A QUICK NOTE ON ORTHOGONAL LIE ALGEBRAS

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EUCLIDEAN ALGEBRAS

Definition 1. The *special orthogonal Lie algebra* of dimension $n \geq 1$ over $\mathbb{R}$ is defined as

$$\mathfrak{so}(n, \mathbb{R}) \doteq \{A \in \mathfrak{gl}(n, \mathbb{R}) \mid A^\top + A = 0\}.$$

It is a vector subspace of the space $\mathfrak{gl}(n, \mathbb{R})$ of all $n \times n$ real matrices, and its Lie algebra structure comes from the commutator of matrices, $[A, B] \doteq AB - BA$.

For $n = 1$ we have $\mathfrak{so}(1, \mathbb{R}) = \{0\}$, and for $n = 2$ we have $\mathfrak{so}(2, \mathbb{R}) \cong \mathbb{R}$. *So, from here on, assume that* $n \geq 3$. On dimension $n = 3$, the first interesting case, we recall that in the space $\mathbb{R}^3$ we have a cross product operation $\times$. Recall the double cross product identity

$$\boldsymbol{x} \times (\boldsymbol{y} \times \boldsymbol{z}) = \langle \boldsymbol{x}, \boldsymbol{z} \rangle \boldsymbol{y} - \langle \boldsymbol{x}, \boldsymbol{y} \rangle \boldsymbol{z},$$

for any $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^3$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^3$. A simple way to remember it is that $\boldsymbol{x} \times (\boldsymbol{y} \times \boldsymbol{z})$ must be orthogonal to $\boldsymbol{y} \times \boldsymbol{z}$, and so it should be a linear combination of $\boldsymbol{y}$ and $\boldsymbol{z}$. The only natural possibility for the coefficients of $\boldsymbol{y}$ and $\boldsymbol{z}$ is the inner product of the remaining vectors, while the negative sign on the $\boldsymbol{z}$-coefficient accounts for skew-symmetry of $\times$. As a consequence:

Theorem 2. $(\mathbb{R}^3, \times)$ *is a Lie algebra, which is isomorphic to* $\mathfrak{so}(3, \mathbb{R})$*.*

Proof: We have already mentioned that $\times$ is skew-symmetric. Moreover, the cyclic sum over $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ of the double cross product identity vanishes, but this is the Jacobi identity for $(\mathbb{R}^3, \times)$. Thus it is a Lie algebra. Now, given any $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, we may consider the skew-symmetric linear operator $\boldsymbol{x} \times \_ \colon \mathbb{R}^3 \to \mathbb{R}^3$. Its matrix relative to the standard basis of $\mathbb{R}^3$ is

$$\mathsf{A}_{\boldsymbol{x}} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}.$$

The map $\mathsf{A} \colon \mathbb{R}^3 \to \mathfrak{so}(3, \mathbb{R})$ given by $\boldsymbol{x} \mapsto \mathsf{A}_{\boldsymbol{v}}$ is clearly a vector space isomorphism, but it is also a Lie algebra isomorphism, as the relation $[\mathsf{A}_{\boldsymbol{x}}, \mathsf{A}_{\boldsymbol{y}}] = \mathsf{A}_{\boldsymbol{x} \times \boldsymbol{y}}$ follows from a straightforward computation, for all vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$. $\square$

Remark. Note that $\ker \mathsf{A}_{\boldsymbol{x}} = \mathbb{R}\boldsymbol{x}$ for all non-zero $\boldsymbol{x} \in \mathbb{R}^3$, and $\mathsf{A}_{\boldsymbol{x}}^L$ is never diagonalizable, as its characteristic polynomial is $t(t^2 + \|\boldsymbol{x}\|^2)$.

Thus, we may treat $\mathfrak{so}(3, \mathbb{R})$ as $(\mathbb{R}^3, \times)$. This works so nicely because the isomorphism given in the above proof turns out to be nothing more than the adjoint representation of $(\mathbb{R}^3, \times)$. More precisely, we have that $\mathsf{A}_{\boldsymbol{x}} = \mathrm{ad}(\boldsymbol{x})$, for all $\boldsymbol{x} \in \mathbb{R}^3$. This allows us to quickly obtain several properties of $\mathfrak{so}(3, \mathbb{R})$.

Proposition 3. $\mathfrak{so}(3, \mathbb{R})' = \mathfrak{so}(3, \mathbb{R})$. *In particular,* $\mathfrak{so}(3, \mathbb{R})$ *is not solvable nor nilpotent.*

Proof: All the elements of the standard basis of $\mathbb{R}^3$ are obtained as cross products of the remaining ones. □

Remark. That $\mathfrak{so}(3, \mathbb{R})$ is not nilpotent also follows from Engel's theorem, as we have already obtained the characteristic polynomials of all the $\mathsf{A}_{\boldsymbol{x}}$.

Proposition 4. *The Cartan-Killing form of* $\mathfrak{so}(3, \mathbb{R})$ *is given by* $\kappa = -2\langle \cdot, \cdot \rangle$, *and so* $\mathfrak{so}(3, \mathbb{R})$ *is semi-simple.*

Proof: The double cross product formula may be recast as

$$\mathrm{ad}(\boldsymbol{x}) \circ \mathrm{ad}(\boldsymbol{y}) = \boldsymbol{x} \otimes \boldsymbol{y} - \langle \boldsymbol{x}, \boldsymbol{y} \rangle \mathrm{Id}_{\mathbb{R}^3},$$

and so we obtain that $\kappa(\boldsymbol{x}, \boldsymbol{y}) = \mathrm{tr}(\mathrm{ad}(\boldsymbol{x}) \circ \mathrm{ad}(\boldsymbol{y})) = \langle \boldsymbol{x}, \boldsymbol{y} \rangle - 3\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -2\langle \boldsymbol{x}, \boldsymbol{y} \rangle$, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$. Thus, $\kappa = -2\langle \cdot, \cdot \rangle$ is negative-definite. In particular, it is non-degenerate, so *Cartan's Second Criterion* now says that $\mathfrak{so}(3, \mathbb{R})$ is semi-simple. □

Remark. The fact that κ is negative-definite can be traced back to the fact that $\mathfrak{so}(3, \mathbb{R})$ is the Lie algebra of the *compact* Lie group $\mathrm{SO}(3, \mathbb{R})$ — compact Lie groups admit bi-invariant Riemannian metrics, for which every $\mathrm{ad}(X)$ is a skew-symmetric map, thus having all (complex) eigenvalues purely imaginary. It follows that the trace of the composition of two such maps is never positive. Of course, this is true not only in dimension 3, but for general n.

In any case, with a bit more of work, we can obtain something even better:

Theorem 5. $\mathfrak{so}(3, \mathbb{R})$ *is a simple Lie algebra.*

Proof: Let $\mathfrak{i} \lhd \mathfrak{so}(3, \mathbb{R})$ be an ideal. Our goal is to show that the dimension of $\mathfrak{i}$ cannot be 1 or 2. We proceed by cases.

- The dimension of $\mathfrak{i}$ cannot be 1. By contradiction, if we have $\mathfrak{i} = \mathbb{R}\boldsymbol{v}$ for some non-zero vector $\boldsymbol{v} \in \mathbb{R}^3$, we may choose any non-zero $\boldsymbol{w} \in \boldsymbol{v}^\perp$ and use that $\mathfrak{i}$ is an ideal to write $\boldsymbol{v} \times \boldsymbol{w} = \lambda \boldsymbol{v}$ for some $\lambda \in \mathbb{R}$. Thus $\lambda \boldsymbol{v}$ is both proportional and orthogonal to the non-zero vector $\boldsymbol{v}$, leading to $\lambda \boldsymbol{v} = \boldsymbol{0}$ and thus $\lambda = 0$. Now, $\boldsymbol{v} \times \boldsymbol{w} = \boldsymbol{0}$ says that $\boldsymbol{v}$ and $\boldsymbol{w}$ are both proportional and orthogonal, which is impossible as both vectors are non-zero.

- The dimension of $\mathfrak{i}$ cannot be 2. By contradiction, if $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is a basis for $\mathfrak{i}$, using that $\mathfrak{i}$ is an ideal gives that $\boldsymbol{v}_1 \times \boldsymbol{v}_2 \in \mathfrak{i} \cap \mathfrak{i}^\perp = \{\boldsymbol{0}\}$, but $\boldsymbol{v}_1 \times \boldsymbol{v}_2 = \boldsymbol{0}$ contradicts linear independence of $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

□

Remark. The above proof in fact shows the stronger statement: $\mathfrak{so}(3,\mathbb{R})$ does not even have bidimensional subalgebras.

The strategy for dealing with $\mathfrak{so}(n,\mathbb{R})$ for arbitrary $n \geq 3$ is, in general, pretty different. For that, we'll use exterior algebra instead of cross products to do it. This generalization is to be expected, as $(\mathbb{R}^3, \times)$ satisfies the universal property of $(\mathbb{R}^3)^{\wedge 2}$. Recall that given any vector space V, the exterior power $V^{\wedge k}$ is the space generated by all *k-blades* $v_1 \wedge \cdots \wedge v_k$, where $v_1, \ldots, v_k \in V$ and $\wedge$ multilinear and alternating. We also have that $v_1 \wedge \cdots \wedge v_k = 0$ if and only if $\{v_1, \ldots, v_k\}$ is linearly dependent. Moreover, if V is equipped with an inner product $\langle \cdot, \cdot \rangle$, $V^{\wedge 2}$ may be identified with $\mathfrak{so}(V, \langle \cdot, \cdot \rangle)$.

Theorem 6. $\mathfrak{so}(n,\mathbb{R})$ *is simple,* unless $n = 4$.

Proof: Let $\mathfrak{i} \lhd \mathfrak{so}(n,\mathbb{R})$ be a non-zero ideal. Our goal is to show that $\mathfrak{i} = \mathfrak{so}(n,\mathbb{R})$. So, take a non-zero matrix $A \in \mathfrak{so}(n,\mathbb{R})$ and write it as

$$A = \lambda_1 e_1 \wedge e_2 + \lambda_2 e_3 \wedge e_4 + \cdots + \lambda_r e_{2r-1} \wedge e_{2r}$$

for some $r \geq 1$, where $(e_1, \ldots, e_n)$ is a positive orthonormal basis of $\mathbb{R}^n$, and the coefficients $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$ are non-zero real scalars[1]. Since $\mathfrak{i}$ is an ideal, we get that the expression on the right side of the above is in $\mathfrak{i}$ for *any* positive orthonormal basis of $\mathbb{R}^n$. Let's show that unless $n = 4$, the minimal possible r for a non-zero element of $\mathfrak{i}$ is $r = 1$. This will be enough to conclude that $\mathfrak{i} = \mathfrak{so}(n,\mathbb{R})$ because $\mathfrak{so}(n,\mathbb{R})$ is generated by 2-blades of orthonormal vectors.

- If n is odd. There is a vector e_k which does not appear in the expression for A. Since flipping signs $e_1 \mapsto -e_1$ and $e_k \mapsto -e_k$ is an orientation preserving orthogonal map, we have that $-\lambda_1 e_1 \wedge e_2 + \lambda_2 e_3 \wedge e_4 + \cdots + \lambda_r e_{2r-1} \wedge e_{2r} \in \mathfrak{i}$. Adding that to A, we obtain that $2\lambda_2 e_3 \wedge e_4 + \cdots + 2\lambda_r e_{2r-1} \wedge e_{2r} \in \mathfrak{i}$.

- If n is even and $n \neq 4$. Similarly to what was done above, we have that the reflection $e_1 \mapsto -e_1$ and $e_3 \mapsto -e_3$ is an orientation preserving orthogonal map, so we get that $-\lambda_1 e_1 \wedge e_2 - \lambda_2 e_3 \wedge e_4 + \cdots + \lambda_r e_{2r-1} \wedge e_{2r} \in \mathfrak{i}$. Adding that to A, we have that $2\lambda_3 e_5 \wedge e_6 + \cdots + 2\lambda_r e_{2r-1} \wedge e_{2r} \in \mathfrak{i}$.

In any case we obtain, by rescaling, that $\boldsymbol{x} \wedge \boldsymbol{y} \in \mathfrak{i}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, and so $\mathfrak{i} = \mathfrak{so}(n,\mathbb{R})$ as wanted. $\square$

[1] The complex eigenvalues of a skew-symmetric operator A in an Euclidean vector space $(V, \langle \cdot, \cdot \rangle)$ are necessarily purely imaginary. The real and imaginary parts of an associated complex eigenvector (in the complexified $V^{\mathbb{C}}$) span an A-invariant plane in V. Restricted to this plane, our skew-symmetric operator acts as a rotation, as the complex number λi is identified with the matrix

$$\begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix},$$

so the expression $A = \lambda_1 e_1 \wedge e_2 + \lambda_2 e_3 \wedge e_4 + \cdots + \lambda_r e_{2r-1} \wedge e_{2r}$ says that $Ae_1 = \lambda_1 e_2$, $Ae_2 = -\lambda_1 e_1$, and so on. Note that the assumption that $\langle \cdot, \cdot \rangle$ is positive-definite is crucial to diagonalize A over $\mathbb{C}$.

When $n = 4$, the argument given in the second bullet above fails, since by adding A to the modified operator would give the zero operator, and so one could not conclude that the minimal r could be reduced. In fact, $e_1 \wedge e_2 + e_3 \wedge e_4$ spans the ideal of *self-dual* operators and $e_1 \wedge e_2 - e_3 \wedge e_4$ spans the ideal of *anti-self-dual* operators, and it turns out that $\mathfrak{so}(4,\mathbb{R}) \cong \mathfrak{so}(3,\mathbb{R}) \oplus \mathfrak{so}(3,\mathbb{R})$.

Corollary 7. $\mathfrak{so}(n,\mathbb{R})' = \mathfrak{so}(n,\mathbb{R})$ *for all* $n \neq 4$. *In particular,* $\mathfrak{so}(n,\mathbb{R})$ *is not solvable nor nilpotent.*

Proof: $\mathfrak{so}(n,\mathbb{R})'$ is a non-zero ideal of $\mathfrak{so}(n,\mathbb{R})$. □

Remark. Of course, the above is true replacing $\mathfrak{so}(n,\mathbb{R})$ by any simple Lie algebra $\mathfrak{g}$.

Comments about Lorentzian algebras

Definition 8. The *special pseudo-orthogonal Lie algebra* of dimension $n \geq 1$ and index ν over $\mathbb{R}$ is defined as

$$\mathfrak{so}_\nu(n,\mathbb{R}) \doteq \{A \in \mathfrak{gl}(n,\mathbb{R}) \mid A^\top \mathrm{Id}_{n-\nu,\nu} + \mathrm{Id}_{n-\nu,\nu} A = 0\},$$

where $\mathrm{Id}_{n-\nu,\nu} = \mathrm{Id}_{n-\nu} \oplus (-\mathrm{Id}_\nu)$. Again, it is a vector subspace of the space $\mathfrak{gl}(n,\mathbb{R})$ of all $n \times n$ real matrices, and its Lie algebra structure comes from the commutator of matrices, $[A,B] \doteq AB - BA$.

Remark. Our convention for $\mathrm{Id}_{n-\nu,\nu}$ reflects the choice of convention $(+\cdots+-\cdots-)$ for indefinite products. We may always assume that $\nu \leq \lfloor n/2 \rfloor$, by switching the sign of the metric if needed.

Let's focus on the case $\nu = 1$, i.e., on the case where our scalar product instead of being Euclidean, are Lorentzian. Again, the dimensions $n = 1$ and $n = 2$ are completely uninteresting. For dimension $n = 3$, we use the natural cross product $\times_L$ in Lorentz-Minkowski space $\mathbb{L}^3 = \mathbb{R}^3_1$, defined in a similar way to the usual cross product $\times$, but flipping the sign of the timelike component. Namely, if (e_1, e_2, e_3) is the standard basis of $\mathbb{L}^3$, one has that

$$e_1 \times_L e_2 = -e_3, \quad e_2 \times_L e_3 = e_1 \quad \text{and} \quad e_3 \times_L e_1 = e_2,$$

with an arbitrary $\boldsymbol{x} \times_L \boldsymbol{y}$ being computed from the above via bilinearity and skew-symmetry of $\times_L$. The Lorentzian version of the double cross product formula is

$$\boldsymbol{x} \times_L (\boldsymbol{y} \times_L \boldsymbol{z}) = -\langle \boldsymbol{x}, \boldsymbol{z} \rangle_L \boldsymbol{y} + \langle \boldsymbol{x}, \boldsymbol{y} \rangle_L \boldsymbol{z},$$

for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{L}^3$. With this in place, the situation here mirrors what happened in the Euclidean case:

Theorem 9. *$(\mathbb{L}^3, \times_L)$ is a Lie algebra, which is isomorphic to $\mathfrak{so}_1(3, \mathbb{R})$.*

Proof: Again, we have that $\times_L$ is skew-symmetric and the cyclic sum over $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ of the Lorentzian double cross product identity vanishes, establishing the Jacobi identity for $(\mathbb{L}^3, \times_L)$. Thus it is a Lie algebra. As before, given any $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{L}^3$, we may consider the (Lorentzian) skew-symmetric map $\boldsymbol{x} \times_L \_ \colon \mathbb{L}^3 \to \mathbb{L}^3$, whose matrix relative to the standard basis of $\mathbb{L}^3$ is

$$\mathsf{A}^L_{\boldsymbol{x}} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}.$$

And the map $\mathsf{A}^L \colon \mathbb{L}^3 \to \mathfrak{so}_1(3, \mathbb{R})$ given by $\boldsymbol{x} \mapsto \mathsf{A}^L_{\boldsymbol{x}}$ is the desired Lie algebra isomorphism, as a quick computation shows that the relation $[\mathsf{A}^L_{\boldsymbol{x}}, \mathsf{A}^L_{\boldsymbol{y}}] = \mathsf{A}^L_{\boldsymbol{x} \times_L \boldsymbol{y}}$ still holds for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{L}^3$ (even though, *a priori*, one might fear the appearance of a minus sign due to some sort of causal character interference). $\square$

Remark. Again, the kernel of $\mathsf{A}^L_{\boldsymbol{x}}$ is just $\ker \mathsf{A}^L_{\boldsymbol{x}} = \mathbb{R}\boldsymbol{x}$ for all non-zero $\boldsymbol{x} \in \mathbb{L}^3$. But this time, the characteristic polynomial of $\mathsf{A}^L_{\boldsymbol{x}}$ is $t(t^2 - \langle \boldsymbol{x}, \boldsymbol{x} \rangle_L)$, from where we see that:

- If $\boldsymbol{x}$ is spacelike, $\mathsf{A}^L_{\boldsymbol{x}}$ is diagonalizable (to wit, the timelike plane $\boldsymbol{x}^\perp$ cuts the lightcone of $\mathbb{L}^3$ along two lightrays, which are the eigenspaces of $\mathsf{A}^L_{\boldsymbol{x}}$).

- If $\boldsymbol{x}$ is lightlike, then $\mathsf{A}^L_{\boldsymbol{x}}$ is nilpotent, by Cayley-Hamilton. More precisely, making $\boldsymbol{y} = \boldsymbol{x}$ in the the Lorentzian double cross product formula gives that $(\mathsf{A}^L_{\boldsymbol{x}})^2 \neq 0$, so t^3 is in fact the *minimal* polynomial of $\mathsf{A}^L_{\boldsymbol{x}}$. It is also not hard to see that if $\boldsymbol{y} \in \mathbb{L}^3$ is orthogonal to $\boldsymbol{x}$ (i.e., $\boldsymbol{y}$ lies in the lightlike plane containing $\boldsymbol{x}$), then $\boldsymbol{x} \times_L \boldsymbol{y}$ is always proportional to $\boldsymbol{x}$, vanishing when $\boldsymbol{y}$ is lightlike (hence itself[2] proportional to $\boldsymbol{x}$) and being lightlike when $\boldsymbol{y}$ is spacelike.

- If $\boldsymbol{x}$ is timelike, then $\mathsf{A}^L_{\boldsymbol{x}}$ is never diagonalizable, just as in the Euclidean case (this is hinted at by the fact that the spacelike plane $\boldsymbol{x}^\perp$ has trivial intersection with the lightcone of $\mathbb{L}^3$).

So, treating $\mathfrak{so}_1(3, \mathbb{R})$ as $(\mathbb{L}^3, \times_L)$, we may repeat the strategy adopted in the beginning of this note to obtain properties of $\mathfrak{so}_1(3, \mathbb{R})$. The same argument given in the Euclidean case gives us the:

Proposition 10. $\mathfrak{so}_1(3, \mathbb{R})' = \mathfrak{so}_1(3, \mathbb{R})$. *In particular, $\mathfrak{so}_1(3, \mathbb{R})$ is not solvable nor nilpotent.*

Using causal characters, we see the first striking difference between the algebras $\mathfrak{so}(3, \mathbb{R})$ and $\mathfrak{so}_1(3, \mathbb{R})$:

Proposition 11. $\mathfrak{so}_1(3, \mathbb{R})$ *is not simple, and the non-trivial ideals of $(\mathbb{L}^3, \times_L)$ are precisely the lightlike planes in $\mathbb{L}^3$.*

[2] In any Lorentzian vector space, two lightlike vectors are orthogonal if and only if they are proportional.

Proof: Fix a non-zero $\boldsymbol{v} \in \mathbb{L}^3$ and consider the line $\mathbb{R}\boldsymbol{v}$. We argue that $\mathbb{R}\boldsymbol{v}$ is never an ideal. If $\boldsymbol{v}$ is spacelike or timelike, take a spacelike $\boldsymbol{w} \in \mathbb{L}^3$ orthogonal to $\boldsymbol{v}$, so that $\boldsymbol{v} \times_L \boldsymbol{w}$ has the "opposite" causal type of $\boldsymbol{v}$, and thus cannot be in $\mathbb{R}\boldsymbol{v}$. If $\boldsymbol{v}$ is lightlike, take a lightlike $\boldsymbol{w} \in \mathbb{L}^3$ with $\langle \boldsymbol{v}, \boldsymbol{w}\rangle_L = 1$, so that $\boldsymbol{v} \times_L \boldsymbol{w}$ is spacelike, and again cannot be in $\mathbb{R}\boldsymbol{v}$. Now consider a plane Π in $\mathbb{L}^3$, passing through the origin. If Π is spacelike or timelike, the cross product of any two vectors in a basis for Π will have the "opposite" causal type of Π, and thus cannot be in Π. The last thing to verify is that all lightlike planes are ideals. This was done in the previous remark. $\square$

In view of the previous result, the next best thing we could ask ourselves is whether $\mathfrak{so}_1(3,\mathbb{R})$ is semi-simple.

Proposition 12. *The Cartan-Killing form of $\mathfrak{so}_1(3,\mathbb{R})$ is given by $\kappa = 2\langle\cdot,\cdot\rangle_L$, and so $\mathfrak{so}_1(3,\mathbb{R})$ is semi-simple.*

Proof: Like before, we have that

$$\mathrm{ad}(\boldsymbol{x}) \circ \mathrm{ad}(\boldsymbol{y}) = -\boldsymbol{x} \otimes \boldsymbol{y} + \langle \boldsymbol{x}, \boldsymbol{y}\rangle_L \mathrm{Id}_{\mathbb{L}^3},$$

and so we obtain that $\kappa(\boldsymbol{x},\boldsymbol{y}) = \mathrm{tr}(\mathrm{ad}(\boldsymbol{x}) \circ \mathrm{ad}(\boldsymbol{y})) = -\langle \boldsymbol{x}, \boldsymbol{y}\rangle_L + 3\langle \boldsymbol{x}, \boldsymbol{y}\rangle_L = 2\langle \boldsymbol{x}, \boldsymbol{y}\rangle_L$, for all $\boldsymbol{x},\boldsymbol{y} \in \mathbb{L}^3$. Thus, $\kappa = 2\langle\cdot,\cdot\rangle_L$ is non-degenerate. The conclusion follows from *Cartan's Second Criterion*. $\square$

Remark. Despite κ being non-degenerate in this case, we see that it is indefinite. This can be traced back to the fact that $\mathfrak{so}_1(3,\mathbb{R})$ is the Lie algebra of the Lorentz group $\mathrm{O}_1(3,\mathbb{R})$, which is non-compact — this is easily seen, for example, by noting that $\mathrm{O}_1(3,\mathbb{R})$ contains unbounded 1-parameter subgroups consisting of Lorentz boosts.

We'll conclude the discussion by giving another identification of $\mathfrak{so}_1(3,\mathbb{R})$ with a better known algebra. Namely, we'll show that there is a two-fold (homomorphic) covering map $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SO}_1(3,\mathbb{R})$, which will then imply that $\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{so}_1(3,\mathbb{R})$.

For that, the approach will be mostly coordinate-free. Let Π be a plane equipped with an area form $\alpha \in (\Pi^*)^{\wedge 2}$, and consider the three dimensional space $V = (\Pi^*)^{\odot 2}$. In other words, elements of V may be seen as symmetric bilinear forms $\tau\colon \Pi \times \Pi \to \mathbb{R}$. One quadratic form one would like to consider on V, in some sense, is $\det \tau$. However, this does not make sense without any extra structure, and this is where the area form enters the fray. One defines $Q\colon V \to \mathbb{R}$ by setting $Q(\tau) = \det[\tau]_{\mathcal{B}}$, where $\mathcal{B} = (e_1, e_2)$ is any basis for Π with $\alpha(e_1,e_2) = 1$. To see that this is well-defined, let $\widetilde{\mathcal{B}} = (\widetilde{e}_1, \widetilde{e}_2)$ be another basis for Π with $\alpha(\widetilde{e}_1, \widetilde{e}_2) = 1$. This condition means that if we write $\widetilde{e}_j = \sum_{i=1}^2 a^i{}_j e_i$, then the change of basis matrix $A = (a^i{}_j)_{i,j=1}^2$ is in $\mathrm{SL}(2,\mathbb{R})$. Taking determinants on both sides of the relation $[\tau]_{\widetilde{\mathcal{B}}} = A^\top [\tau]_{\mathcal{B}} A$, well-definedness of Q follows, where by $[\tau]_{\mathcal{B}}$ we mean the Gram matrix $(\tau(e_i,e_j))_{i,j=1}^2$. To avoid unimodular bases, one must pay the price and normalize the expression defining Q in an adequate way. Namely, one may also write

$$Q(\tau) = \frac{\det[\tau]_{(v_1,v_2)}}{\alpha(v_1,v_2)^2},$$

where (v_1, v_2) is *any* basis for Π — the square in the denominator is crucial to maintain well-definedness. Clearly Q is a quadratic form.

Proposition 13. *Q induces a Lorentzian scalar product $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ on V.*

Proof: Fix once and for all a unimodular basis $\mathscr{B} = (e_1, e_2)$ for Π. Then we simply have that

$$[\tau]_{\mathscr{B}} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \implies Q(\tau) = ac - b^2.$$

With suggestive notation, polarizing Q we obtain

$$\langle\!\langle \tau_1, \tau_2 \rangle\!\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\Big((a_1 + ta_2)(c_1 + tc_2) - (b_1 + tb_2)^2\Big) = \frac{a_1c_2 + a_2c_1}{2} - b_1b_2.$$

From the above expression, it's clear that $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ is Lorentzian. $\square$

Remark. With the notation above: the a-axis and c-axis are both lightlike, and the b-axis is timelike.

To understand better the geometry of $(V, \langle\!\langle\cdot,\cdot\rangle\!\rangle)$, we consider the *group of unimodular automorphisms* $\mathrm{Aut}(\Pi, \alpha) \doteq \{T \in \mathrm{GL}(\Pi) \mid T^*\alpha = \alpha\}$. Each linear map $T\colon \Pi \to \Pi$ induces a pull-back map $T^*\colon V \to V$, via $(T^*\tau)(v, w) = \tau(Tv, Tw)$.

Theorem 14. *For any $T \in \mathrm{Aut}(\Pi,\alpha)$, we have that $T^* \in \mathrm{SO}(V, \langle\!\langle\cdot,\cdot\rangle\!\rangle)$. Moreover, the map $\mathrm{Aut}(\Pi, \alpha) \to \mathrm{SO}(V, \langle\!\langle\cdot,\cdot\rangle\!\rangle)$ given by $T \mapsto T^*$ is a two-fold homomorphic covering.*

Remark. Once a basis for Π has been chosen and fixed, we get isomorphisms

$$\mathrm{Aut}(\Pi, \alpha) \cong \mathrm{SL}(2, \mathbb{R}) \quad \text{and} \quad \mathrm{SO}(V, \langle\!\langle\cdot,\cdot\rangle\!\rangle) \cong \mathrm{SO}_1(3, \mathbb{R}),$$

and hence a two-fold covering $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SO}_1(3,\mathbb{R})$. It induces, via derivatives, an isomorphism $\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{so}_1(3,\mathbb{R})$.

Proof: Pick a unimodular basis $\mathscr{B}$ for Π and note that if $\tau \in V$, we have the relation $[T^*\tau]_{\mathscr{B}} = [T]_{\mathscr{B}}^\top[\tau]_{\mathscr{B}}[T]_{\mathscr{B}}$. Now take the determinant of both sides to conclude that $Q(T^*\tau) = Q(\tau)$. By polarization, T^* preserves $\langle\!\langle\cdot,\cdot\rangle\!\rangle$. With this, we have that the map $\mathrm{Aut}(\Pi,\alpha) \to \mathrm{SO}(V, \langle\!\langle\cdot,\cdot\rangle\!\rangle)$ is now defined. Clearly it is smooth and a group homomorphism. To argue that the kernel of this map is $\{\pm\mathrm{Id}_\Pi\}$, assume that for all $\tau \in V$, we have $[\tau]_{\mathscr{B}} = [T]_{\mathscr{B}}^\top[\tau]_{\mathscr{B}}[T]_{\mathscr{B}}$. Then, in purely matricial terms, our goal amounts to showing that if $\det A = 1$ and $S = A^\top SA$ for every symmetric matrix S, then $A = \pm\mathrm{Id}_2$. Choosing $S = \mathrm{Id}_2$, we immediately obtain that $A \in \mathrm{SO}(2,\mathbb{R})$. Then writing

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad \text{with } a^2 + b^2 = 1,$$

we have that

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \implies a^2 = 1 \quad \text{and} \quad b = 0.$$

It follows that this map is surjective[3]. Finally, a surjective Lie group homomorphism with discrete kernel is a covering map, concluding the argument. $\square$

[3]A Lie group homomorphism $F : G \to H$ between connected groups with small enough kernel, i.e., such that $\dim\ker F \leq \dim G - \dim H$, is surjective: F has constant rank, and the rank-nullity theorem gives that F is a submersion. Thus F is open and $F[G]$ is an open connected subgroup of H — hence $F[G] = H$.