} A,$$

and this implies that $W = W'$. For the next step, we will compute the derivative $\mathrm{d}\Gamma_W\colon T_W\mathrm{Gr}_k(V) \to T_{\Gamma(W)}\mathrm{P}(V^{\wedge k})$, as follows: given $H \in T_W\mathrm{Gr}_k(V)$, it suffices to compute $\mathrm{d}\Gamma_W(H)(x_1 \wedge \cdots \wedge x_k)$, and the result will be correct provided the result we obtain is multilinear and alternating in $(x_1, \ldots, x_k)$ (due to the universal property of the exterior power $V^{\wedge k}$). So, we have that if $t \mapsto W(t)$ is a curve in $\mathrm{Gr}_k(V)$ with $W'(0) = H$, then $Hx_i = x_i'(0)$, where $t \mapsto x_i(t)$ is any smooth curve with $x_i(0) = x_i$

and $x_i(t) \in W(t)$, for all $i = 1, \ldots, k$. Then $t \mapsto x_1(t) \wedge \cdots \wedge x_k(t)$ is a curve in $\mathrm{P}(V^{\wedge k})$ that at 0 gives $x_1 \wedge \cdots \wedge x_k$ and also satisfies $x_1(t) \wedge \cdots \wedge x_k(t) \in \Gamma(W(t))$ for all t. For this reason, we are justified in writing

$$\begin{aligned} \mathrm{d}\Gamma_W(H)(x_1 \wedge \cdots \wedge x_k) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} x_1(t) \wedge \cdots \wedge x_k(t) \\ &= \sum_{i=1}^k x_1(0) \wedge \cdots \wedge x_{i-1}(0) \wedge x_i'(0) \wedge x_{i+1}(0) \wedge \cdots \wedge x_k(0) \\ &= \sum_{i=1}^k x_1 \wedge \cdots \wedge x_{i-1} \wedge Hx_i \wedge x_{i+1} \wedge \cdots \wedge x_k. \end{aligned}$$

And indeed, we also check that

$$\left\langle \sum_{i=1}^k (x_1 \wedge \cdots \wedge x_{i-1} \wedge Hx_i \wedge x_{i+1} \wedge \cdots \wedge x_k), x_1 \wedge \cdots \wedge x_k \right\rangle = 0,$$

since each term in the left side equals the determinant of the matrix obtained from $(\langle x_i, x_j \rangle)_{i,j=1}^k$ by replacing the i-th row with

$$\begin{pmatrix} \langle Hx_i, x_1 \rangle & \cdots & \langle Hx_i, x_k \rangle \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix},$$

so indeed we have that $\mathrm{d}\Gamma_W(H)(x_1 \wedge \cdots \wedge x_k) \in \Gamma(W)^\perp$. We'll conclude this proof (from here on, using only $\langle \cdot, \cdot \rangle_{\mathbb{R}}$) by showing that $\|\mathrm{d}\Gamma_W(H)\| = \|H\|$ for all maps $H \in T_W\mathrm{Gr}_k(V)$ for, if this is the case, the metrics being positive-definite implies that $\mathrm{d}\Gamma_W$ is injective, meaning that Γ is an immersion, and hence an (isometric) embedding since $\mathrm{Gr}_k(V)$ is compact. So, assume now that the chosen basis $(x_1, \ldots, x_k)$ for W is orthonormal. We will deal with squared quantities and show that

$$\|\mathrm{d}\Gamma_W(H)(x_1 \wedge \cdots \wedge x_k)\|^2 = \sum_{i=1}^k \|Hx_i\|^2$$

instead. To wit, the right side equals the sum over i and j of the determinant of the matrix described as follows: take the $k \times k$ identity matrix, replace the i-th row and the j-th column by zeros, and then insert in the (i, j)-th slot the product $\langle Hx_i, Hx_j \rangle_{\mathbb{R}}$. Only the terms corresponding to $i = j$ survive and the corresponding determinant is $\|Hx_i\|^2$. So, the conclusion follows. $\square$

Remark.

- When $V = \mathbb{C}^n$, we may consider a subspace $W \subseteq \mathbb{C}^n$ as an equivalence class of $n \times k$ matrices under the equivalence relation given by saying two matrices are equivalent if their row-span is the same. The Plücker embedding takes such a class to the vector
$$(p_{i_1 \ldots i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n)$$
where $p_{i_1 \cdots i_k}$ is the order k minor subdeterminant obtained from selecting columns i_1 through i_k. Changing the class of the input matrix rescales the above vector by

the determinant of the matrix that changes one row-basis to the other, and so this defines a point in a projective space. Not surprisingly, the $p_{i_1\cdots i_k}$ are called *Plücker coordinates* for $\mathrm{Gr}_k(\mathbb{C}^n)$. For instance, for $\mathrm{Gr}_2(\mathbb{C}^4)$, the embedding takes a matrix to its projective point of minors $[p_{11} : p_{12} : p_{13} : p_{14} : p_{23} : p_{24} : p_{34}] \in \mathbb{C}\mathrm{P}^5$. A brute force calculation then shows that $\mathrm{Gr}_k(\mathbb{C}^n)$ is a projective subvariety of a projective space, as it may be described by the zero-set of a collection of homogeneous polynomials. For instance, since $\dim \mathrm{Gr}_2(\mathbb{C}^4) = 4$ and $\dim \mathbb{C}\mathrm{P}^5 = 5$, we will be able to describe $\mathrm{Gr}_2(\mathbb{C}^4)$ as the zero set of a single polynomial. Namely, $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$.

- The dimension of $\mathrm{P}(V^{\wedge k})$ is much larger than the dimension of $\mathrm{Gr}_k(V)$, so that the Plücker embedding is not, in some sense, optimal (compare with Nash's embedding theorem). To get more intuition on the image of $\mathrm{Gr}_k(V)$ inside $\mathrm{P}(V^{\wedge k})$, one has to abandon differential geometry for the time being and focus on algebraic geometry instead. It turns out that generally, $\mathrm{Gr}_k(V)$ is *Cremona linearizable*, i.e., it is birationally equivalent to a projective subspace of $\mathrm{P}(V^{\wedge k})$ under a Cremona transformation (that is, a birational map $\mathbb{C}\mathrm{P}^r \dashrightarrow \mathbb{C}\mathrm{P}^r$ of the form

$$[x_0 : \cdots : x_r] \mapsto [F_0(x_1, \ldots, x_r) : \cdots : F_r(x_1, \ldots, x_r)],$$

where the F_j's are coprime homogeneous polynomials with the same degree). This is explored in [3]. Obviously such a Cremona transformation cannot be an isometry, as $\mathrm{Gr}_k(V)$ is not totally geodesic in $\mathrm{P}(V^{\wedge k})$.

For another example of embedding, consider now two vector spaces V_1 and V_2, equipped with hermitian products (both denoted by) $\langle\cdot,\cdot\rangle$ and its real parts $\langle\cdot,\cdot\rangle_{\mathbb{R}}$. The tensor product $V_1 \otimes V_2$ inherits products

$$\langle v_1 \otimes w_1, v_2 \otimes w_2\rangle = \langle v_1, v_2\rangle\langle w_1, w_2\rangle \text{ and } \langle v_1 \otimes w_1, v_2 \otimes w_2\rangle_{\mathbb{R}} = \langle v_1, v_2\rangle_{\mathbb{R}}\langle w_1, w_2\rangle_{\mathbb{R}}.$$

In particular, $\mathrm{P}(V \otimes W)$ gains a Fubini-Study metric, and we may consider the product metric on $\mathrm{P}V \times \mathrm{P}W$.

Theorem 91 (The Segre Embedding). *The map $\sigma\colon \mathrm{P}V_1 \times \mathrm{P}V_2 \to \mathrm{P}(V_1 \otimes V_2)$ given by $\sigma(L_1, L_2) = L_1 \otimes L_2$ is an isometric embedding.*

Proof: Let's argue that σ is injective: assume that $L_1 \otimes L_2 = L_1' \otimes L_2'$ for $L_1, L_1' \in \mathrm{P}V_1$ and $L_2, L_2' \in \mathrm{P}V_2$. If $L_1 \neq L_2$ and $L_2 \neq L_2$, we may pick $x_1 \in L_1$, $x_1' \in L_1'$, $x_2 \in L_2$ and $x_2' \in L_2'$ with $x_1 \otimes x_2 = \lambda x_1' \otimes x_2'$ for some $\lambda \in \mathbb{C}$, but with $\{x_1, x_1'\}$ and $\{x_2, x_2'\}$ both linearly independent. This allows us to choose two linear functionals $h_1 \in V_1^*$ and $h_2 \in V_2^*$ with $h_1(x_1) = 1$, $h_1(x_1') = 0$, $h_2(x_2) = 1$ and $h_2(x_2') = 0$. Apply $h_1 \otimes h_2$ on both sides of $x_1 \otimes x_2 = \lambda x_1' \otimes x_2'$ to get $1 = 0$. Thus we conclude that $L_1 = L_1'$ and $L_2 = L_2'$, as required. Let's proceed: by now it is easy to see that the derivative of σ is given by

$$\mathrm{d}\sigma_{(L_1,L_2)}(H_1, H_2) = H_1 \otimes \mathrm{Id}_{L_2} + \mathrm{Id}_{L_1} \otimes H_2.$$

Choose unit vectors $x_1 \in L_1$ and $x_2 \in L_2$. Note that $x_1 \otimes x_2 \in L_1 \otimes L_2$ is also an unit vector. We compute

$$
\begin{aligned}
(g^{\mathrm{FS}})_{L_1\otimes L_2}(\mathrm{d}\sigma_{(L_1,L_2)}(H_1,H_2),\mathrm{d}\sigma_{(L_1,L_2)}(H_1',H_2')) &= \\
&= \langle H_1x_1, H_1'x_1\rangle_{\mathbb{R}}\langle x_2,x_2\rangle_{\mathbb{R}} + \langle H_1x_1,x_1\rangle_{\mathbb{R}}\langle x_2,H_2'x_2\rangle_{\mathbb{R}} \\
&\quad + \langle x_1,H_1'x_1\rangle_{\mathbb{R}}\langle H_2x_2,x_2\rangle_{\mathbb{R}} + \langle x_1,x_1\rangle_{\mathbb{R}}\langle H_2x_2,H_2'x_2\rangle_{\mathbb{R}} \\
&= \langle H_1x_1,H_1'x_1\rangle_{\mathbb{R}} + \langle H_2x_2,H_2'x_2\rangle_{\mathbb{R}} \\
&= g^{\mathrm{FS}}_{L_1}(H_1,H_1') + g^{\mathrm{FS}}_{L_2}(H_2,H_2').
\end{aligned}
$$

Thus σ is an isometric immersion. Since $\mathrm{P}V_1 \times \mathrm{P}V_2$ is compact, σ is an isometric embedding. $\square$

Remark. Clearly if $V_1,\ldots,V_r$ are vector spaces with hermitian products, the above generalizes to give an isometric embedding $\prod_{i=1}^r \mathrm{P}V_i \hookrightarrow \mathrm{P}\left(\bigotimes_{i=1}^r V_i\right)$.

Example 92 (The Veronese embedding)**.** Let $d \geq 1$ be some integer degree. The composition $\nu_d\colon \mathrm{P}V \to \mathrm{P}(V^{\odot d})$ given by the composition

$$
d\cdot \mathrm{P}V \xhookrightarrow{\quad\Delta\quad} (\mathrm{P}V)^{\times d} \xrightarrow{\quad\sigma\quad} \mathrm{P}(V^{\odot d}),
$$

where $d\cdot \mathrm{P}V$ denotes $\mathrm{P}V$ equipped with the metric $d\cdot g^{\mathrm{FS}}$ and Δ is the diagonal embedding, is an isometric embedding. It is called the *Veronese embedding of degree d*. In this case, we could replace $\mathrm{P}(V^{\otimes d})$ by $\mathrm{P}(V^{\odot d})$ due to the presence of Δ.

Example 93. The embedding given in Theorem 72 (p. 37) is actually a homothetic embedding. One may compute $\langle \mathrm{d}\Phi_W(H),\mathrm{d}\Phi_W(H)\rangle = 2n^2\,\mathrm{Re}\,\mathrm{tr}\,(H^*H) = 2n^2\, g^{\mathrm{G}}_W(H,H)$ for all $H \in T_W\mathrm{Gr}_k(V)$.

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