A GUIDE TO SYMPLECTIC GEOMETRY

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These are lecture notes for the SYMPLECTIC GEOMETRY CRASH COURSE held at The Ohio State University during the summer term of 2021, as our first attempt for a series of mini-courses run by graduate students for graduate students. Due to time and space constraints, many things will have to be omitted, but this should serve as a quick introduction to the subject, as courses on Symplectic Geometry are not currently offered at OSU. There will be many exercises scattered throughout these notes, most of them routine ones or just really remarks, not only useful to give the reader a working knowledge about the basic definitions and results, but also to serve as a self-study guide. And as far as references go, arXiv.org links as well as links for authors' webpages were provided whenever possible.

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1 Symplectic Linear Algebra

1.1 Symplectic spaces and their subspaces

There is nothing more natural than starting a text on *Symplecic Geometry*[1] with the definition of a symplectic vector space. All the vector spaces considered will be real and finite-dimensional unless otherwise specified.

Definition 1

A **symplectic vector space** is a pair (V, Ω), where:

- V is a vector space, and;
- $\Omega\colon V \times V \to \mathbb{R}$ is a non-degenerate[a] skew-symmetric bilinear form.

We say that Ω is a **linear symplectic form**.

[a] That is, if $\Omega(v, w) = 0$ for all $w \in V$, then $v = 0$.

Before anything else, a quick observation: every symplectic vector space (V, Ω) is even-dimensional. To wit, let $m = \dim V$ and pick any basis $(v_1, \ldots, v_m)$ for V. Identify Ω with the matrix $(\Omega(v_i, v_j))_{i,j=1}^m$. Non-degeneracy of Ω says that $\det \Omega \neq 0$, while skew-symmetry leads to $\det \Omega = (-1)^m \det \Omega$, and thus $(-1)^m = 1$. Thus m is even. Our first example will be manifestly even-dimensional:

Example 1

The mother of all examples (for precise reasons we'll see later in the chapter) is $(\mathbb{R}^{2n}, \Omega_{2n})$, where $\Omega_{2n}\colon \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ is given by

$$\Omega_{2n}((\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{x}', \boldsymbol{y}')) \doteq \langle \boldsymbol{x}, \boldsymbol{y}' \rangle - \langle \boldsymbol{x}', \boldsymbol{y} \rangle.$$

Here, we write $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ stands for the standard Euclidean inner product in $\mathbb{R}^n$. Here's a concrete instance (up to perhaps relabeling some coordinates): in $\mathbb{R}^4$, the symplectic form Ω_4 is given by

$$\Omega_4((x, y, z, w), (x', y', z', w')) = xy' - x'y + zw' - z'w.$$

The matrix representing Ω_{2n} relative to the standard basis of $\mathbb{R}^{2n}$, in block form, is simply

$$-J_{2n} \doteq \begin{pmatrix} \mathbf{0} & \mathrm{Id}_n \\ -\mathrm{Id}_n & \mathbf{0} \end{pmatrix}.$$

We call $(\mathbb{R}^{2n}, \Omega_{2n})$ the **canonical symplectic prototype**.

[1] By the way, the name "symplectic" is due to Hermann Weyl, changing the Latin *com/plex* to the Greek *sym/plectic*.

Here's another example, more abstract in spirit.

Example 2

Let L be a vector space, and consider $V \doteq L \oplus L^*$, where L^* is the dual space of L. We define $\Omega_L \colon V \times V \to \mathbb{R}$ by

$$\Omega_L((x,f),(y,g)) \doteq g(x) - f(y).$$

This Ω_L is called the **canonical symplectic structure of $L \oplus L^*$**.

Exercise 1

(a) Check that the symplectic forms given in the previous two examples are indeed non-degenerate.

(b) In the previous example, let $(e_1, \ldots, e_n)$ be a basis for L, take the dual basis $(e^1, \ldots, e^n)$ in L^*, and compute the matrix of Ω relative to the joint basis $((e_1,0), \ldots, (e_n,0), (0,e^1), \ldots, (0,e^n))$ of $L \oplus L^*$. Does the result surprise you?

When dealing with spaces equipped with inner products, we have orthogonal complements. There's a natural analogue here:

Definition 2

Let (V,Ω) be a symplectic vector space and $S \subseteq V$ be a subspace. The **symplectic complement** of S is defined to be $S^\Omega \doteq \{x \in V \mid \Omega(x,y) = 0 \text{ for all } y \in S\}$.

However, the behavior of symplectic complements is not exactly the same as the one for orthogonal complements. The most striking difference is that the sum of S and S^Ω does not need to be direct, nor equal to V, in general. But there's a saving grace:

Proposition 1

Let (V,Ω) be a symplectic vector space, and $S \subseteq V$ a subspace. Then:

(i) $\dim S + \dim S^\Omega = \dim V$.

(ii) $(S^\Omega)^\Omega = S$.

(iii) $S \oplus S^\Omega = V \iff S \cap S^\Omega = \{0\} \iff (S, \Omega|_{S\times S})$ is a symplectic vector space.

Proof:

(i) The map $V \ni x \mapsto \Omega(x,\cdot)|_S \in S^*$ is surjective with kernel S^Ω, so the formula follows from the rank-nullity theorem, noting that $\dim S = \dim S^*$.

(ii) Clearly $S \subseteq (S^\Omega)^\Omega$, so equality follows since $\dim S = \dim (S^\Omega)^\Omega$ by the previous item applied twice.

(iii) Since $\dim(S + S^\Omega) = \dim S + \dim S^\Omega - \dim(S \cap S^\Omega)$, item (i) now says that $\dim(S + S^\Omega) = \dim V$ if and only if $\dim(S \cap S^\Omega) = 0$.

□

In particular, this proposition shows that it is not true that restricting Ω to subspaces of V gives symplectic spaces. This is a very special condition. There are other types of subspaces worthy of attention.

Definition 3

Let (V, Ω) be a symplectic vector space and $S \subseteq V$ be a subspace. We say that S is:

(i) **symplectic** if $S \cap S^\Omega = \{0\}$.

(ii) **isotropic** if $S \subseteq S^\Omega$.

(iii) **coisotropic** if $S^\Omega \subseteq S$.

(iv) **Lagrangian** if $S = S^\Omega$.

As immediate examples, all lines passing through the origin are isotropic, and all hyperplanes passing through the origin are coisotropic. And Lagrangian subspaces are the isotropic subspaces which are maximal (relative to inclusion) — it is not hard to see that Lagrangian subspaces are always mid-dimensional. Here are more concrete examples:

Example 3

(1) Let $S \subseteq \mathbb{R}^n$ be any subspace. Then $S \times \{\mathbf{0}\}$ and $\{\mathbf{0}\} \times S$ are isotropic subspaces of $(\mathbb{R}^{2n}, \Omega_{2n})$, and they're Lagrangian if and only if $S = \mathbb{R}^n$. Moreover, $S \times S^\perp$ is always Lagrangian.

(2) Let L be a vector space, and consider the canonical symplectic structure Ω_L on $V = L \oplus L^*$. If $S \subseteq L$ and $X \subseteq L^*$ are vector subspaces, then $S \oplus \{\mathbf{0}\}$ and $\{\mathbf{0}\} \oplus X$ are both isotropic, and Lagrangian if and only if $S = L$ and $X = L^*$, respectively. If $\mathrm{Ann}(S) \subseteq L^*$ is the annihilator of S, then $S \oplus \mathrm{Ann}(S)$ is Lagrangian.

(3) If (V_1, Ω_1) and (V_2, Ω_2) are symplectic vector spaces, so is their (external) direct sum $(V_1 \oplus V_2, \Omega_1 \oplus \Omega_2)$, where

$$(\Omega_1 \oplus \Omega_2)((v_1, v_2), (v_1', v_2')) \doteq \Omega_1(v_1, v_1') + \Omega_2(v_2, v_2').$$

If $S_1 \subseteq V_1$ and $S_2 \subseteq V_2$ are subspaces, then $(S_1 \times S_2)^{\Omega_1 \oplus \Omega_2} = S_1^{\Omega_1} \oplus S_2^{\Omega_2}$, so that if S_1 and S_2 are both of the same "type" (i.e., symplectic, isotropic, coisotropic, Lagrangian), then $S_1 \times S_2$ will also have the same type.

Exercise 2

Explore the above example further: if $S_1, S_2 \subseteq \mathbb{R}^n$ are two subspaces, consider $(\mathbb{R}^{2n}, \Omega_{2n})$ and show that $(S_1 \times S_2)^{\Omega_{2n}} = S_2^{\perp} \times S_1^{\perp}$. Conclude that $S_1 \times S_2$ is isotropic if and only if $S_1 \subseteq S_2^{\perp}$, coisotropic if and only if $S_1 \supseteq S_2^{\perp}$, and Lagrangian if and only if $S_1 = S_2^{\perp}$. State and prove a similar result replacing $(\mathbb{R}^{2n}, \Omega_{2n})$ with $(L \oplus L^*, \Omega_L)$ and orthogonal complements with annihilators. Check the claims made in item (3).

Here's a less obvious example of a symplectic vector space, coming from a differential equation:

Example 4

Let $(L, \langle \cdot, \cdot \rangle)$ be a pseudo-Euclidean space (i.e., $\langle \cdot, \cdot \rangle$ is a non-degenerate symmetric bilinear form on L), $I \subseteq \mathbb{R}$ be an open interval, and $\Phi\colon I \to \operatorname{End}(L)$ be a smooth curve of self-adjoint operators. Let V be the space of smooth solutions $v\colon I \to L$ of the ordinary differential equation $\ddot{v}(t) = \Phi(t)v(t)$ (note that $\dim V = 2\dim L$ is even). Then, define $\Omega\colon V \times V \to \mathbb{R}$ by $\Omega(v, w) = \langle \dot{v}, w \rangle - \langle v, \dot{w} \rangle$ (observe that the right side is indeed constant). Clearly Ω is skew-symmetric, but we claim that it is non-degenerate as well: if $v \in V \setminus \{0\}$, there is $t_0 \in I$ such that $v(t_0) \neq 0$, so non-degeneracy of $\langle \cdot, \cdot \rangle$ provides a vector $w_0 \in L$ with $\langle v(t_0), w_0 \rangle \neq 0$. Then the unique $w \in V$ with $w(t_0) = 0$ and $\dot{w}(t_0) = w_0$ satisfies $\Omega(v, w) \neq 0$. For each $t \in I$, we have the operators $\delta_t, \dot{\delta}_t\colon V \to L$ given by $\delta_t(v) = v(t)$ and $\dot{\delta}_t(v) = \dot{v}(t)$, and both $\ker \delta_t$ and $\ker \dot{\delta}_t$ are Lagrangian.

In particular, the same idea shows that the space $\mathcal{J}(\gamma)$ of Jacobi fields along a geodesic γ on a pseudo-Riemannian manifold (M, g) has a natural symplectic form, since $J \in \mathcal{J}(\gamma)$ if and only if

$$\frac{\mathrm{D}^2 J}{\mathrm{d}t}(t) = R(\dot{\gamma}(t), J_t)\dot{\gamma}(t),$$

where $\mathrm{D}/\mathrm{d}t$ is the covariant derivative along γ induced by the Levi-Civita connection of (M, g), and each $\Phi(t) = R(\dot{\gamma}(t), \cdot)\dot{\gamma}(t)$ is self-adjoint due to symmetries of the curvature tensor.

The following exercise is also useful to practice using the definitions:

Exercise 3

Let (V, Ω) be a symplectic vector space, and $S_1, S_2 \subseteq V$ be two subspaces. Show that:

(a) $(S_1 + S_2)^{\Omega} = S_1^{\Omega} \cap S_2^{\Omega}$;

(b) $(S_1 \cap S_2)^{\Omega} = S_1^{\Omega} + S_2^{\Omega}$.

Conclude that if S is any subspace of V, then $S \cap S^{\Omega}$ is always isotropic and $S + S^{\Omega}$ is always coisotropic. **Hint:** how does (b) follow from (a)?

Let's make sure you're comfortable with everything seen so far.

Exercise 4 (Challenge #1)

Let V and W be two vector spaces. A map $B\colon V \times W \to \mathbb{R}$ is called a **perfect pairing** if both maps $V \to W^*$ and $W \to V^*$ induced by B are isomorphisms. Define $\Omega_B\colon (V \times W) \times (V \times W) \to \mathbb{R}$ by $\Omega_B((v,w),(v',w')) = B(v,w') - B(v',w)$.

(a) Show that $(V \times W, \Omega_B)$ is a symplectic vector space and explain how the two main examples we have been dealing with so far fit here. Namely, what are the perfect pairings?

(b) Let $T\colon V \to W$ be a linear map. Show that the graph $\mathrm{gr}(T)$ is a Lagrangian subspace of $V \times W$ if and only if the diagram

$$\begin{array}{ccc} V & \xrightarrow{\;T\;} & W \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ W^* & \xrightarrow[\;T^*\;]{} & V^* \end{array}$$

commutes.

(c) Conclude that if L is a vector space and $T\colon L \to L^*$ is linear, then $\mathrm{gr}(T)$ is a Lagrangian subspace of $(L \oplus L^*, \Omega_L)$ if and only if the bilinear map induced by T, $(x,y) \mapsto T(x)(y)$, is symmetric.

We move on. Spaces with inner products have orthonormal bases, but we have seen that orthogonal complements and symplectic complements are not quite the same. Are there symplectic analogues of orthonormal bases? You shouldn't be surprised that the answer is "yes".

Theorem 1

Let (V,Ω) be a symplectic vector space, with dimension $\dim V = 2n$. There is a basis $\mathscr{B} = (e_1,\ldots,e_n,f_1,\ldots,f_n)$ for V such that for all $i,j = 1,\ldots,n$, we have $\Omega(e_i,e_j) = 0$, $\Omega(f_i,f_j) = 0$ and $\Omega(e_i,f_j) = \delta_{ij}$. Or, in other words, the matrix of Ω relative to $\mathscr{B}$ is $-J_{2n}$. Such a basis $\mathscr{B}$ is called a **Darboux basis** for (V,Ω).

Proof: The proof goes by (strong) induction on n. If $\dim V = 2$, then pick any $e_1 \in V$, non-zero. Since Ω is non-degenerate, there is $f_1 \in V$ such that $\Omega(e_1,f_1) \neq 0$. Rescaling f_1, we may assume that $\Omega(e_1,f_1) = 1$, which establishes the base of the induction. For the inductive step, assume that $\dim V = 2n$, take $e_1, f_1 \in V$ just as above, and let $S = \mathrm{span}(e_1,f_1)$. Clearly S is symplectic, so S^{Ω} is also symplectic and $S \oplus S^{\Omega} = V$. By the induction assumption, we may take a Darboux basis $(e_2,\ldots,e_n,f_2,\ldots,f_n)$ for S^{Ω}. Throw in e_1 and f_1 to get a Darboux basis for V. $\square$

Example 5

Let (V, Ω) be a symplectic vector space and $\mathscr{B} = (e_1, \ldots, e_n, f_1, \ldots, f_n)$ be a Darboux basis for V. Let $I \subseteq [n]$, where $[n] = \{1, \ldots, n\}$. Then $S_{\mathscr{B},e,I} = \operatorname{span}(e_i : i \in I)$ and $S_{\mathscr{B},f,I} = \operatorname{span}(f_i : i \in I)$ are isotropic subspaces, Lagrangian if and only if $I = [n]$. And the subspace $S_{\mathscr{B},e,f,I} = \operatorname{span}(e_i, f_i : i \in I)$ is symplectic, with complement $S^{\Omega}_{\mathscr{B},e,f,I} = S_{\mathscr{B},e,f,[n]\setminus I}$.

Let's conclude this section with a lemma that will be useful later.

Lemma 1

Let (V, Ω) be a symplectic vector space, and $L \subseteq V$ a Lagrangian subspace. Then L has a Lagrangian complement, that is, there is a second Lagrangian subspace $L' \subseteq V$ such that $V = L \oplus L'$.

Proof: Start taking any complementary subspace $W \subseteq V$ for L, i.e., a subspace such that $V = L \oplus W$. There's no reason this W should already be Lagrangian, so we correct it as follows: for any $w \in W$, we look at the linear functional $f_w \colon V \to \mathbb{R}$ given by $f_w(x) = \Omega(w, \operatorname{pr}_W(x))/2$, where $\operatorname{pr}_W \colon V \to W$ is the projection defined by the direct sum. Note that f_w annihilates L, so we may write $f_w = \Omega(w', \cdot)$ for some $w' \in L^{\Omega} = L$ which is uniquely determined by w. Now we let $L' = \{w - w' \mid w \in W\}$, and claim that L' is Lagrangian and $V = L \oplus L'$. If $w_1, w_2 \in W$, compute

$$\begin{aligned}\Omega(w_1 - w_1', w_2 - w_2') &= \Omega(w_1, w_2) - \Omega(w_1, w_2') - \Omega(w_1', w_2) - \Omega(w_1', w_2') \\ &= \Omega(w_1, w_2) + f_{w_2}(w_1) - f_{w_1}(w_2) - 0 \\ &= \Omega(w_1, w_2) + \frac{1}{2}\Omega(w_2, w_1) - \frac{1}{2}\Omega(w_1, w_2) \\ &= 0.\end{aligned}$$

So L' is isotropic and $V = L + L'$, hence L' is Lagrangian and $V = L \oplus L'$ by a dimension count. The equality $V = L + L'$ is justified as follows: if $v \in V$, then $v = x + w$ with $x \in L$ and $w \in W$, so we have $v = (x + w') + (w - w')$. $\square$

1.2 Symplectomorphisms

In the previous section, we have seen symplectic analogues of orthogonal complements and of orthonormal bases. Naturally, there are analogues of isometries.

Definition 4

Let (V_1, Ω_1) and (V_2, Ω_2) be two symplectic vector spaces, and $\varphi \colon V_1 \to V_2$ be linear. The map φ is called **symplectic** if $\Omega_1(x, y) = \Omega_2(\varphi(x), \varphi(y))$ for all vectors $x, y \in V$. A symplectic isomorphism is then called a **symplectomorphism**.

Symplectic maps should behave well relative to symplectic complements, just like isometries behave well relative to orthogonal complements. There are some minor things one should pay attention to, though. The next exercise will clarify them.

Exercise 5

Let (V_1, Ω_1) and (V_2, Ω_2) be two symplectic vector spaces, and $\varphi\colon V_1 \to V_2$ be a symplectic map.

(a) Show that φ is necessarily injective. Give an example showing that φ does not need to be surjective.

(b) Show that if $S \subseteq V_1$ is a subspace, then $\varphi[S^{\Omega_1}] \subseteq \varphi[S]^{\Omega_2}$. Give an example showing that the reverse inclusion need not hold.

(c) With the same notation as in (b), show that $\varphi[S^{\Omega_1}] = \varphi[S]^{\Omega_2} \cap \mathrm{Ran}(\varphi)$.

With this in place, let's see some examples.

Example 6

Let L be a vector space and consider the canonical symplectic space $(L \oplus L^*, \Omega_L)$. If $T \in \mathrm{GL}(L)$, then $\varphi \doteq T \oplus (T^*)^{-1}\colon L \oplus L^* \to L \oplus L^*$ is a symplectomorphism, since

$$\begin{aligned}\Omega_L(\varphi(x,f), \varphi(y,g)) &= (T^*)^{-1}(g)(Tx) - (T^*)^{-1}(f)(Ty) \\ &= T^* \circ (T^*)^{-1}(g)(x) - T^* \circ (T^*)^{-1}(f)(y) \\ &= g(x) - f(y) \\ &= \Omega_L((x,f),(y,g)).\end{aligned}$$

As a particular example, we may take the dilations $T(x) = \mathrm{e}^a x$, where $a \in \mathbb{R}$ is fixed. This construction gives $\varphi_a(x,f) = (\mathrm{e}^a x, \mathrm{e}^{-a} f)$.

Exercise 6

Show that the above example is optimal, in the following sense: if $T\colon L \to L$ and $\hat{T}\colon L^* \to L^*$ are any linear maps, then $T \oplus \hat{T}$ is a symplectomorphism of $L \oplus L^*$ if and only if $\hat{T} = (T^*)^{-1}$.

Example 7

Let (V, Ω) be a symplectic vector space and $\mathscr{B} = (e_1, \ldots, e_n, f_1, \ldots, f_n)$ be a Darboux basis for V.

- The linear map $J\colon V \to V$ defined by setting $J(e_i) = f_i$ and $J(f_i) = -e_i$, for all $i = 1, \ldots, n$, is a symplectomorphism. Note that $J^2 = -\mathrm{Id}_V$ (that is, J is a linear complex structure on V).

- If $\sigma \in S_n$ is a permutation, then $\varphi_\sigma\colon V \to V$ given by $\varphi_\sigma(e_i) = e_{\sigma(i)}$ and $\varphi_\sigma(f_i) = f_{\sigma(i)}$, for all $i = 1, \ldots, n$, is a symplectomorphism.
- Let's generalize the above item. Let $A \in \mathrm{GL}_n(\mathbb{R})$, and define $\varphi_A\colon V \to V$ by

$$\varphi(e_j) = \sum_{i=1}^n (A^\top)^i{}_j e_i \quad \text{and} \quad \varphi(f_j) = \sum_{i=1}^n (A^{-1})^i{}_j f_i.$$

Then φ_A is a symplectomorphism. It's clear that $\Omega(\varphi_A(e_i), \varphi_A(e_j)) = 0$ and $\Omega(\varphi_A(f_i), \varphi_A(f_j)) = 0$ for all choices of i and j, so let's only check the last Darboux relation:

$$\begin{aligned}
\Omega(\varphi_A(e_i), \varphi_A(e_j)) &= \Omega\left(\sum_{k=1}^n (A^\top)^k{}_i e_k, \sum_{\ell=1}^n (A^{-1})^\ell{}_j f_\ell\right) \\
&= \sum_{k,\ell=1}^n (A^\top)^k{}_i (A^{-1})^\ell{}_j \Omega(e_k, f_\ell) \\
&= \sum_{k,\ell=1}^n (A^\top)^k{}_i (A^{-1})^\ell{}_j \delta_{k\ell} \\
&= \sum_{k=1}^n (A^\top)^k{}_i (A^{-1})^k{}_j \\
&= \delta_{ij}.
\end{aligned}$$

Exercise 7

Show that the last bullet of the previous example is optimal, in the following sense (with same notations as above and before): if $\varphi\colon V \to V$ is a symplectomorphism which leaves $S_{\mathscr{B},e,[n]}$ and $S_{\mathscr{B},f,[n]}$ invariant, there are two matrices $B, C \in \mathrm{GL}_n(\mathbb{R})$, $B = (b^i{}_j)_{i,j=1}^n$ and $C = (c^i{}_j)_{i,j=1}^n$, such that

$$\varphi(e_j) = \sum_{i=1}^n b^i{}_j e_i \quad \text{and} \quad \varphi(f_j) = \sum_{i=1}^n c^i{}_j f_i.$$

Then there is $A \in \mathrm{GL}_n(\mathbb{R})$ such that $B = A^\top$ and $C = A^{-1}$.

Example 8

As in Example 4 (p. 4), let $(L, \langle \cdot, \cdot \rangle)$ be a pseudo-Euclidean space and (V, Ω) be the vector space of smooth solutions of $\ddot{v}(t) = \Phi(t)v(t)$ on some interval $I \subseteq \mathbb{R}$, for a curve of self-adjoint endomorphisms of L. For every $t \in I$, the evaluation map $V \ni v \mapsto (v(t), \dot{v}(t)) \in L \oplus L$ is a symplectomorphism, where $L \oplus L$ is considered with the symplectic form from Example 1 (p. 1, with L playing the role of $\mathbb{R}^n$) or, alternatively, Exercise 4 (p. 5) with the perfect pairing $B = \langle \cdot, \cdot \rangle$.

Now we are ready to discuss the symplectic analogue of $\mathrm{O}(V, \mathrm{g})$, where (V, g) is an inner product space.

Definition 5

Let (V, Ω) be a symplectic vector space. The **symplectomorphism group** of (V, Ω), denoted by $\mathrm{Sp}(V, \Omega)$, is the collection of all symplectomorphisms from V to itself.

Clearly $\mathrm{Sp}(V, \Omega)$ is a group when equipped with composition of functions. In particular, we write $\mathrm{Sp}_{2n}(\mathbb{R}) \doteq \mathrm{Sp}(\mathbb{R}^{2n}, \Omega_{2n})$ when discussing things on the matrix level. Namely, we have that

$$\mathrm{Sp}_{2n}(\mathbb{R}) = \{R \in \mathrm{GL}_{2n}(\mathbb{R}) \mid R^\top J_{2n} R = J_{2n}\},$$

and from there we see that $\mathrm{Sp}_{2n}(\mathbb{R})$ is closed under matrix transposition (this is not true for every matrix group). Writing matrices in block-form, we can explictly write what it means for R to be in $\mathrm{Sp}_{2n}(\mathbb{R})$.

Exercise 8 (The Luneburg Relations)

Write $R \in \mathrm{GL}_{2n}(\mathbb{R})$ as

$$R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Show that $R \in \mathrm{Sp}_{2n}(\mathbb{R})$ if and only if we have that $A^\top C$ and $B^\top D$ are symmetric and $A^\top D - C^\top B = \mathrm{Id}_n$. In this case, also show that

$$R^{-1} = \begin{pmatrix} D^\top & -B^\top \\ -C^\top & A^\top \end{pmatrix}.$$

We will not be able to go into much detail about $\mathrm{Sp}(V, \Omega)$, so we will avoid technical proofs (references will be given) and settle for a few comments regarding:

- a distinguished set of generators $\langle \tau_{v,\lambda} \mid (v, \lambda) \in (V \setminus \{0\}) \times \mathbb{R} \rangle$.
- the spectrum $\sigma(\varphi)$ of a symplectomorphism $\varphi \in \mathrm{Sp}(V, \Omega)$;
- the fundamental group $\pi_1 \mathrm{Sp}(V, \Omega)$.

We will start on the group-theoretic side of things.

Definition 6

Let (V, Ω) be a symplectic vector space, and fix $v \in V$, $v \neq 0$, and $\lambda \in \mathbb{R}$. The **symplectic transvection** in the direction of v with intensity λ is the linear map $\tau_{v,\lambda}\colon V \to V$ given by $\tau_{v,\lambda}(x) \doteq x + \lambda\Omega(x, v)v$.

A symplectic transvection $\tau_{v,\lambda}$ is essentially a "shear-like reflection" about the hyperplane $(\mathbb{R}v)^{\Omega}$. It is instructive to establish some of their basic properties:

Exercise 9

Let (V, Ω) be a symplectic vector space, and $v \in V$ a non-zero vector. Then:

(a) $\tau_{v,\lambda} \in \mathrm{Sp}(V, \Omega)$, for all $\lambda \in \mathbb{R}$;

(b) $\tau_{v,\lambda_1} \circ \tau_{v,\lambda_2} = \tau_{v,\lambda_1+\lambda_2}$, for all $\lambda_1, \lambda_2 \in \mathbb{R}$;

(c) $\tau_{v,\lambda} = \mathrm{Id}_V$ if and only if $\lambda = 0$;

(d) if $\varphi \in \mathrm{Sp}(V, \Omega)$, then $\varphi \circ \tau_{v,\lambda} \circ \varphi^{-1} = \tau_{\varphi(v),\lambda}$, for all $\lambda \in \mathbb{R}$;

(e) $\tau_{av,\lambda} = \tau_{v,a^2\lambda}$, for all $a, \lambda \in \mathbb{R}$, $a \neq 0$.

The takeaway here is that if $v \neq 0$, the curve $\mathbb{R} \ni \lambda \mapsto \tau_{v,\lambda} \in \mathrm{Sp}(V, \Omega)$ is an injective group homomorphism, and the images of any two such curves are conjugate in $\mathrm{Sp}(V, \Omega)$.

The properties given above are the key for proving the big theorem on the group structure of $\mathrm{Sp}(V, \Omega)$:

Theorem 2 (Symplectic "Cartan-Dieudonné")

For any symplectic vector space (V, Ω), the group $\mathrm{Sp}(V, \Omega)$ is generated by symplectic transvections.

See [24] for a proof. From the definition of $\mathrm{Sp}_{2n}(\mathbb{R})$ and using Darboux bases, we can see that the determinant of any symplectomorphism equals 1 or -1. This theorem allows us to obtain the optimal conclusion:

Corollary 1

If (V, Ω) is a symplectic vector space and $\varphi \in \mathrm{Sp}(V, \Omega)$, then $\det \varphi = 1$.

Proof: By the symplectic version of Cartan-Dieudonné, it suffices to check that given a non-zero vector $v \in V$ and $\lambda \in \mathbb{R}$, then $\det \tau_{v,\lambda} = 1$. But it is easy to see that $\det \tau_{v.\lambda} = 1 + \lambda\Omega(v, v) = 1$. □

Now, we change gears. It turns out that knowing that every symplectomorphism is unimodular is exactly what we need to understand its spectrum.

Theorem 3 (The Symplectic Eigenvalue Theorem)

Let (V,Ω) be a symplectic vector space and $\varphi \in \mathrm{Sp}(V,\Omega)$. If $\lambda \in \mathbb{C}$ is an eigenvalue of φ, so will be λ^{-1}, $\overline{\lambda}$ and $\overline{\lambda}^{-1}$. Moreover, the algebraic multiplicities of λ and λ^{-1} are the same, and the multiplicities of 1 and -1 (if they occur at all) are even.

For a proof, see [1]. The key idea, though, is to show that if $p(\lambda)$ is the characteristic polynomial of φ, the relation $p(\lambda) = \lambda^{2n}p(\lambda^{-1})$ holds — and here enters the condition $\det\varphi = 1$ (try to prove it). And to conclude this section, we observe that $\mathrm{Sp}(V,\Omega)$ is (perhaps not surprisingly) a Lie group (with dimension $\dim \mathrm{Sp}(V,\Omega) = n(2n+1)$, if $2n = \dim V$, by Luneburg's relations), and its Lie algebra $\mathfrak{sp}(V,\Omega)$ consists of all linear operators with are Ω-skew (those are called **Hamiltonian operators**). There is a version of the above theorem for $\mathfrak{sp}(V,\Omega)$, as well as "infinitesimal Luneburg relations". Now, every Lie group has, first and foremost, a topology. And what can we say about the topology of $\mathrm{Sp}(V,\Omega)$? The simplest (functorial) topological invariant is the fundamental group. It turns out that $\pi_1\mathrm{Sp}(V,\Omega) \cong \mathbb{Z}$, but the surprising fact is that there is a very specific isomorphism, called the **Maslov index** — defined from loops in the Grassmannian of Lagrangian subspaces of (V,Ω) and the usual notion of degree between self-maps of the circle $\mathbb{S}^1$. See [28] or [35] for more details.

1.3 Local linear forms

The reason we're starting these lecture notes discussing linear algebra is not only for pedagogical reasons, but really because symplectic vector spaces are the linear prototypes of symplectic manifolds we'll study in the next chapters. More precisely, the tangent spaces to symplectic manifolds are symplectic vector spaces, so (essentially) all linear algebra tools can be smoothly applied pointwise on a manifold. And frequent questions are "what does a symplectic manifold look like locally" and "how does a given submanifold fit into a symplectic manifold?". Answering these questions in the linear setting is an easy task, and we'll do this now.

Theorem 4 (Linear Darboux)

Let (V,Ω) be a symplectic vector space with $2n = \dim V$. Then (V,Ω) is symplectomorphic to the canonical symplectic prototype $(\mathbb{R}^{2n},\Omega_{2n})$.

Proof: Take a Darboux basis $\mathcal{B} = (e_1,\ldots,e_n,f_1,\ldots,f_n)$ for V and define $\varphi\colon \mathbb{R}^{2n} \to V$ by

$$\varphi(\boldsymbol{x},\boldsymbol{y}) = \sum_{i=1}^{n}(x^i e_i + y^i f_i).$$

Clearly φ is a symplectomorphism. □

This is underwhelming, and we could have established this theorem in the last section (perhaps the reader even guessed the statement by then). One drawback should be pointed: the symplectomorphism is not natural (in a categorical sense), and depends on a choice of Darboux basis for V — which may be completely arbitrary. Despite this, it has somewhat deep consequences (e.g., the only invariant of a symplectic vector space is its dimension), which will become more evident when we encounter its non-linear version in the next chapter. We will see that every symplectic manifold locally looks like $(\mathbb{R}^{2n}, \Omega_{2n})$, not only at a given point, but at a *neighborhood* of each point. For example, the symplectomorphism $(V, \Omega) \cong (\mathbb{R}^{2n}, \Omega_{2n})$ should be thought of as a *global* Darboux coordinate system for V. More on this later. From the several "types" of subspaces (V, Ω) may have, Lagrangian subspaces seem to be the most important ones, and thus one could ask whether there is a local form for Lagrangian submanifolds of a symplectic manifold. We'll start to answer this question now, in the linear setting — the proof, as opposed to the previous one, will be slightly more interesting. The naturality problem pointed out for the linear Darboux theorem will not be an issue now due to this simple fact from linear algebra: if V is a vector space, $L \subseteq V$ is a subspace, and $L', L'' \subseteq V$ are two subspaces complementary to L, then the diagram

$$\begin{array}{ccc} & V/L & \\ {}^{\widetilde{\mathrm{pr}}_{L'}}\swarrow & & \searrow^{\widetilde{\mathrm{pr}}_{L''}} \\ L' & \xrightarrow[\mathrm{pr}_{L''}|_{L'}]{} & L'' \end{array}$$

commutes where all the pr stand for the corresponding projections given by direct sum decompositions, and the $\widetilde{\mathrm{pr}}$ stand for maps induced on V/L.

Theorem 5 (Linear Weinstein)

Let (V, Ω) be a symplectic vector space, and $L \subseteq V$ a Lagrangian subspace. Then there is a natural symplectomorphism between (V, Ω) and $(L \oplus L^*, \Omega_L)$ such that the diagram

$$\begin{array}{ccc} & & V \\ & \nearrow & \| \wr \\ L & & \| \cong \\ & \searrow & \| \\ & & L \oplus L^* \end{array}$$

commutes.

Proof: Choose a complementary Lagrangian subspace L' for L, and write $V = L \oplus L'$. We have the following chain of equalities and isomorphisms:

$$V = L \oplus L' \xrightarrow[\cong]{\mathrm{Id}_L \oplus \widetilde{\mathrm{pr}}_{L'}^{\,-1}} L \oplus \frac{V}{L} = L \oplus \frac{V}{L^\Omega} \xrightarrow[\cong]{\mathrm{Id}_L \oplus \Omega} L \oplus L^*$$

Above, $\widetilde{\mathrm{pr}_{L'}}\colon V/L \to L'$ is the map induced by the projection $\mathrm{pr}_{L'}\colon V \to L'$ given by the direct sum decomposition — which also yields $\mathrm{pr}_L\colon V \to L$. The given diagram obviously commutes, so we only have to check that the full composition φ is symplectic. Write $\varphi(x) = (\mathrm{pr}_L(x), f_x)$, where $f_x = \Omega(\cdot, x)|_L$ (be careful here: doing $\Omega(x,\cdot)|_L$ as usual will create a negative sign later). First, note that for all $y \in V$, we have that

$$\begin{aligned} f_x(\mathrm{pr}_L(y)) &= \Omega(\mathrm{pr}_L(y), x) = \Omega(y - \mathrm{pr}_{L'}(y), x) \\ &= \Omega(y,x) - \Omega(\mathrm{pr}_{L'}(y), x) \stackrel{(1)}{=} \Omega(y,x) - \Omega(\mathrm{pr}_{L'}(y), \mathrm{pr}_L(x)) \\ &\stackrel{(2)}{=} \Omega(y,x) - \Omega(y, \mathrm{pr}_L(x)) = \Omega(\mathrm{pr}_L(x) - x, y), \end{aligned}$$

where in (1) we use that L' is Lagrangian, and in (2) that L is Lagrangian. Hence $\Omega(\mathrm{pr}_L(x) - x, \cdot) = f_x \circ \mathrm{pr}_L$. With this in place, for all $x, y \in V$, we may compute

$$\begin{aligned} \Omega_L(\varphi(x), \varphi(y)) &= \Omega_L((\mathrm{pr}_L(x), f_x), (\mathrm{pr}_L(y), f_y)) \\ &= f_y(\mathrm{pr}_L(x)) - f_x(\mathrm{pr}_L(y)) \\ &= \Omega(\mathrm{pr}_L(y) - y, x) - \Omega(\mathrm{pr}_L(x) - x, y) \\ &= \Omega(\mathrm{pr}_L(y), x) - \Omega(y,x) - \Omega(\mathrm{pr}_L(x), y) + \Omega(x,y) \\ &= 2\Omega(x,y) - \Omega(x, \mathrm{pr}_L(y)) - \Omega(\mathrm{pr}_L(x), y) \\ &= \Omega(x,y), \end{aligned}$$

as wanted, since $\Omega(x,y) = \Omega(\mathrm{pr}_L(x), \mathrm{pr}_{L'}(y)) + \Omega(\mathrm{pr}_{L'}(x), \mathrm{pr}_L(y))$, as both L and L' are Lagrangian. $\square$

2 Symplectic Manifolds

2.1 Definitions and examples

The most important definition in this text requires some motivation. Let's think of the canonical symplectic prototype not as a vector space, but as a manifold. Each tangent space to $\mathbb{R}^{2n}$ is naturally isomorphic to $\mathbb{R}^{2n}$ itself, so let $\omega_{2n} \in \Omega^2(\mathbb{R}^{2n})$ assign to each point in $\mathbb{R}^{2n}$, the linear symplectic form Ω_{2n}. If we write the coordinates in $\mathbb{R}^{2n}$ as $(x^1, \ldots, x^n, y_1, \ldots, y_n)$, then we have that

$$\omega_{2n} = \sum_{k=1}^{n} \mathrm{d}x^k \wedge \mathrm{d}y_k = -\mathrm{d}\left(\sum_{k=1}^{k} y_k \, \mathrm{d}x^k\right).$$

Thus ω_{2n} is an exact form (the reason for writing it like this, with the negative sign, will become clear soon), and it is non-degenerate at each point. This would suggest defining a symplectic manifold as a smooth manifold equipped with a non-degenerate exact 2-form. It turns out that this is *not* a good definition. Here's the reason:

Proposition 2

Let M be a smooth manifold and $\omega \in \Omega^2(M)$ be exact and non-degenerate. Then M is not compact.

Proof: Assume by contradiction that M is compact. Since ω is non-degenerate, the dimension of M must be even, say $\dim M = 2n$. And non-degeneracy also gives that $\omega^{\wedge n}$ is a volume form on M. But if ω is exact, write $\omega = \mathrm{d}\eta$ for some $\eta \in \Omega^1(M)$. Now Stokes's theorem gives that

$$0 \neq \int_M \omega^{\wedge n} = \int_M \mathrm{d}\eta \wedge \omega^{\wedge(n-1)} = \int_M \mathrm{d}(\eta \wedge \omega^{\wedge(n-1)}) = 0,$$

as $\partial M = \varnothing$. □

This means that our first guess of what a symplectic manifold should be would exclude all compact manifolds from the theory. Thus, we must move on to the next best thing. If we cannot require ω to be exact, let's settle for locally exact. By Poincaré's lemma, this is equivalent to being closed.

Definition 7

A **symplectic manifold** is a pair (M, ω), where M is a smooth manifold and ω is a closed and non-degenerate 2-form on M. We'll say that (M, ω) is **exact** if ω is exact, and we'll call ω a **symplectic form**.

Remark. Non-degeneracy is an algebraic condition, while closedness is an analytic condition. It turns out that closedness is crucial for the development of the theory, and we will have several chances to appreciate this as we proceed.

Rephrasing Proposition 2 above, we already have our very first result[2] about symplectic manifolds:

Proposition 3

If (M, ω) is a compact symplectic manifold, then $H^2_{\mathrm{dR}}(M) \neq 0$.

This means that unlike what happens with Riemannian metrics, for example, it is not true that every manifold has a symplectic form, even if one assumes from the start that it already has even dimension. With a minor modification of what we just did above, we obtain a slightly stronger topological restriction.

Exercise 10

(a) Let (M^{2n}, ω) be a compact symplectic manifold. Show that $H^{2k}_{\mathrm{dR}}(M) \neq 0$ for all $1 \leq k \leq n$. **Hint:** Show that $[\omega^{\wedge k}] \neq 0$ in $H^{2k}_{\mathrm{dR}}(M)$ using Stokes' theorem.

(b) Show that $H^{2k}_{\mathrm{dR}}(\mathbb{S}^2 \times \mathbb{S}^4) = \mathbb{R}$ for $k = 0, 1, 2, 3$, but yet $\mathbb{S}^2 \times \mathbb{S}^4$ does not admit any symplectic form. **Hint:** To compute cohomologies, use the Künneth formula. Then assume by contradiction that $\mathbb{S}^2 \times \mathbb{S}^4$ has a symplectic form ω, and write $[\omega] = c[\pi^*\alpha]$, where $c \neq 0$, $[\alpha]$ generates $H^2_{\mathrm{dR}}(\mathbb{S}^2)$, and $\pi\colon \mathbb{S}^2 \times \mathbb{S}^4 \to \mathbb{S}^2$ is the projection. What happens next?

Time for some examples.

Example 9

As mentioned in the beginning of this section, if (V, Ω) is a symplectic vector space, we may use that $T_xV \cong V$ to let $\omega_x = \Omega$, for all $x \in V$, so that (V, ω) becomes a symplectic manifold.

Example 10

Let $M \subseteq \mathbb{R}^3$ be an orientable surface, with unit normal field $\boldsymbol{N}\colon M \to \mathbb{S}^2$. Define $\omega \in \Omega^2(M)$ by $\omega_x(\boldsymbol{v}, \boldsymbol{w}) = \langle \boldsymbol{N}(x), \boldsymbol{v} \times \boldsymbol{w} \rangle$, where $\langle \cdot, \cdot \rangle$ is the standard Euclidean product and $\times$ is the associated cross product. Clearly ω is skew-symmetric, and it is closed by dimension reasons. Here's a proof for non-degeneracy: fix $x \in M$, $\boldsymbol{v} \in T_xM$, and assume that $\omega_x(\boldsymbol{v}, \cdot) = 0$. So $\langle \boldsymbol{v}, \boldsymbol{w} \times \boldsymbol{N}(x) \rangle = 0$ for all $\boldsymbol{w} \in T_xM$, but the values $\boldsymbol{N}(x) \times \boldsymbol{w}$ fill up T_xM, which says that $\boldsymbol{v} \in (T_xM)^\perp$ — hence $\boldsymbol{v} = \boldsymbol{0}$. Note that this ω is the standard area form of M.

The next example is more of an useful construction.

[2] Second result, if you consider that: ω being non-degenerate gives that $\omega^{\wedge n}$ is a volume form, so that M must be orientable. Explicitly: every symplectic manifold is orientable.

Example 11

If (M_1, ω_1) and (M_2, ω_2) are symplectic manifolds, so is $(M_1 \times M_2, \omega_1 \oplus \omega_2)$. Here, $\omega_1 \oplus \omega_2$ is, of course, a shorthand for $\pi_1^*\omega_1 + \pi_2^*\omega_2$, where for $i = 1, 2$, $\pi_i \colon M_1 \times M_2 \to M_i$ is the projection. Since d commutes with pull-backs, we directly have $\mathrm{d}(\omega_1 \oplus \omega_2) = \mathrm{d}\omega_1 \oplus \mathrm{d}\omega_2 = 0$, and non-degeneracy follows from $\det(\omega_1 \oplus \omega_2) = \det \omega_1 \det \omega_2 \neq 0$, where this equality is to be understood locally and identifying each ω_i with its matrix relative to a coordinate system for M_i.

Exercise 11 (Hinting at future ideas)

Consider the prototype $(\mathbb{R}^{2n}, \omega_{2n})$. There are (at least?) two possible realizations of the torus $\mathbb{T}^{2n}$. The first, as $\mathbb{T}^{2n} = (\mathbb{S}^1 \times \mathbb{S}^1)^n$ — this inherits a product symplectic structure — and the second one as the quotient by a lattice $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ — this also has a symplectic structure, as ω_{2n} is translation-invariant, and thus survives in the quotient. What is the relation between the two structures?

The rest of this section is dedicated to one of the fundamental examples of a symplectic manifold — perhaps even more fundamental than $(\mathbb{R}^{2n}, \omega_{2n})$. Let Q be a smooth manifold (we're using Q instead of M because Q itself will not be the symplectic manifold we're after). Recall that its **cotangent bundle** T^*Q is the disjoint union of all the **cotangent spaces** T_x^*Q, as x ranges over Q, and we denote an element of T^*Q by a pair (x, p), where $p \in T_x^*Q$. Local coordinates $(q^1, \ldots, q^n)$ induce **cotangent coordinates** $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ for T^*Q, via

$$(x, p) \mapsto \left(q^1(x), \ldots, q^n(x), p\left(\left.\frac{\partial}{\partial q^1}\right|_x \right), \ldots, p\left(\left.\frac{\partial}{\partial q^n}\right|_x \right) \right).$$

In classical mechanics, Q is taken to be the configuration space of a mechanical system, while T^*Q is seen as the phase space of positions and momenta. The tangent bundle TQ, in turn, would be seen as the phase space of positions and velocities. In some sense (which can be made precise), momenta and velocities are the same thing up to isomorphism (once a Riemannian metric or a hyperregular Lagrangian function has been introduced on Q). But T^*Q has some natural structure that TQ does not, and we'll exploit that.

Definition 8

Let Q be a smooth manifold. The **tautological form** (also called the **Liouville form** of T^*Q is the 1-form $\lambda \in \Omega^1(T^*Q)$ defined by $\lambda_{(x,p)}(\boldsymbol{X}) \doteq p(\mathrm{d}\pi_{(x,p)}(\boldsymbol{X}))$, for all $\boldsymbol{X} \in T_{(x,p)}(T^*Q)$, where $\pi \colon T^*Q \to Q$ is the natural projection. The **canonical symplectic structure** of T^*Q is $\omega_{\mathrm{can}} \doteq -\mathrm{d}\lambda$.

Remark. Note that if $\pi \colon T^*Q \to Q$, then $\mathrm{d}\pi_{(x,p)} \colon T_{(x,p)}(T^*Q) \to T_xQ$, for all elements $(x, p) \in T^*Q$. Some texts define ω_{can} to be $\mathrm{d}\lambda$ without the negative sign. You should always pay attention to sign conventions throughout the literature.

Example 12

If Q is a smooth manifold, then $(T^*Q, \omega_{\text{can}})$ is a symplectic manifold. The only non-trivial thing to be checked is that ω_{can} is indeed non-degenerate. This can be done locally, noting that in terms of cotangent coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$, we have that

$$\lambda = \sum_{k=1}^n p_k \, \mathrm{d}q^k \quad \text{and} \quad \omega_{\text{can}} = \sum_{k=1}^n \mathrm{d}q^k \wedge \mathrm{d}p_k.$$

To wit, we have that

$$\lambda = \sum_{k=1}^n \lambda\left(\frac{\partial}{\partial q^k}\right) \mathrm{d}q^k + \sum_{k=1}^n \lambda\left(\frac{\partial}{\partial p_k}\right) \mathrm{d}p_k,$$

but

$$\lambda_{(x,p)}\left(\left.\frac{\partial}{\partial q^k}\right|_{(x,p)}\right) = p\left(\mathrm{d}\pi_{(x,p)}\left(\left.\frac{\partial}{\partial q^k}\right|_{(x,p)}\right)\right) = p\left(\left.\frac{\partial}{\partial q^k}\right|_x\right) = p_k(x,p)$$

and

$$\lambda_{(x,p)}\left(\left.\frac{\partial}{\partial p_k}\right|_{(x,p)}\right) = p\left(\mathrm{d}\pi_{(x,p)}\left(\left.\frac{\partial}{\partial p_k}\right|_{(x,p)}\right)\right) = p(\mathbf{0}) = 0.$$

Exercise 12

Let Q be a smooth manifold. Here's another justificative for the name "tautological", given to $\lambda \in \Omega^1(T^*Q)$, in two steps:

(a) Show that for every $\sigma \in \Omega^1(Q)$, we have that $\sigma^*\lambda = \sigma$.

(b) Conversely, show that if $\alpha \in \Omega^1(T^*Q)$ has the property that $\sigma^*\alpha = \sigma$ for all $\sigma \in \Omega^1(Q)$, then $\alpha = \lambda$.

Hint: Here, you should think of $\sigma \in \Omega^1(Q)$ as a map $\sigma\colon Q \to T^*Q$ (which just happens to be a section of $\pi\colon T^*Q \to Q$), so pull-backs make sense.

Here's one last example for this section:

Example 13 (Magnetic symplectic forms)

Let Q be a smooth manifold, and let $B \in \Omega^2(Q)$ be a closed. Define $\omega_B \in \Omega^2(T^*Q)$ by $\omega_B = \omega_{\text{can}} + \pi^*B$, where $\pi\colon T^*Q \to Q$ is the natural projection. Note that $\mathrm{d}(\omega_B) = \pi^*(\mathrm{d}B) = 0$ since d commutes with pull-backs and, conversely, that ω_B is not closed if B is not (because since π is a surjective submersion, π^* is injective). Moreover, taking cotangent coordinates and identifying B, ω, and ω_B with their matrices relative to those coordinates, we have that

$$\omega_B = \begin{pmatrix} B & \mathrm{Id}_n \\ -\mathrm{Id}_n & \mathbf{0} \end{pmatrix},$$

which has full-rank (no matter what B is), so ω_B is non-degenerate. We call ω_B the **magnetic symplectic form** associated to B. The reason for this name is that the equations which dictate the motion of a particle on Q subject to the action of a magnetic field B are the equations describing the flow of a certain vector field associated to a Hamiltonian function (i.e., the "total energy" classical observable) via ω_B.

Some general advice: you should keep all of those examples in mind as we proceed, as they're so useful to gain intuition for new definitions and theorems.

2.2 Symplectomorphisms (*redux*)

What is the correct notion of equivalence for symplectic manifolds? Equivalently, what are the symplectic analogues of isometries? Or yet, what are the isomorphisms in the category of symplectic manifolds?

Definition 9

Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds. A **symplectomorphism** is a diffeomorphism $\varphi\colon M_1 \to M_2$ such that $\varphi^*\omega_2 = \omega_1$ or, more explictly, such that

$$(\omega_1)_x(\boldsymbol{v}, \boldsymbol{w}) = (\omega_2)_{\varphi(x)}(\mathrm{d}\varphi_x(\boldsymbol{v}), \mathrm{d}\varphi_x(\boldsymbol{w})),$$

for all $x \in M_1$, $\boldsymbol{v}, \boldsymbol{w} \in T_x M_1$. The **symplectomorphism group** of a symplectic manifold (M, ω), denoted by $\mathrm{Sp}(M, \omega)$, is the collection of all symplectomorphisms from M to itself.

As before, $\mathrm{Sp}(M, \omega)$ is a group when equipped with the composition of functions. Of course that symplectomorphisms between symplectic vector spaces are symplectomorphisms in the above sense (the total derivative of a linear map is itself). But in general, symplectomorphisms may be highly non-linear.

Exercise 13

Consider in $\mathbb{R}^4$, with coordinates (x, y, u, v), the standard symplectic form

$$\omega = \mathrm{d}x \wedge \mathrm{d}u + \mathrm{d}y \wedge \mathrm{d}v.$$

Show that $\varphi\colon \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$\varphi(x, y, u, v) = (x, y, u + \mathrm{e}^x(\cos y + \sin y), v + \mathrm{e}^x(\cos y - \sin y))$$

is a a symplectomorphism. Note that identifying $\mathbb{R}^4$ with $\mathbb{C}^2$, we have that this map is just $\varphi(z,w) = (z, w + (1-\mathrm{i})\mathrm{e}^{iz})$. The complex Jacobian has determinant 1, so maybe it shouldn't surprise you that φ is a symplectomorphism.

Here's a friendlier example:

Example 14

Consider $(\mathbb{S}^2, \omega)$, where $\omega \in \Omega^2(\mathbb{S}^2)$ is the standard area form, given (as before) by $\omega_x(\boldsymbol{v}, \boldsymbol{w}) = \langle x, \boldsymbol{v} \times \boldsymbol{w} \rangle$. Restrict a linear isometry $A \in \mathrm{SO}(3)$ to the sphere, $A\colon \mathbb{S}^2 \to \mathbb{S}^2$. Then $A \in \mathrm{Sp}(\mathbb{S}^2, \omega)$. Indeed, we have that

$$\langle Ax, A\boldsymbol{v} \times A\boldsymbol{w} \rangle = \langle Ax, A(\boldsymbol{v} \times \boldsymbol{w}) \rangle = \langle x, \boldsymbol{v} \times \boldsymbol{w} \rangle,$$

since $\det A = 1$ and A preserves $\langle \cdot, \cdot \rangle$. This shows that $\mathrm{SO}(3) \subseteq \mathrm{Sp}(\mathbb{S}^2, \omega)$. We will see later (after Theorem 8, p. 29) that the inclusion is strict.

However, the symplectic manifold really rich in "symmetries" are cotangent bundles.

Example 15 (Fiberwise translations)

Let Q be a smooth manifold, and consider $(T^*Q, \omega_{\mathrm{can}})$. Given $\sigma \in \Omega^1(Q)$, define $\tau_\sigma\colon T^*Q \to T^*Q$ by $\tau_\sigma(x,p) = (x, p + \sigma_x)$. When is τ_σ a symplectomorphism? A general rule of thumb is that when dealing with cotangent bundles, always look at what happens with the tautological form $\lambda \in \Omega^1(T^*Q)$ first — if λ is preserved, so will be ω_{can}. Note that if $\pi\colon T^*Q \to Q$ is the standard projection, then $\pi \circ \tau_\sigma = \pi$. Now take $(x,p) \in T^*Q$ and compute

$$\begin{aligned}
(\tau_\sigma^*\lambda)_{(x,p)} &= \lambda_{(x,p+\sigma_x)} \circ \mathrm{d}(\tau_\sigma)_{(x,p)} \\
&= (p + \sigma_x) \circ \mathrm{d}\pi_{(x,p+\sigma_x)} \circ \mathrm{d}(\tau_\sigma)_{(x,p)} \\
&= (p + \sigma_x) \circ \mathrm{d}(\pi \circ \tau_\sigma)_{(x,p)} \\
&= (p + \sigma_x) \circ \mathrm{d}\pi_{(x,p)} \\
&= p \circ \mathrm{d}\pi_{(x,p)} + \sigma_x \circ \mathrm{d}\pi_{(x,p)} \\
&= \lambda_{(x,p)} + (\pi^*\sigma)_{(x,p)}.
\end{aligned}$$

So $\tau_\sigma^*\lambda = \lambda + \pi^*\sigma$ leads to $\tau_\sigma^*\omega_{\mathrm{can}} = \omega_{\mathrm{can}} - \pi^*(\mathrm{d}\sigma)$, and we conclude that τ_σ is a symplectomorphism for $(T^*Q, \omega_{\mathrm{can}})$ if and only if σ is closed. Note that τ_σ is indeed a diffeomorphism because $(\tau_\sigma)^{-1} = \tau_{-\sigma}$. If we choose a closed form $B \in \Omega^2(Q)$ to form the associated magnetic symplectic form ω_B, the same computation shows that τ_σ with σ closed is a symplectomorphism for (T^*Q, ω_B) as well.

You can explore this situation a bit further in the next exercise.

Exercise 14

Let Q be a smooth manifold and take two closed forms $B, B' \in \Omega^2(Q)$. Show that if $[B] = [B']$ in $H^2_{\mathrm{dR}}(Q)$, then (T^*Q, ω_B) and $(T^*Q, \omega_{B'})$ are symplectomorphic under a fiberwise translation. In other words, a magnectic symplectic structure depends only on the cohomology class of the chosen magnetic form.

It turns out that cotangent bundles have another natural family of (this time non-linear) symplectomorphisms. It relies on the following concept:

Definition 10

Let Q_1 and Q_2 be two smooth manifold, and $f\colon Q_1 \to Q_2$ be a diffeomorphism. The **cotangent lift** of f is the map $\widehat{f}\colon T^*Q_1 \to T^*Q_2$ defined by

$$\widehat{f}(x,p) \doteq (f(x), p \circ (\mathrm{d}f_x)^{-1}).$$

At a first sight, the definition of a cotangent lift might seem a bit awkward, due to the presence of the *inverse* of the derivative of f. The reason for that is not only to make the compositions work. The next exercise should give you some more context.

Exercise 15 (The cotangent T^* functor)

Let $\mathsf{Core}(\mathsf{Man})$ be the category whose objects are smooth manifolds, and morphisms are diffeomorphisms[a]. Define $T^*\colon \mathsf{Core}(\mathsf{Man}) \to \mathsf{Core}(\mathsf{Man})$ as follows:

- on objects: $Q \mapsto T^*Q$;
- on morphisms: $(h\colon Q_1 \to Q_2) \mapsto (T^*h\colon T^*Q_2 \to T^*Q_1)$, where T^*h is defined by $(T^*h)(y,p) \doteq (h^{-1}(y), (\mathrm{d}h_{h^{-1}(y)})^*(p))$.

Show that T^* is a contravariant (faithful) functor. With the notation of the previous definition, we have that $\widehat{f} = T^*(f^{-1})$, to make $\widehat{\ }$ a covariant functor.

[a]The *core* of a category C is the category (in fact, groupoid) $\mathsf{Core}(\mathsf{C})$ whose objects are the same as the objects in C, but the morphisms in $\mathsf{Core}(\mathsf{C})$ are only the morphisms in C which are isomorphisms. One could (should?) expect $\mathsf{Core}\colon \mathsf{Cat} \to \mathsf{Grpd}$ to be a functor too, where Cat and Grpd are exactly what you think. Is it, though? If yes, how? If not, why?

Proposition 4

Let Q_1 and Q_2 be two smooth manifolds, with tautological forms λ_1 and λ_2 on their cotangent bundles. If $f\colon Q_1 \to Q_2$ is a diffeomorphism, then $(\widehat{f})^*\lambda_2 = \lambda_1$, and thus $\widehat{f}\colon T^*Q_1 \to T^*Q_2$ is a symplectomorphism.

Proof: Cotangent lifts are natural, in the sense that the diagram

$$\begin{array}{ccc} T^*Q_1 & \xrightarrow{\widehat{f}} & T^*Q_2 \\ \Big\downarrow \pi & & \Big\downarrow \pi \\ Q_1 & \xrightarrow{f} & Q_2 \end{array}$$

commutes. Now, take $(x,p) \in T^*Q_1$ and repeatedly use the chain rule to compute:

$$\begin{aligned} ((\widehat{f})^*\lambda_2)_{(x,p)} &= (\lambda_2)_{\widehat{f}(x,p)} \circ \mathrm{d}\widehat{f}_{(x,p)} \\ &= (p \circ (\mathrm{d}f_x)^{-1}) \circ \mathrm{d}\pi_{\widehat{f}(x,p)} \circ \mathrm{d}\widehat{f}_{(x,p)} \\ &= p \circ (\mathrm{d}f_x)^{-1} \circ \mathrm{d}(\pi \circ \widehat{f})_{(x,p)} \\ &= p \circ (\mathrm{d}f_x)^{-1} \circ \mathrm{d}(f \circ \pi)_{(x,p)} \\ &= p \circ (\mathrm{d}f_x)^{-1} \circ \mathrm{d}f_x \circ \mathrm{d}\pi_{(x,p)} \\ &= p \circ \mathrm{d}\pi_{(x,p)} \\ &= (\lambda_1)_{(x,p)}. \end{aligned}$$

□

Corollary 2

Let Q_1 and Q_2 be smooth manifolds equipped with closed 2-forms $B_1 \in \Omega^2(Q_1)$ and $B_2 \in \Omega^2(Q_2)$, and consider the magnetic cotangent bundles (T^*Q_1, ω_{B_1}) and (T^*Q_2, ω_{B_2}). If $f\colon Q_1 \to Q_2$ is a diffeomorphism, then $\widehat{f}\colon T^*Q_1 \to T^*Q_2$ is a symplectomorphism if and only if $f^*B_2 = B_1$.

Now, cotangent lifts are a very special type of symplectomorphism, because not only they preserve the canonical symplectic form, they also preserve the tautological form, which is a much stronger condition to require. It turns out that this condition is characteristic of cotangent lifts, in the sense of the following:

Theorem 6

Let Q_1 and Q_2 be smooth manifolds, with tautological forms λ_1 and λ_2 on their cotangent bundles. If $\varphi\colon T^*Q_1 \to T^*Q_2$ is a diffeomorphism such that $\varphi^*\lambda_2 = \lambda_1$, then there is a diffeomorphism $f\colon Q_1 \to Q_2$ such that $\varphi = \widehat{f}$.

The proof essentially consists on showing that the condition $\varphi^*\lambda_2 = \lambda_1$ implies that φ takes fibers to fibers, that is, if $x \in Q_1$, then the image $\varphi[T_x^*Q_1]$ actually equals $T_y^*Q_2$ for some (unique) $y \in Q_2$ — one then sets $f(x) \doteq y$ and checks directly that $\widehat{f} = \varphi$, as desired. See [14], [15], or even [43] for more details.

2.3 Hamiltonian fields

On Riemannian manifolds, smooth functions have gradients. Namely, gradients are vector fields equivalent to derivatives under the metric. The key fact here is not that Riemannian metrics are positive-definite, but instead that they are non-degenerate. Well, symplectic forms are non-degenerate. So it should not be surprising that we will also have symplectic analogues of gradient fields.

Definition 11

Let (M, ω) be a symplectic manifold, and $H: M \to \mathbb{R}$ be a smooth function. The **Hamiltonian field** (or **symplectic gradient**) of H is the vector field $\boldsymbol{X}_H \in \mathfrak{X}(M)$ characterized by the relation $\omega(\boldsymbol{X}_H, \cdot) = \mathrm{d}H$.

Remark. Again there's an ambiguity regarding sign conventions here: some texts replace $\mathrm{d}H$ with $-\mathrm{d}H$, or $\omega(\boldsymbol{X}_H, \cdot)$ with $\omega(\cdot, \boldsymbol{X}_H)$ in the above definition. Our convention should agree with most of the Physics literature.

As usual, let's start with examples.

Example 16

Let $M \subseteq \mathbb{R}^3$ be an orientable surface with unit normal field $\boldsymbol{N}: M \to \mathbb{S}^2$ and area form $\omega \in \Omega^2(M)$. Let $H: M \to \mathbb{R}$ be smooth. The standard Euclidean inner product of $\mathbb{R}^3$ restricts to a Riemannian metric on M, and so we may consider the gradient field $\operatorname{grad} H \in \mathfrak{X}(M)$. What is the relation between $\operatorname{grad} H$ and $\boldsymbol{X}_H$? Using the definitions and invariance under cyclic permutations of the triple product, we see that for each $x \in M$ and $\boldsymbol{v} \in T_xM$, we have

$$\langle (\operatorname{grad} H)_x, \boldsymbol{v} \rangle = \mathrm{d}H_x(\boldsymbol{v}) = \omega_x(\boldsymbol{X}_H|_x, \boldsymbol{v}) = \langle \boldsymbol{N}(x), \boldsymbol{X}_H\big|_x \times \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{N}(x) \times \boldsymbol{X}_H\big|_x \rangle,$$

so that $\operatorname{grad} H = \boldsymbol{N} \times \boldsymbol{X}_H$. With this in place, we use double cross product formulas to obtain

$$\boldsymbol{N} \times \operatorname{grad} H = \boldsymbol{N} \times (\boldsymbol{N} \times \boldsymbol{X}_H) = \langle \boldsymbol{N}, \boldsymbol{X}_H \rangle \boldsymbol{N} - \langle \boldsymbol{N}, \boldsymbol{N} \rangle \boldsymbol{X}_H = -\boldsymbol{X}_H,$$

and so $\boldsymbol{X}_H = \operatorname{grad} H \times \boldsymbol{N}$.

Example 17

Let (V, ω) be a symplectic vector space, and let $H \in V^*$. Here are two ways to see $\boldsymbol{X}_H$, since $\mathrm{d}H_x = H$ for all $x \in V$:

- Identify $V \cong V^*$ using ω. So $\boldsymbol{X}_H = H$, simply because $H = \omega(\boldsymbol{X}_H, \cdot)$.
- Identify $V \cong V^*$ using a Darboux basis $\mathscr{B} = (e_1, \ldots, e_n, f_1, \ldots, f_n)$ (that is, send $\mathscr{B}$ to its dual basis and extend linearly). Also consider the linear complex structure J determined by $\mathscr{B}$, that is, defined by $J(e_i) = f_i$ and

$J(f_i) = -e_i$. Writing H and X_H as linear combinations of elements in $\mathscr{B}$ and evaluating both sides of $H = \omega(X_H, \cdot)$ at e_j and f_j, it will follow that $X_H = -JH$.

The next example has such a fundamental importance in mechanics, that we'll phrase it as a theorem.

Theorem 7 (Hamilton's equations)

Let Q be a smooth manifold, consider its cotangent bundle $(T^*Q, \omega_{\mathrm{can}})$, and let $H\colon T^*Q \to \mathbb{R}$ be a smooth function. In T^*Q, take local cotangent coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$. Then:

(i) the Hamiltonian field X_H of H is given by

$$X_H = \sum_{k=1}^{n} \left(\frac{\partial H}{\partial p_k} \frac{\partial}{\partial q^k} - \frac{\partial H}{\partial q^k} \frac{\partial}{\partial p_k} \right)$$

(ii) if $\gamma\colon I \to T^*Q$ is an integral curve of the field X_H, described in coordinates by $\gamma(t) = (q^1(t), \ldots, q^n(t), p_1(t), \ldots, p_n(t))$, it satisfies **Hamilton's equations**:

$$\begin{cases} \dfrac{\mathrm{d}q^k}{\mathrm{d}t}(t) = \dfrac{\partial H}{\partial p_k}(\gamma(t)) \\ \dfrac{\mathrm{d}p_k}{\mathrm{d}t}(t) = -\dfrac{\partial H}{\partial q^k}(\gamma(t)). \end{cases}$$

Proof: Item (b) follows trivially from (a). So let's check (a). Write

$$X_H = \sum_{k=1}^{n} \left(a^k \frac{\partial}{\partial q^k} + b_k \frac{\partial}{\partial p_k} \right),$$

where the a^k and b^k are to be determined. We have that

$$\frac{\partial H}{\partial q^k} = \mathrm{d}H\left(\frac{\partial}{\partial q^k}\right) = \omega_{\mathrm{can}}\left(X_H, \frac{\partial}{\partial q^k}\right) = -b_k$$

and

$$\frac{\partial H}{\partial p_k} = \mathrm{d}H\left(\frac{\partial}{\partial p_k}\right) = \omega_{\mathrm{can}}\left(X_H, \frac{\partial}{\partial p_k}\right) = a^k,$$

since for all choices of i and j we have that

$$\omega_{\mathrm{can}}\left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}\right) = 0, \quad \omega_{\mathrm{can}}\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}\right) = 0 \quad \text{and} \quad \omega_{\mathrm{can}}\left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial p_j}\right) = \delta_i^j.$$

□

Remark. We'll see in the next section that every symplectic manifold admits coordinates for which the symplectic form looks like the canonical symplectic prototype. This means that any local computations we make, for instance, in a cotangent bundle, will be locally true for any symplectic manifold.

Exercise 16

Let Q be a smooth manifold, $B \in \Omega^2(Q)$ a closed form, and consider the magnetic cotangent bundle (T^*Q, ω_B). Let $H\colon T^*Q \to \mathbb{R}$ be a smooth function and denote by $\boldsymbol{X}_H^B$ its Hamiltonian field computed with ω_B. Show that relative to cotangent coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$, we have that:

(a) the expression for $\boldsymbol{X}_H^B$ is

$$\boldsymbol{X}_H^B = \sum_{k=1}^n \left(\frac{\partial H}{\partial p_k} \frac{\partial}{\partial q^k} + \left(-\frac{\partial H}{\partial q^k} + \sum_{j=1}^n B_{jk} \frac{\partial H}{\partial p_j} \right) \frac{\partial}{\partial p_k} \right),$$

where the functions B_{jk} are the components of B relative to the coordinate system $(q^1, \ldots, q^n)$ on Q.

(b) if $\gamma\colon I \to T^*Q$ is an integral curve of the field $\boldsymbol{X}_H^B$, described in coordinates by $\gamma(t) = (q^1(t), \ldots, q^n(t), p_1(t), \ldots, p_n(t))$, it satisfies the magnetic version of Hamilton's equations:

$$\begin{cases} \dfrac{\mathrm{d}q^k}{\mathrm{d}t}(t) = \dfrac{\partial H}{\partial p_k}(\gamma(t)) \\ \dfrac{\mathrm{d}p_k}{\mathrm{d}t}(t) = -\dfrac{\partial H}{\partial q^k}(\gamma(t)) + \displaystyle\sum_{j=1}^n B_{jk}(\gamma(t)) \dfrac{\partial H}{\partial p_j}(\gamma(t)). \end{cases}$$

(c) if $\mathbb{R} \times T^*Q$ is the *extended phase space* equipped with the (non-symplectic, for dimensional reasons) 2-form $\eta = \omega_{\mathrm{can}} + \mathrm{d}H \wedge \mathrm{d}t$ (we identify ω_{can} and $\mathrm{d}H$ with their pull-backs under $\mathbb{R} \times T^*Q \to T^*Q$), then $\boldsymbol{X}_H^B$ is the unique field on T^*Q such that $\eta(\boldsymbol{X}_H^B + \partial_t, \cdot) = 0$.

Hint: Item (c) is independent from (a) and (b) and, again, (b) follows trivially from (a) (why?).

Exercise 17 (Variational characterizations of Hamilton's equations)

Let Q be a smooth manifold and $H\colon T^*Q \to \mathbb{R}$ be a smooth function. Define the **action integral of** H by sending every curve $\gamma = (x, p)\colon [a, b] \to T^*Q$ to the number

$$\mathcal{A}^H[\gamma] = \int_a^b p(t)(\dot{x}(t)) - H(x(t), p(t))\,\mathrm{d}t.$$

Show that critical points of $\mathcal{A}^H$ appear, in cotangent coordinates, as solutions to Hamilton's equations. **Hint:** see [38] if needed.

Hamilton's equations have far-reaching consequences in Physics and Geometry. We'll illustrate one situation from each area in what follows. Having some more vocabulary before proceeding will be helpful.

Definition 12

An **autonomous Hamiltonian system** is a triple (M, ω, H), where (M, ω) is a symplectic manifold, and $H\colon M \to \mathbb{R}$ is smooth.

- The function H is called the **Hamiltonian** of the system;
- The flow of the vector field $\boldsymbol{X}_H$ is called the **Hamiltonian flow** of the system;
- For each $e \in \mathbb{R}$, the inverse image $\Sigma_e \doteq H^{-1}(e)$ is called an **energy level** (or **energy surface**) of the system.
- A smooth function $f\colon M \to \mathbb{R}$ is called a **constant of motion** if it is constant along the integral curves of $\boldsymbol{X}_H$.

Example 18 (The Harmonic Oscillator)

Consider a unidimensional simple **harmonic oscillator**:

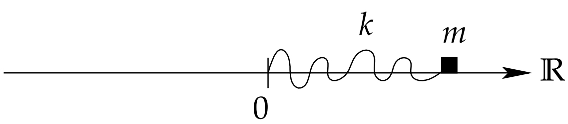


Figure 1: Simple Harmonic Oscillator.

Suppose that the mass of the particle attached to the string is $m > 0$, that the elasticity constant of the string is $k > 0$, and that the particle moves without friction. The configuration space here is $Q = \mathbb{R}$, and the phase space of positions and momenta is $T^*\mathbb{R}$ — so we'll use the position x and the momentum p of the particle as coordinates (in particular, the symplectic form to be used will be simply $\mathrm{d}x \wedge \mathrm{d}p$). By Hooke's Law, the force needed to extend or compress the spring by some distance x is $F(x) = -kx$. Moreover, since the total energy is the sum of the kinetic and potential energies, and velocity and momentum are linked through $p = mv$, the Hamiltonian function controlling the time evolution of this system is $H\colon T^*\mathbb{R} \cong \mathbb{R}^2 \to \mathbb{R}$ given by

$$H(x, p) = \frac{p^2}{2m} + \frac{kx^2}{2}.$$

Thus we have a Hamiltonian system $(T^*\mathbb{R}, \mathrm{d}x \wedge \mathrm{d}p, H)$. In matrix form, Hamilton's equations become just

$$\begin{pmatrix} \dot{x}(t) \\ \dot{p}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1/m \\ -k & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ p(t) \end{pmatrix}$$

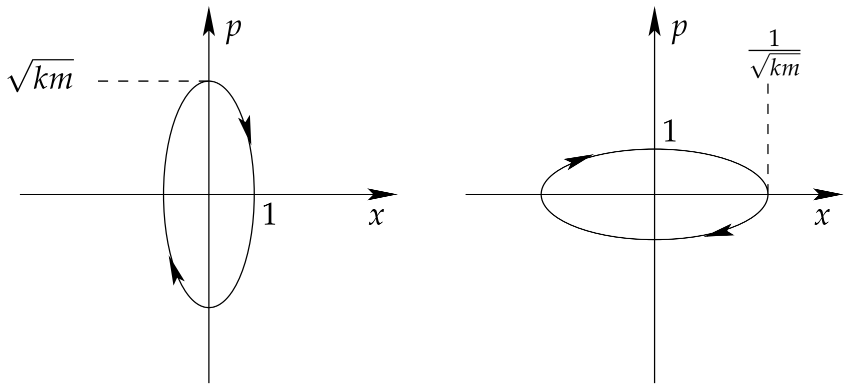
Let's denote by A the constant matrix of coefficients above. A straightforward computation (for instance, using that $A^2 = (-k/m)\mathrm{Id}_2$ and a series expansion) gives that

$$\exp(tA) = \cos\left(t\sqrt{\frac{k}{m}}\right)\mathrm{Id}_2 + \sin\left(t\sqrt{\frac{k}{m}}\right)\sqrt{\frac{m}{k}}A.$$

This motivates us to define the **angular frequency** of the system by $\omega \doteq \sqrt{k/m}$. Thus, it follows that the solution (x, p) to Hamilton's equations with initial condition (x_0, p_0) is given by

$$x(t) = x_0\cos(\omega t) + \frac{p_0}{\sqrt{km}}\sin(\omega t) \quad \text{and} \quad p(t) = -x_0\sqrt{km}\sin(\omega t) + p_0\cos(\omega t).$$

Observe that if we set $H_0 \doteq H(x_0, y_0)$, then $H(x(t), y(t)) = H_0$ for all $t \in \mathbb{R}$. One can see that $H(x, p) = H_0$ is the equation of an ellipse centered at the origin, in the xp-plane, which is then parametrized by $t \mapsto (x(t), p(t))$. Hence, we may draw the phase portrait for the harmonic oscillator:



(a) Orbit of $(1, 0)$. (b) Orbit of $(0, 1)$.

Figure 2: Periodic orbits with time interval $0 \leq t \leq 2\pi/\omega$.

This phenomenon of conservation of energy is something very general: for every Hamiltonian system, the Hamiltonian itself is a constant of motion — simply because skew-symmetry of the symplectic form implies that $\omega(\boldsymbol{X}_H, \boldsymbol{X}_H) = 0$.

Exercise 18 ("Chain" conservation of energy)

Let's expand on the last comment made on the above example. Let (M, ω, H) be a Hamiltonian system. Show that any function of the Hamiltonian itself is also a constant of motion. Namely, if $\varphi\colon \mathbb{R} \to \mathbb{R}$ is any smooth function, show that $\boldsymbol{X}_{\varphi\circ H} = (\varphi' \circ H)\boldsymbol{X}_H$ and conclude that $\varphi \circ H$ is a constant of motion.

Example 19 (Cogeodesic flow)

Let (Q, g) be a Riemannian manifold. The isomorphism $TQ \cong T^*Q$ induced by g allows us to define a fiber metric on T^*Q, which we'll also denote by g (note that this is not the same as saying that we have a Riemannian metric on the manifold T^*Q). Consider the Hamiltonian $H\colon T^*Q \to \mathbb{R}$ given by

$$H(x,p) \doteq \frac{1}{2}\mathrm{g}_x(p,p).$$

Using the canonical symplectic form ω_{can} on T^*Q, we may consider the associated Hamiltonian field $\boldsymbol{X}_H \in \mathfrak{X}(T^*Q)$. The corresponding Hamiltonian flow is called the **cogeodesic flow** on T^*Q, because its integral curves project to geodesics on (Q, g). Let's understand how this happens, checking that Hamilton's equations are equivalent to the geodesic equations. So, consider tangent and cotangent coordinates for TQ and T^*Q, related under g. Namely, we have

$$(q^1, \ldots, q^n, v^1, \ldots, v^n, p_1, \ldots, p_n),$$

where

$$p_j = \sum_{i=1}^n g_{ij}v^i \quad \text{and} \quad v^j = \sum_{i=1}^n g^{ij}p_i.$$

Here, $(g_{ij})_{i,j=1}^n$ is the matrix of g relative to the chosen coordinate system for Q, and $(g^{ij})_{i,j=1}^n$ is its inverse matrix — such inverse matrix, incidentally, represents the fiber metric induced on T^*Q, so that

$$H = \frac{1}{2}\sum_{i,j=1}^n g^{ij}p_ip_j.$$

The first of Hamilton's equations becomes just $\dot{q}^k = v^k$, as expected. The second equation, in the presence of the first one, will become the geodesic equation. Here's how this happens. First, we compute the derivative

$$\frac{\partial H}{\partial q^k} = \frac{1}{2}\sum_{i,j=1}^n \frac{\partial g^{ij}}{\partial q^k}p_ip_j = -\frac{1}{2}\sum_{i,j,r,s=1}^n g^{ir}\frac{\partial g_{rs}}{\partial q^k}g^{sj}p_ip_j = -\frac{1}{2}\sum_{r,s=1}^n \frac{\partial g_{rs}}{\partial q^k}v^rv^s,$$

and rename back $r, s \to i, j$. Now,

$$\left(\sum_{i=1}^n g_{ik}v^i\right)^{\cdot} = \frac{1}{2}\sum_{i,j=1}^n \frac{\partial g_{ij}}{\partial q^k}v^iv^j$$

becomes

$$\sum_{i=1}^n g_{ik}\dot{v}^i + \sum_{i,j=1}^n \frac{\partial g_{ik}}{\partial q^j}v^i\dot{q}^j = \frac{1}{2}\sum_{i,j=1}^n \frac{\partial g_{ij}}{\partial q^k}v^iv^j.$$

Splitting the middle term as a sum (since it's symmetric in i and j) and writing everything in terms of derivatives of the q^i's yields

$$\sum_{i=1}^n g_{ik}\ddot{q}^i + \frac{1}{2}\sum_{i,j=1}^n \left(\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k}\right)\dot{q}^i\dot{q}^j = 0.$$

Multiply everything by g^{kr}, sum over r, and rename $r \to k$ to get

$$\ddot{q}^k + \sum_{i,j=1}^n \Gamma^k_{ij}\dot{q}^i\dot{q}^j = 0,$$

where the

$$\Gamma^k_{ij} \doteq \frac{1}{2}\sum_{r=1}^n g^{kr}\left(\frac{\partial g_{ir}}{\partial q^j} + \frac{\partial g_{jr}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^r}\right)$$

are the **Christoffel symbols** describing the **Levi-Civita connection** of (Q, g) in coordinates. Thus we have obtained the coordinate description of the geodesic equation

$$\frac{\mathrm{D}\dot{\gamma}}{\mathrm{d}t}(t) = \mathbf{0},$$

as claimed.

Remark. Using the isomorphism $TQ \cong T^*Q$ given by g to convert $\boldsymbol{X}_H$ into a vector field on TQ, one obtains the standard **geodesic field** $\boldsymbol{G} \in \mathfrak{X}(TQ)$, characterized by

$$\boldsymbol{G}_{(x,\boldsymbol{v})} = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\gamma_{x,\boldsymbol{v}}(t), \dot{\gamma}_{x,\boldsymbol{v}}(t)),$$

where $\gamma_{x,\boldsymbol{v}}$ is the unique maximal geodesic on (Q, g) with $\gamma_{x,\boldsymbol{v}}(0) = x$ and $\dot{\gamma}_{x,\boldsymbol{v}}(0) = \boldsymbol{v}$.

Exercise 19 (Challenge #2)

Let (Q, g) be a Riemannian manifold and $B \in \Omega^2(Q)$ be closed. Consider the (skew-adjoint) bundle morphism $\boldsymbol{F}\colon TQ \to TQ$, called the **Lorentz force** associated with B, characterized by the relation $\mathrm{g}_x(\boldsymbol{F}_x(\boldsymbol{v}), \boldsymbol{w}) = B_x(\boldsymbol{v}, \boldsymbol{w})$, for all $x \in Q$ and $\boldsymbol{v}, \boldsymbol{w} \in T_xQ$. Mimic what was done in the previous example and show that integral curves of the Hamiltonian field $\boldsymbol{X}^B_H \in \mathfrak{X}(T^*Q)$ (computed with ω_B) of $H\colon T^*Q \to \mathbb{R}$ given by

$$H(x, p) = \frac{1}{2}\mathrm{g}_x(p, p)$$

project onto solutions of the **magnetic geodesic equation**

$$\frac{\mathrm{D}\dot{\gamma}}{\mathrm{d}t}(t) = \boldsymbol{F}_{\gamma(t)}(\dot{\gamma}(t))$$

on Q. **Hint:** You already did some of the hard work on Exercise 16 (p. 24).

Remark. Recall that the geodesics of (Q, g) are also charaterized as critical points of the **energy functional** $\mathscr{E}$, defined as

$$\mathscr{E}[\gamma] = \frac{1}{2}\int_a^b \mathrm{g}_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))\,\mathrm{d}t,$$

where $\gamma\colon [a,b] \to M$ is smooth (actually, one can get away with assuming that γ is just absolutely continuous — smoothness of minimizing geodesics then follows). If we introduce a magnetic form $B \in \Omega^2(Q)$, and B is exact, that is, $B = \mathrm{d}A$ for some $A \in \Omega^1(Q)$ (called **magnetic potential**), then it turns out that magnetic geodesics appear as critical points of the **magnetic energy functional** $\mathscr{E}_A$, defined as

$$\mathscr{E}_A[\gamma] = \frac{1}{2}\int_a^b \mathrm{g}_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))\,\mathrm{d}t + \int_a^b A_{\gamma(t)}(\dot{\gamma}(t))\,\mathrm{d}t.$$

Note that the choice of magnetic potential A is not unique, as one could replace A with $A + \mathrm{d}f$, for any smooth $f\colon Q \to \mathbb{R}$. Despite this gauge freedom, there is still some physical interpretation for A, justifying its name. Related to this, there's the so-called **Aharonov-Bohm experiment**. You can read more about it, from a mathematical perspective, in [41]. For a crash review on variational calculus on manifolds, in case you want to try and prove the claims made here, see [38].

We continue to explore the importance of flows.

Theorem 8

Let (M, ω) be a compact symplectic manifold and $H\colon M \to \mathbb{R}$ be a smooth function. Then each stage $\Phi_{t,\boldsymbol{X}_H}\colon M \to M$ of the Hamiltonian flow is a symplectomorphism of (M, ω).

Proof: Since M is compact, the field $\boldsymbol{X}_H$ is complete. Noting that the relation characterizing $\boldsymbol{X}_H$ can be rewritten as $\iota_{\boldsymbol{X}_H}\omega = \mathrm{d}H$, where ι stands for the interior derivative (or contraction), we may use Cartan's homotopy formula[3]:

$$\mathscr{L}_{\boldsymbol{X}_H}\omega = \iota_{\boldsymbol{X}_H}(\mathrm{d}\omega) + \mathrm{d}(\iota_{\boldsymbol{X}_H}\omega) = \iota_{\boldsymbol{X}_H}(0) + \mathrm{d}(\mathrm{d}H) = 0 + 0 = 0.$$

□

Remark.

- This theorem is, in some sense, the true reason why we require symplectic forms to be closed. Namely, for any $H\colon M \to \mathbb{R}$, we have that $\mathscr{L}_{\boldsymbol{X}_H}\omega = \iota_{\boldsymbol{X}_H}(\mathrm{d}\omega)$, so requiring that the flows of all $\boldsymbol{X}_H$ preserve ω directly leads to $\mathrm{d}\omega = 0$.

[3] If M is any smooth manifold, $\boldsymbol{X} \in \mathfrak{X}(M)$, and $\alpha \in \Omega^k(M)$, then $\mathscr{L}_{\boldsymbol{X}}\alpha = \iota_{\boldsymbol{X}}\mathrm{d}\alpha + \mathrm{d}(\iota_{\boldsymbol{X}}\alpha)$. This formula says that the operator $\mathscr{L}_{\boldsymbol{X}}$ is nullhomotopic, in the following sense: the contraction $\iota_{\boldsymbol{X}}$ defines a (co)chain homotopy between $\mathscr{L}_{\boldsymbol{X}}$ and the zero map, seen as (co)chain self-maps of the de Rham complex $(\Omega^\bullet(M), \mathrm{d})$ of M. Thus $\mathscr{L}_{\boldsymbol{X}}$ induces the zero map in cohomology.

- We can also show now that the inclusion $\mathrm{SO}(3) \subseteq \mathrm{Sp}(\mathbb{S}^2, \omega)$ seen in Example 14 (p. 19) is strict: let φ be the time 1 of the Hamiltonian flow of some $H\colon \mathbb{S}^2 \to \mathbb{R}$ which vanishes on some proper open subset of $\mathbb{S}^2$. Then φ will act there as the identity, but will do something else on the complement (hence φ is non-linear).

This raises a pertinent question: if (M, ω) is a symplectic manifold, and $\boldsymbol{X} \in \mathfrak{X}(M)$ is any vector field, when does the flow of $\boldsymbol{X}$ consist of symplectomorphisms of (M, ω)? Again, by Cartan's homotopy formula we have that $\mathscr{L}_{\boldsymbol{X}}\omega = \mathrm{d}(\iota_{\boldsymbol{X}}\omega)$, as $\mathrm{d}\omega = 0$. So if $\iota_{\boldsymbol{X}}\omega \in \Omega^1(M)$ is closed (which in particular happens if it is exact), ok.

Definition 13

Let (M, ω) be a symplectic manifold and $\boldsymbol{X} \in \mathfrak{X}(M)$. We say that $\boldsymbol{X}$ is:

- **Hamiltonian**, if there is a smooth function $H\colon M \to \mathbb{R}$ such that $\boldsymbol{X} = \boldsymbol{X}_H$ (such H is then called an **energy function** for $\boldsymbol{X}$);
- **symplectic** (or, **locally Hamiltonian**) if $\iota_{\boldsymbol{X}}\omega$ is closed.

We'll write $\mathcal{S}(M, \omega)$ and $\mathscr{H}(M, \omega)$ for the algebras of symplectic and Hamiltonian fields on (M, ω).

Clearly every Hamiltonian field is symplectic, but for the converse there is the obvious unique obstruction: $H^1_{\mathrm{dR}}(M)$. Thus, finding symplectic fields which are not Hamiltonian is, in spirit, the same as knowing how to find closed 1-forms which are not exact. The next exercise illustrates this.

Exercise 20

Consider the cylinder $\mathbb{S}^1 \times \mathbb{R} \subseteq \mathbb{R}^3$ equipped with its standard symplectic form $\omega = x\,\mathrm{d}y \wedge \mathrm{d}z + y\,\mathrm{d}z \wedge \mathrm{d}x$. Consider the field $\partial/\partial z \in \mathfrak{X}(\mathbb{S}^1 \times \mathbb{R})$.

(a) Show that $\partial/\partial z$ is symplectic.

(b) Compute $\displaystyle\int_{\mathbb{S}^1} \iota_{\partial/\partial z}\omega$ and conclude that $\partial/\partial z$ is not Hamiltonian.

You can also explore more relations between symplectic and Hamiltonian fields here:

Exercise 21

Let (M, ω) be a symplectic manifold. Show that:

(a) for any $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \mathcal{S}(M, \omega)$, the Jacobi-type identity

$$\omega(\boldsymbol{X}, [\boldsymbol{Y}, \boldsymbol{Z}]) + \omega(\boldsymbol{Y}, [\boldsymbol{Z}, \boldsymbol{X}]) + \omega(\boldsymbol{Z}, [\boldsymbol{X}, \boldsymbol{Y}]) = 0$$

holds.

(b) $[\mathcal{S}(M,\omega),\mathcal{S}(M,\omega)] \subseteq \mathcal{H}(M,\omega)$ (in words: the Lie bracket of two symplectic fields is Hamiltonian).

And we'll conclude this section with an instructive exercise:

Exercise 22

Let Q be a smooth manifold, $\boldsymbol{X} \in \mathfrak{X}(Q)$ be a complete vector field, and consider its flow $\Phi_{t,\boldsymbol{X}}\colon Q \to Q$, for each $t \in \mathbb{R}$. Equip T^*Q with ω_{can}. Show that:

(a) there is a complete vector field $\widehat{\boldsymbol{X}} \in \mathfrak{X}(T^*Q)$ such that $\widehat{\Phi_{t,\boldsymbol{X}}} = \Phi_{t,\widehat{\boldsymbol{X}}}$ for all $t \in \mathbb{R}$, where $\widehat{\ }$ on the left side denotes a cotangent lift, as usual. **Hint:** use the functorial behavior of $\widehat{\ }$ to show that $\widehat{\Phi_{t,\boldsymbol{X}}}$ defines a flow on T^*Q.

(b) the field $\widehat{\boldsymbol{X}}$ in found in (a) is Hamiltonian and find an energy function for it. **Hint:** Use Cartan's homotopy formula.

Exercise 23

Let Q be a smooth manifold, $f\colon Q \to \mathbb{R}$ be a smooth function, and consider the cotangent bundle $(T^*Q, \omega_{\mathrm{can}})$. If $\pi\colon T^*Q \to Q$ is the natural projection, we have the pull-back $\pi^*f\colon T^*Q \to \mathbb{R}$. Show that the time 1 of the flow of $\boldsymbol{X}_{\pi^*f}$ maps the zero section $Q \hookrightarrow T^*Q$ to the graph of $-\mathrm{d}f$. **Hint:** show that $\boldsymbol{X}_{\pi^*f}$ is vertical and compute $\Phi_{t,\boldsymbol{X}_{\pi^*f}}(x,0)$ explicitly, for instance, using cotangent coordinates.

2.4 Submanifolds and local forms

This situation will mimic what happened with subspaces of a symplectic vector space (V,Ω). As usual, we'll smoothly apply the all the linear algebra concepts previously seen to all tangent spaces to a manifold. For this, let's register our first definition:

Definition 14

Let (M,ω) be a symplectic manifold and $N \subseteq M$ a submanifold. We'll say that N is **symplectic**, **isotropic**, **coisotropic** or **Lagrangian** if, for every $x \in N$, the tangent space T_xN is a subspace of (T_xM, ω_x) of the corresponding type.

Remark. More abstractly, one can say that an immersion $\iota\colon N \hookrightarrow M$ is Lagrangian if $\iota^*\omega = 0$ and $2\dim N = \dim M$. This is particularly useful to show that certain submanifolds are Lagrangian, when employing convenient identifications.

Two immediate examples: any curve in a symplectic manifold is isotropic, and any hypersurface is coisotropic, for dimensional reasons (of course, these are not all the examples). Still, the main focus will thus be on Lagrangian submanifolds. Mid-dimensional submanifolds appear naturally in several situations, while symplectic submanifolds generally turn out to be studied as symplectic manifolds on their own right. Before going to examples, the following result should further emphasize why should one care about Lagrangian submanifolds.

Proposition 5

Let (M, ω, H) be a Hamiltonian system, and assume that $N \subseteq M$ is a coisotropic submanifold which is contained in some energy level of H. Then N is invariant under the Hamiltonian flow. More precisely, $\Phi_{t,\boldsymbol{X}_H}[N] = N$ for all t for which the flow is defined.

Proof: It suffices to show that for each $x \in N$, we have $\boldsymbol{X}_H|_x \in T_xN$ — the result then follows by integration. Since we assume that N is coisotropic, we have by definition that $(T_xN)^{\omega_x} \subseteq T_xN$. So, let's show that $\boldsymbol{X}_H|_x \in (T_xN)^{\omega_x}$. For any $\boldsymbol{v} \in T_xN$, we have that

$$\omega_x(\boldsymbol{X}_H|_x, \boldsymbol{v}) = \mathrm{d}H_x(\boldsymbol{v}) = 0,$$

using that N is contained in an energy level of H. This concludes the proof. □

Remark. So, coisotropic submanifolds contained in energy levels of a Hamiltonian system are invariant under the dynamics. This holds *a fortiori* for Lagrangian submanifolds. However, in dynamics one often cares about minimal invariant sets. But "Lagrangian" not only means "maximal isotropic", but also "minimal coisotropic".

With this in place, here's our first non-trivial example.

Example 20

Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds, and let $\varphi\colon M_1 \to M_2$ be smooth. Let's see that the graph

$$\mathrm{gr}(\varphi) = \{(x, \varphi(x)) \mid x \in M_1\}$$

is a Lagrangian submanifold of $(M_1 \times M_2, \omega_1 \oplus (-\omega_2))$ if and only if we have $\dim M_1 = \dim M_2$ and $\varphi^*\omega_2 = \omega_1$ (which then implies that φ is a local diffeomorphism). First, note that for every $x \in M_1$ we have that

$$T_{(x,\varphi(x))}\mathrm{gr}(f) = \{(\boldsymbol{v}, \mathrm{d}\varphi_x(\boldsymbol{v})) \mid \boldsymbol{v} \in T_xM_1\}.$$

With this, if $\widetilde{\boldsymbol{v}} = (\boldsymbol{v}, \mathrm{d}\varphi_x(\boldsymbol{v})), \widetilde{\boldsymbol{w}} = (\boldsymbol{w}, \mathrm{d}\varphi_x(\boldsymbol{w})) \in T_{(x,\varphi(x))}\mathrm{gr}(\varphi)$, we may compute

$$\begin{aligned}(\omega_1 \oplus (-\omega_2))_{(x,\varphi(x))}(\widetilde{\boldsymbol{v}}, \widetilde{\boldsymbol{w}}) &= (\omega_1)_x(\boldsymbol{v}, \boldsymbol{w}) - (\omega_2)_{\varphi(x)}(\mathrm{d}\varphi_x(\boldsymbol{v}), \mathrm{d}\varphi_x(\boldsymbol{w})) \\ &= (\omega_1)_x(\boldsymbol{v}, \boldsymbol{w}) - (\varphi^*\omega_2)_x(\boldsymbol{v}, \boldsymbol{w}),\end{aligned}$$

which says that $T_{(x,\varphi(x))}\mathrm{gr}(\varphi)$ is isotropic if and only if $\varphi^*\omega_2 = \omega_1$. As for the dimension, we have that $\operatorname{gr}(\varphi)$ is diffeomorphic to M_1, so that

$$\dim \operatorname{gr}(\varphi) = \frac{1}{2}(\dim M_1 + \dim M_2)$$

if and only if $\dim M_1 = \dim M_2$.

Remark. Taking $(M, \omega) \doteq (M_1, \omega_1) = (M_2, \omega_2)$ and $\varphi = \mathrm{Id}_M$ in the above example, we see that the **diagonal of** M, $\Delta(M) \doteq \{(x, x) \mid x \in M\}$, is a Lagrangian submanifold of $(M \times M, \omega \oplus (-\omega))$. In this case, the projection $\Delta(M) \to M$ is a symplectomorphism.

As usual, cotangent bundles will serve as rich source of examples.

Example 21 (Lagrangian submanifolds of cotangent bundles)

There are three types of submanifolds here which deserve attention:

(1) Fibers. That is, any fixed cotangent space T_x^*Q. We consider the inclsion $\iota\colon T_x^*Q \hookrightarrow T^*Q$, given by $\iota(p) = (x, p)$. We have that

$$(\iota^*\lambda)_p = \lambda_{(x,p)} \circ \mathrm{d}\iota_p = p \circ \mathrm{d}\pi_{(x,p)} \circ \mathrm{d}\iota_p = p \circ \mathrm{d}(\pi \circ \iota)_p = p \circ 0 = 0,$$

since $(\pi \circ \iota)(p) = x$ is a constant. Hence $\iota^*\lambda = 0$ leads to $\iota^*\omega_{\mathrm{can}} = 0$. If $B \in \Omega^2(Q)$ is closed, the fact that $\pi \circ \iota$ again implies that $\iota^*\pi^*B = 0$. So T_x^*Q is a Lagrangian submanifold of both $(T^*Q, \omega_{\mathrm{can}})$ and (T^*Q, ω_B), as it already has the correct dimension.

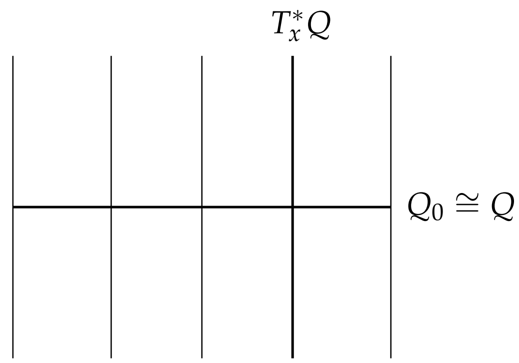


Figure 3: Embedding Q in T^*Q as the zero section.

(2) Ranges of 1-forms. Let $\sigma \in \Omega^1(Q)$ and let $Q_\sigma \doteq \operatorname{Ran}(\sigma) \subseteq T^*Q$. More precisely, Q_σ is the range of σ regarded as a map $\sigma\colon Q \to T^*Q$. Note that Q_σ is automatically mid-dimensional. The inclusion $Q_\sigma \hookrightarrow T^*Q$ is identified with σ itself. By Exercise 12 (p. 17), we have that $\sigma^*\lambda = \sigma$, so that applying $-\mathrm{d}$ we have $\sigma^*\omega_{\mathrm{can}} = -\mathrm{d}\sigma$. Moreover, since $\pi \circ \sigma = \mathrm{Id}_Q$, we have that if $B \in \Omega^2(Q)$ is closed, then $\sigma^*\pi^*B = B$. So, Q_σ is a Lagrangian submanifold of $(T^*Q, \omega_{\mathrm{can}})$ if and only if σ is closed, and it is a Lagrangian submanifold of (T^*Q, ω_B) if and only if $\mathrm{d}\sigma = B$ (i.e., σ is a magnetic potential for B). Note

that $Q_0 = Q$ is always a Lagrangian submanifold of $(T^*Q, \omega_{\rm can})$, but never for (T^*Q, ω_B). Broadly speaking, the introduction of a magnetic term eliminates the choice of a distinguished Lagrangian section of T^*Q — this is akin to what happens when one deals with affine spaces in terms of vector spaces (there is no canonical choice of origin).

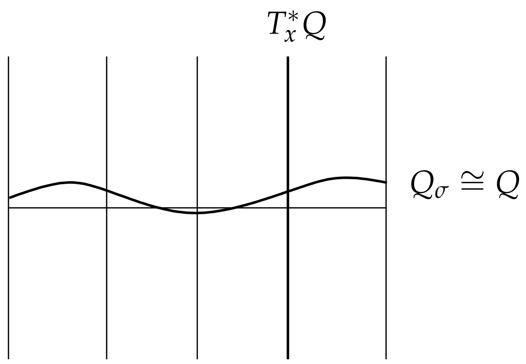


Figure 4: Embedding Q in T^*Q as the graph Q_σ.

(3) Conormal bundles: Let $P \subseteq Q$ be a submanifold. The **conormal bundle** of P (relative to Q) is the vector bundle $\pi^*\colon \nu^*P \to P$ whose fibers are the annihilators of the tangent spaces to P. Namely, for all $x \in P$, we have

$$\nu_x^*P \doteq \operatorname{Ann}(T_xP) = \{p \in T_x^*Q \mid p|_{T_xP} = 0\}.$$

We have that

$$\dim \nu^*P = \dim P + (\dim Q - \dim P) = \dim Q = \frac{1}{2}\dim T^*Q,$$

so the dimension of ν^*P is already correct. Let $\iota\colon N^*P \hookrightarrow T^*Q$ be the inclusion, and write $\pi\colon T^*Q \to Q$ for the projection. Note that $\pi^*\colon \nu^*P \to P$ is valued on P, so its derivative is valued on the tangent spaces to P, which are annihilated by elements in the fibers of ν^*P. So the relation $\pi \circ \iota = \pi^*$ gives that

$$(\iota^*\lambda)_{(x,p)} = p \circ \mathrm{d}\pi^*{}_{(x,p)} = 0,$$

provided $(x,p) \in \nu^*P$. We conclude that ν^*P is a Lagrangian submanifold of $(T^*Q, \omega_{\rm can})$. One can also easily see that ν^*P will be a Lagrangian submanifold of (T^*Q, ω_B), for closed $B \in \Omega^2(Q)$, if and only if B pulls back to the zero 2-form on P.

The main idea we'll discuss here is called **Moser's trick**. It is crucial to the study of deformation of symplectic structures, to establishing local form theorems for submanifolds of symplectic manifolds and, in particular, has Darboux's theorem as a consequence. For that, fix a manifold M and recall that an **isotopy** is a family $(\varphi_t)_{t\in\mathbb{R}}$ of diffeomorphisms of M with φ_0 equal to the identity map, and such that the joint map $\mathbb{R} \times M \ni (t,x) \mapsto \varphi_t(x) \in M$ is smooth. Note that an isotopy does not necessarily satisfy the group-like property that flows of vector fields satisfy, but for each such isotopy there is an associated **time-dependent vector field** on M, that is, a family $(\mathbf{X}_t)_{t\in\mathbb{R}}$ of vector fields on M such that the joint map $\mathbb{R}\times M \ni (t,x)\mapsto \mathbf{X}_{t,x}\in T_xM\subseteq TM$ is smooth. Namely, the isotopy defines the time-dependent vector field with the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(x) = \mathbf{X}_{t,\varphi_t(x)}.$$

When M is compact, a time-dependent vector field gives rise to an isotopy, via "integration".

Lemma 2 (Moser's trick)

Let M be a compact manifold with symplectic forms ω_0 and ω_1. Assume that $(\omega_t)_{t\in[0,1]}$ is a smooth family of symplectic forms joining ω_0 and ω_1, such that the cohomology class $[\omega_t]$ is independent of t. Then there is an isotopy $(\varphi_t)_{t\in\mathbb{R}}$ such that $\varphi_t^*\omega_t = \omega_0$ for all $t\in[0,1]$ — in particular, $\varphi_1^*\omega_1=\omega_0$.

Proof: The idea essentially consists in reverse engineering what $(\varphi_t)_{t\in\mathbb{R}}$ must be. Since the cohomology class $[\omega_t]$ in $H^2_{\mathrm{dR}}(M)$ is independent of t, we may get a smooth family of 1-forms $(\alpha_t)_{t\in[0,1]}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_t = \mathrm{d}\alpha_t, \qquad \text{for all } t\in[0,1].$$

Now, if the isotopy exists and we denote by $(\mathbf{X}_t)_{t\in\mathbb{R}}$ its associated time-dependent vector field, we may take the derivative of $\varphi_t^*\omega_t$ using a "product rule" and Cartan's homotopy formula to obtain

$$\begin{aligned}
0 = \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t^*\omega_t &= \varphi_t^*\mathscr{L}_{\mathbf{X}_t}\omega_t + \varphi_t^*\left(\frac{\mathrm{d}}{\mathrm{d}t}\omega_t\right)\\
&= \varphi_t^*\left(\mathrm{d}(\iota_{\mathbf{X}_t}\omega_t) + \iota_{\mathbf{X}_t}(\mathrm{d}\omega_t) + \frac{\mathrm{d}}{\mathrm{d}t}\omega_t\right)\\
&= \varphi_t^*\mathrm{d}\left(\iota_{\mathbf{X}_t}\omega_t + \alpha_t\right).
\end{aligned}$$

So, provided we can solve **Moser's equation** $\iota_{\mathbf{X}_t}\omega_t + \alpha_t = 0$ for $\mathbf{X}_t$, we can extend it from $[0,1]$ to $\mathbb{R}$ and then integrate it to the desired isotopy, by compactness of M. But we can clearly solve Moser's equation: it suffices to take, for each $(t,x)\in[0,1]\times M$, the vector $\mathbf{X}_{t,x}$ to be the image of $-\alpha_{t,x}$ under the isomorphism $T_x^*M\to T_xM$ induced by ω_t. Hence $\varphi_t^*\omega_t$ is independent of t, and it equals its value for $t=0$, namely, $\varphi_0^*\omega_0 = (\mathrm{Id}_M)^*\omega_0 = \omega_0$. ☐

Compactness of M is a somewhat strong assumption above. But we can work around it, to some extent. One useful tool we'll need for this is a more powerful version of Poincaré's Lemma:

Lemma 3 (Poincaré's Lemma — relative version)

Let M be a smooth manifold, $N\subseteq M$ be a compact submanifold, and $\alpha\in\Omega^k(M)$ a closed k-form such that $\iota^*\alpha = 0$, where $\iota\colon N\hookrightarrow M$ is the inclusion. Then there is a (tubular) neighborhood of N in M and a $(k-1)$-form β defined on such neighborhood such that $\alpha = \mathrm{d}\beta$ holds there, with $\beta_x = 0$ for all $x \in N$.

Remark. If $\alpha_x = 0$ for all $x \in N$, one can also arrange for the first order partial derivatives of β, relative to any adapted coordinate system to N, to also vanish along N. Note that when N is a single point, this is the usual Poincaré's Lemma. See [26] or [43] for a proof.

With this in place:

Theorem 9 (Moser stability)

Let M be a smooth manifold, $N \subseteq M$ be a compact submanifold, and assume that $\omega_0, \omega_1 \in \Omega^2(M)$ are two symplectic forms such that $(\omega_0)_x = (\omega_1)_x$ for all $x \in N$. Then there are (tubular) neighborhoods U_0 and U_1 of N in M and a diffeomorphism $\varphi\colon U_0 \to U_1$ such that $\varphi^*\omega_1 = \omega_0$ and $\varphi(x) = x$ for all $x \in N$.

Proof: (Sketch) By Poincaré's Lemma there is a 1-form β defined on a tubular neighborhood U_0 of N such that $\omega_1 - \omega_0 = \mathrm{d}\beta$ holds there. Define $\omega_t \doteq \omega_0 + t\,\mathrm{d}\beta$ on U_0, for all $t \in [0,1]$. By compactness of N and $[0,1]$, reducing U_0 if necessary, we may assume that all ω_t are symplectic. Now, since

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_t = \mathrm{d}\beta$$

does not depend on t, by reducing U_0 further, Moser's trick gives an isotopy φ_t of U_0 such that $\varphi_t^*\omega_t = \omega_0$ for all $t \in [0,1]$. Let $\varphi = \varphi_1$ and $U_1 = \varphi[U_0]$ — we may arrange for $\varphi(x) = x$ for all $x \in N$ by doing a choice of gauge $\beta \mapsto \beta + \mathrm{d}f$, with f smooth on U_0, according to the remark after Poincaré's Lemma above, before running Moser's argument. □

Theorem 10 (Darboux)

Let (M,ω) be a symplectic manifold. Around each point in (M,ω) there are coordinates $(x^1,\ldots,x^n,y_1,\ldots,y_n)$ for which ω is expressed as

$$\omega = \sum_{k=1}^n \mathrm{d}x^k \wedge \mathrm{d}y_k.$$

We'll call such coordinates **Darboux coordinates** for (M,ω).

Proof: Let N be a singleton in Theorem 9 above. More precisely, fix $x_0 \in M$ and pick coordinates $(\widetilde{x}^1,\ldots,\widetilde{x}^n,\widetilde{y}_1,\ldots,\widetilde{y}_n)$ around x_0 such that

$$\left(\frac{\partial}{\partial \widetilde{x}^1}\bigg|_{x_0},\ldots,\frac{\partial}{\partial \widetilde{x}^n}\bigg|_{x_0},\frac{\partial}{\partial \widetilde{y}_1}\bigg|_{x_0},\ldots,\frac{\partial}{\partial \widetilde{y}_n}\bigg|_{x_0}\right)$$

is a Darboux basis for $(T_{x_0}M, \omega_{x_0})$. On this coordinate domain, consider ω itself and

$$\widetilde{\omega} = \sum_{k=1}^{n} \mathrm{d}\widetilde{x}^k \wedge \mathrm{d}\widetilde{y}_k.$$

Since $\omega_{x_0} = \widetilde{\omega}_{x_0}$ by construction, reducing the coordinate domain if necessary, by Moser we obtain a diffeomorphism φ of this domain for which $\varphi^*\widetilde{\omega} = \omega$. Then we set $x^i \doteq \widetilde{x}^i \circ \varphi$ and $y_j \doteq \widetilde{y}_j \circ \varphi$ for all $i, j = 1, \ldots, n$. By definition of pull-back, $(x^1, \ldots, x^n, y_1, \ldots, y_n)$ are the desired Darboux coordinates. □

Note that in the same way that every symplectic vector space (V, Ω) is symplectomorphic to the prototype $(\mathbb{R}^{2n}, \Omega_{2n})$ with the correct dimension (due to the existence of Darboux bases), this theorem is saying that every point in a symplectic manifold (M, ω) has a neighborhood symplectomorphic to an open subset of $(\mathbb{R}^{2n}, \omega_{2n})$ (due to the existence of Darboux coordinates). In particular, we know that cotangent coordinates for cotangent bundles are Darboux coordinates, so pretty much everything we have established for a cotangent bundle holds locally in an arbitrary symplectic manifold — for instance, Theorem 7 (p. 23) tells us how to describe Hamiltonian flows in Darboux coordinates.

More importantly, this theorem says that there are **no local invariants in symplectic geometry**, such as curvature. We know that curvatures come from a connection on the manifold. But even that becomes slightly tricky, as there is no "Koszul formula" in this setting.

Proposition 6 ([34])

Let M be a smooth manifold, and $\omega \in \Omega^2(M)$ be a non-degenerate 2-form. There is a torsionfree connection on M which parallelizes ω if and only if ω is closed. And if this is the case, there are infinitely many such connections.

Proof: If ω is closed, then (M, ω) is symplectic. Cover M with Darboux coordinates, on each such domain define a local connection declaring all the Christoffel symbols to vanish, and then "glue" everything into a single connection using a partition of unity. Conversely, if there is such a torsionfree connection ∇ parallelizing ω, recall that (up to a constant multiple depending on conventions), $\mathrm{d}\omega$ is the alternator of $\nabla\omega$, which is zero by assumption. □

We'll conclude this section by stating local form results for symplectic and Lagrangian submanifolds. They require two brief concepts:

- If M is a smooth manifold and N is a submanifold, we can define the **normal bundle** of N relative to M even without a Riemannian metric on M. Namely, since for all $x \in N$ we may regard T_xN as a subspace of T_xM, we can just consider the quotient $\nu_xN = T_xM/T_xN$. This of course defines $\nu N \to N$, we have an isomorphism $TN \oplus \nu N \cong TM|_N$, and when M does have a Riemannian metric, we have $\nu N \cong TN^\perp$. If we have a smooth map $f \colon M_1 \to M_2$ between two manifolds which restricts to a smooth map $N_1 \to N_2$ between submanifolds, for each $x \in N_1$ we have a "derivative" $\nu f_x \colon \nu_x N_1 \to \nu_{f(x)} N_2$ given by

$$\nu f_x(\boldsymbol{v} + T_x N_1) = \mathrm{d}f_x(\boldsymbol{v}) + T_{f(x)} N_2.$$

If f is a diffeomorphism, this bundle morphism $\nu N_1 \to \nu N_2$ is an isomorphism and, conversely, given an isomorphism $\nu N_1 \to \nu N_2$ covering a diffeomorphism $N_1 \to N_2$, such diffeomorphism extends to a diffeomorphism on a tubular neighborhood of N_1 which induces the given bundle isomorphism via derivative (the gist of it is to take Riemannian metrics on M_1 and M_2 and use normal exponential maps).

- A **symplectic vector bundle**, denoted by $(E, \omega) \to M$, is nothing more than vector bundle $E \to M$ equipped with (smoothly varying) linear symplectic forms ω_x on each fiber E_x. Note that the requirement $\mathrm{d}\omega = 0$ is meaningless in this setting. We already have plenty of examples: the tangent bundle of a symplectic manifold, every symplectic vector space is a symplectic vector bundle over a point, the direct sum of any vector bundle with its dual, normal bundles (in the above sense) of symplectic submanifolds of symplectic manifolds, and so on. In particular, a **morphism of symplectic vector bundles** is exactly what you're thinking — a bundle morphism preserving the symplectic structures.

Theorem 11 (Weinstein's symplectic normal form theorem)

Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds, with symplectic submanifolds N_1 and N_2, and $\nu N_1 \to \nu N_2$ be an isomorphism of symplectic vector bundles covering a symplectomorphism $N_0 \to N_1$. Then such symplectomorphism extends to a symplectomorphism between tubular neighborhoods of N_0 and N_1 which induces the given isomorphism via derivative.

To motivate the next result, consider a symplectic manifold (M, ω) and a Lagrangian submanifold $Q \subseteq M$. We know that for each point $x \in Q$, $T_x Q \cong (T_x Q)^{\omega_x}$ holds, and the isomorphism $T_x M \cong T_x^* M$ induced by ω_x directly restricts to an isomorphism $(T_x Q)^{\omega_x} \cong \mathrm{Ann}(T_x Q)$. Dualizing, it follows that $\nu_x Q \cong T_x^* Q$. Gathering those isomorphisms as x ranges over Q, we have an isomorphism $\nu Q \cong T^* Q$ of symplectic vector bundles.

Theorem 12 (Weinstein's Lagrangian neighboorhood theorem)

Let (M, ω) be a symplectic manifold and $Q \subseteq M$ be a Lagrangian submanifold. Then there is a tubular neighborhood U of Q in M and a neighborhood U_0 of Q as the zero section in $T^* Q$, and a diffeomorphism $\varphi \colon U_0 \to U$ such that $\varphi(x, 0) = x$ for all $x \in Q$ and $\varphi^*(\omega) = \omega_{\mathrm{can}}$.

For proofs of the last two results, see [28]. Let's explore one last consequence of this. We have seen after Example 20 (p. 32) that if (M, ω) is a symplectic manifold, then the diagonal $\Delta(M)$ is a Lagrangian submanifold of $(M \times M, \omega \oplus (-\omega))$, but now we know that the projection $\Delta(M) \to M$ actually extends to a symplectomorphism φ_Δ in some neighborhood of $\Delta(M)$, taking values in a neighborhood of M as the zero section of T^*M:

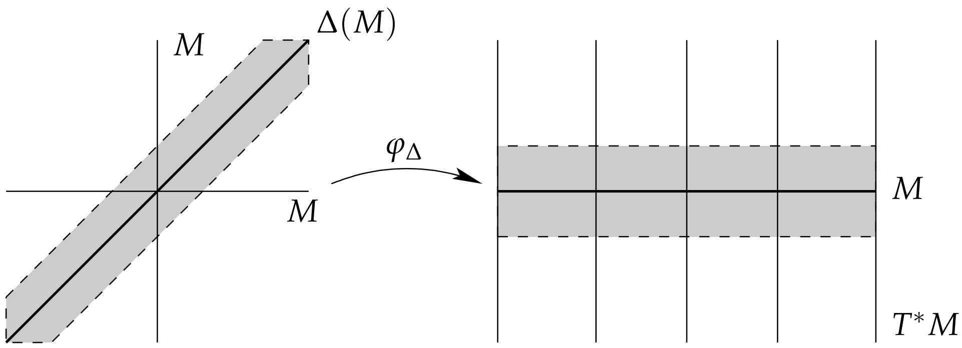


Figure 5: Illustrating the symplectomorphism φ_Δ.

Now, if $\varphi \in \mathrm{Sp}(M, \omega)$, we know by Example 20 (p. 32), again, that the graph $\mathrm{gr}(\varphi)$ is a Lagrangian submanifold of $(M \times M, \omega \oplus (-\omega))$. If φ is $\mathscr{C}^1$-close to the identity, then $\mathrm{gr}(\varphi)$ is close enough to $\Delta(M)$, and so the image $L \doteq \varphi_\Delta[\mathrm{gr}(\varphi)] \subseteq T^*M$ is close enough to the zero section, thus being equal to M_σ for some closed $\sigma \in \Omega^1(M)$, as in Example 21 (p. 33).

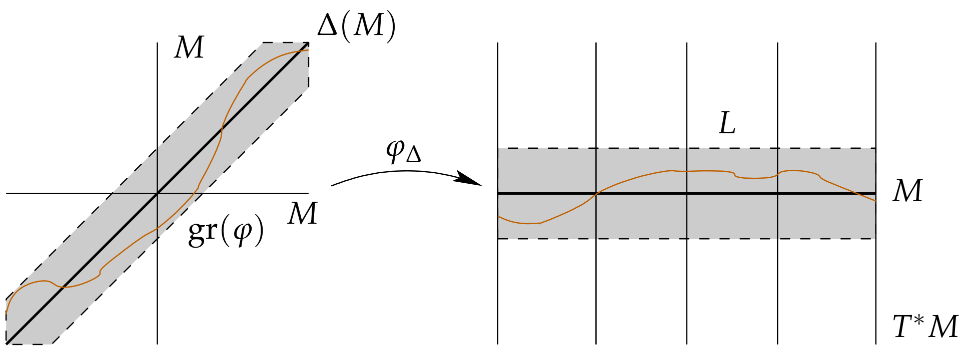


Figure 6: Sending $\mathrm{gr}(\varphi)$ to T^*M as $L = M_\sigma$.

In particular, we see that fixed points of φ are in bijective correspondence with zeros of σ.

Corollary 3

Let (M, ω) be a compact symplectic manifold with $H^1_{\mathrm{dR}}(M) = 0$ and $\varphi \in \mathrm{Sp}(M, \omega)$ be $\mathscr{C}^1$-close to the identity function. Then φ has at least two fixed points.

Proof: As per the above discussion, consider the closed 1-form σ corresponding to the graph $\mathrm{gr}(\varphi)$. Since $H^1_{\mathrm{dR}}(M) = 0$, write $\sigma = \mathrm{d}f$ for some $f \colon M \to \mathbb{R}$. By compactness of M, f achieves its maximum and minimum values at some points in M — those correspond to zeros of σ and hence to fixed points of φ. □

3 Hamiltonian Actions

3.1 Poisson Manifolds

Let (M, ω, H) be a Hamiltonian system. One usually would like to find constants of motion for the system to understand it better. We know that $f \in \mathscr{C}^\infty(M)$ is a constant of motion if f is constant along integral curves of $\boldsymbol{X}_H$, and this is equivalent to $\mathrm{d}f(\boldsymbol{X}_H)$ vanishing. By definition of Hamiltonian field, this is the same as requiring $\omega(\boldsymbol{X}_f, \boldsymbol{X}_H)$ to vanish.

Definition 15

Let (M, ω) be a symplectic manifold. The **Poisson bracket associated with** ω is the operation $\{\cdot,\cdot\}_\omega \colon \mathscr{C}^\infty(M) \times \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$ given by $\{f, g\}_\omega \doteq \omega(\boldsymbol{X}_f, \boldsymbol{X}_g)$.

Example 22 ($\{\cdot,\cdot\}_\omega$ in coordinates)

Let (M, ω) be a symplectic manifold, take $f, g \in \mathscr{C}^\infty(M)$, and (x^k, y_k) be Darboux coordinates for M. By Theorem 7 (p. 23), we know that

$$\boldsymbol{X}_f = \sum_{k=1}^n \left(\frac{\partial f}{\partial y_k} \frac{\partial}{\partial x^k} - \frac{\partial f}{\partial x^k} \frac{\partial}{\partial y_k} \right) \quad \text{and} \quad \boldsymbol{X}_g = \sum_{k=1}^n \left(\frac{\partial g}{\partial y_k} \frac{\partial}{\partial x^k} - \frac{\partial g}{\partial x^k} \frac{\partial}{\partial y_k} \right).$$

Hence, we obtain that

$$\{f, g\}_\omega = \sum_{k=1}^n \left(\frac{\partial f}{\partial x^k} \frac{\partial g}{\partial y_k} - \frac{\partial g}{\partial x^k} \frac{\partial f}{\partial y_k} \right),$$

by applying $\mathrm{d}f$ to $\boldsymbol{X}_g$. The Hamiltonian fields and Poisson brackets of coordinate functions also make sense where they are defined. So, in particular, we have that

$$\boldsymbol{X}_{x^k} = -\frac{\partial}{\partial y_k} \quad \text{and} \quad \boldsymbol{X}_{y_k} = \frac{\partial}{\partial x^k},$$

as well as the relations

$$\{x^i, x^j\}_\omega = 0, \quad \{x^i, y_j\}_\omega = \delta^i_j, \quad \text{and} \quad \{y_i, y_j\}_\omega = 0,$$

for all $i, j = 1, \ldots, n$. Hamilton's equations, in turn, read simply

$$\dot{x}^k = \{x^k, H\}_\omega \quad \text{and} \quad \dot{y}^k = \{y^k, H\}_\omega.$$

In the same way that the symplectic form determines the Poisson bracket, the converse actually holds.

Exercise 24

Let M be a smooth manifold and $\omega, \omega_1, \omega_2 \in \Omega^2(M)$ be symplectic forms.

(a) Show that if $\{\cdot,\cdot\}_{\omega_1} = \{\cdot,\cdot\}_{\omega_2}$, then $\omega_1 = \omega_2$. **Hint:** Show that for all functions $f, g \in \mathscr{C}^\infty(M)$, we'll have that $\boldsymbol{X}_f^{\omega_1}(g) = \boldsymbol{X}_f^{\omega_2}(g)$, so that $\boldsymbol{X}_f^{\omega_1} = \boldsymbol{X}_f^{\omega_2}$. Go from there.

(b) Conclude that $\varphi \in \mathrm{Sp}(M,\omega)$ if and only if $\{f\circ\varphi, g\circ\varphi\}_\omega = \{f,g\}_\omega \circ \varphi$ for all $f, g \in \mathscr{C}^\infty(M)$.

However, Poisson brackets are more than just a convenient device to describe Hamiltonian dynamics. It will turn out that the Poisson bracket alone is enough to do Hamiltonian dynamics, even when a symplectic form is not present! Of course, a few definitions are still required to make sense of this, but we'll continue to develop more intuition with examples and basic results first.

Example 23

Let $M \subseteq \mathbb{R}^3$ be an orientable surface, with unit normal field $\boldsymbol{N}\colon M \to \mathbb{S}^2$, and standard area form $\omega \in \Omega^2(M)$. We have seen in Example 16 (p. 22) that for any $f \in \mathscr{C}^\infty(M)$, we have that $\boldsymbol{X}_f = \operatorname{grad} f \times \boldsymbol{N}$. With this, we may compute the Poisson bracket on (M,ω) using double cross product identities:

$$\begin{aligned}
\{f,g\}_\omega &= \langle \boldsymbol{N}, \boldsymbol{X}_f \times \boldsymbol{X}_g\rangle \\
&= \langle \boldsymbol{N}, (\operatorname{grad} f \times \boldsymbol{N}) \times (\operatorname{grad} g \times \boldsymbol{N})\rangle \\
&= \langle \boldsymbol{N} \times (\operatorname{grad} f \times \boldsymbol{N}), \operatorname{grad} g \times \boldsymbol{N}\rangle \\
&= \langle \langle \boldsymbol{N},\boldsymbol{N}\rangle \operatorname{grad} f - \langle \boldsymbol{N}, \operatorname{grad} f\rangle \boldsymbol{N}, \operatorname{grad} g \times \boldsymbol{N}\rangle \\
&= \langle \operatorname{grad} f, \operatorname{grad} g \times \boldsymbol{N}\rangle \\
&= \langle \boldsymbol{N}, \operatorname{grad} f \times \operatorname{grad} g\rangle.
\end{aligned}$$

In particular, since $\operatorname{grad} f \times \operatorname{grad} g$ is always normal to the surface, we conclude that $\{f,g\}_\omega = 0$ if and only if $\operatorname{grad} f$ and $\operatorname{grad} g$ are always proportional.

Exercise 25

Let (M,ω) be a symplectic manifold. Show that for any $f, g \in \mathscr{C}^\infty(M)$, we have $\boldsymbol{X}_{\{f,g\}_\omega} = -[\boldsymbol{X}_f, \boldsymbol{X}_g]$. This negative sign is just collateral damage from our choices of sign conventions[a]. **Hint:** Use Cartan's homotopy formula together with the general identity $\iota_{[\boldsymbol{X},\boldsymbol{Y}]} = \mathscr{L}_{\boldsymbol{X}} \circ \iota_{\boldsymbol{Y}} - \iota_{\boldsymbol{X}} \circ \mathscr{L}_{\boldsymbol{Y}}$ for interior derivatives.

[a] Perhaps the "correct" definition of our Poisson bracket should have had the opposite sign instead. Arguably, it wouldn't look so natural. But we would have to pay the price somewhere. To try to avoid this sort of thing, some texts (like [28], for example), even define the Lie bracket with the opposite sign!

Next, let's finally justify the name "bracket" (which should have made you think of a Lie bracket):

Theorem 13

Let (M,ω) be a symplectic manifold. Then the Poisson bracket $\{\cdot,\cdot\}_\omega$ is skew-symmetric, $\mathbb{R}$-bilinear, satisfies the **Jacobi identity**

$$\{f,\{g,h\}_\omega\}_\omega + \{g,\{h,f\}_\omega\}_\omega + \{h,\{f,g\}_\omega\}_\omega = 0,$$

and the **Leibniz rule**

$$\{f,gh\}_\omega = g\{f,h\}_\omega + h\{f,g\}_\omega,$$

for all $f,g,h \in \mathscr{C}^\infty(M)$.

Proof: Skew-symmetry and bilinearity over real scalars are obvious. For the Jacobi identity, we will use for the second time in this text the condition $\mathrm{d}\omega = 0$. First, we have that

$$\begin{aligned} \boldsymbol{X}_f(\omega(\boldsymbol{X}_g,\boldsymbol{X}_h)) &= \mathrm{d}\{g,h\}_\omega(\boldsymbol{X}_f) = \omega(\boldsymbol{X}_{\{g,h\}_\omega},\boldsymbol{X}_f) \\ &= \{\{g,h\}_\omega,f\}_\omega = -\{f,\{g,h\}_\omega\}_\omega. \end{aligned}$$

Next, compute (with the result from the previous exercise) that

$$\omega([\boldsymbol{X}_g,\boldsymbol{X}_h],\boldsymbol{X}_f) = \omega(-\boldsymbol{X}_{\{g,h\}_\omega},\boldsymbol{X}_f) = \omega(\boldsymbol{X}_f,\boldsymbol{X}_{\{g,h\}_\omega}) = \{f,\{g,h\}_\omega\}_\omega.$$

Finally, we use that

$$\begin{aligned} 0 = \mathrm{d}\omega(\boldsymbol{X}_f,\boldsymbol{X}_g,\boldsymbol{X}_h) &= \boldsymbol{X}_f(\omega(\boldsymbol{X}_g,\boldsymbol{X}_h)) - \boldsymbol{X}_g(\omega(\boldsymbol{X}_f,\boldsymbol{X}_h)) + \boldsymbol{X}_h(\omega(\boldsymbol{X}_f,\boldsymbol{X}_g)) \\ &\quad - \omega([\boldsymbol{X}_f,\boldsymbol{X}_g],\boldsymbol{X}_h) + \omega([\boldsymbol{X}_f,\boldsymbol{X}_h],\boldsymbol{X}_g) - \omega([\boldsymbol{X}_g,\boldsymbol{X}_h],\boldsymbol{X}_f). \end{aligned}$$

This becomes exactly the Jacobi identity for $\{\cdot,\cdot\}_\omega$ (in fact, $\mathrm{d}\omega = 0$ is equivalent to the Jacobi identity). □

Corollary 4

Let (M,ω,H) be a Hamiltonian system, and $f,g \in \mathscr{C}^\infty(M)$ be constants of motion. Then $\{f,g\}_\omega$ is also a constant of motion.

Being a Lie bracket, it is natural to expect the Poisson bracket to have relations with integrability and submanifolds. The next two results should illustrate this.

Proposition 7

Let (M,ω) be a symplectic manifold, and $N \subseteq M$ be a submanifold. We define the **annihilator ideal** of N to be $\mathscr{C}^\infty(M)_N = \{f \in \mathscr{C}^\infty(M) \mid f|_N = 0\}$. Then, the following are equivalent:

(i) N is coisotropic.

(ii) For each $f \in \mathscr{C}^\infty(M)_N$, $\boldsymbol{X}_f$ is tangent to N along its points.

(iii) $\mathscr{C}^\infty(M)_N$ is closed under $\{\cdot,\cdot\}_\omega$.

Proof: First, note that if $f \in \mathscr{C}^\infty(M)_N$, then for each $x \in N$ and $\boldsymbol{v} \in T_xN$, we have that $0 = \mathrm{d}f_x(\boldsymbol{v}) = \omega_x(\boldsymbol{X}_f|_x, \boldsymbol{v})$, so $\boldsymbol{X}_f|_x \in (T_xN)^{\omega_x}$. In fact, we also have that such values fill the space: $(T_xN)^{\omega_x} = \{\boldsymbol{X}_f|_x \mid f \in \mathscr{C}^\infty(M)_N\}$. With this in place, we move on to the implications.

(i) $\implies$ (ii)**:** Clear from the above as $(T_xN)^{\omega_x} \subseteq T_xN$ for all $x \in N$ by (i).

(ii) $\implies$ (iii)**:** If $f, g \in \mathscr{C}^\infty(M)_N$, then $\{f,g\}_\omega|_N = \mathrm{d}f(\boldsymbol{X}_g)|_N = 0$ because for all $x \in N$, $\mathrm{d}f_x$ annihilates T_xN and $\boldsymbol{X}_g|_x \in T_xN$ by (ii).

(iii) $\implies$ (i)**:** Take $x \in N$, and let's show that $(T_xN)^{\omega_x} \subseteq (T_xN^{\omega_x})^{\omega_x}$. For that, we may just use Hamiltonian fields of functions coming from $\mathscr{C}^\infty(M)_N$. By (iii), we have that $\omega_x(\boldsymbol{X}_f|_x, \boldsymbol{X}_g|_x) = \{f,g\}_\omega(x) = 0$ if $f, g \in \mathscr{C}^\infty(M)_N$.

□

Corollary 5

Let (M,ω) be a symplectic manifold and $f_1, \ldots, f_k \in \mathscr{C}^\infty(M)$ be functions in **involution**, that is, $\{f_i, f_j\}_\omega = 0$ for all $i, j = 1, \ldots, k$. Gather them into a map $\boldsymbol{f}\colon M \to \mathbb{R}^k$ and assume it is a submersion. Then the fibers of $\boldsymbol{f}$ are coisotropic submanifolds of M.

Proof: Since f is a submersion, the fibers of $\boldsymbol{f}$ are submanifolds of M, and the Hamiltonian fields $\{\boldsymbol{X}_{f_i}|_x\}_{i=1}^k$ form a basis for $(T_xN)^{\omega_x}$, for all $x \in N$. The involution assumption says that these vectors are "orthogonal", and so all of them are, in fact, in T_xN. □

Remark. If $2k = n$, we obtain a foliation by Lagrangian submanifolds, and this is the usual setup to study **integrable systems**: a Hamiltonian system (M, ω, H) with dimension $2n$ admiting n constants of motion $f_1 = H, f_2, \ldots, f_n$ which are in involution, whose differentials are linearly independent in some open dense subset of M.

Exercise 26

The converse to the previous corollary holds: if (M,ω) is a symplectic manifold and $N \subseteq M$ is a codimension k coisotropic submanifold, then every point in N has a neighborhood U in M and a submersion $\boldsymbol{f}\colon U \to \mathbb{R}^k$ whose coordinate functions are in involution, and $U \cap N = \boldsymbol{f}^{-1}(\boldsymbol{0})$. **Hint:** Use induction on k and see [30].

Poisson brackets can also be related to a mild generalization of Hamiltonian fields.

Exercise 27 (Challenge #3)

Let (M,ω) be a symplectic manifold. For each $\alpha \in \Omega^1(M)$, let $\mathbf{Z}_\alpha \in \mathfrak{X}(M)$ be the unique field characterized by the relation $\omega(\mathbf{Z}_\alpha,\cdot) = \alpha$. Now, define a bracket $[\![\cdot,\cdot]\!]\colon \Omega^1(M)\times\Omega^1(M) \to \Omega^1(M)$ by the relation $\mathbf{Z}_{[\![\alpha,\beta]\!]} \doteq -[\mathbf{Z}_\alpha,\mathbf{Z}_\beta]$. Fix 1-forms $\alpha,\beta \in \Omega^1(M)$. Show that:

(a) $[\![\alpha,\beta]\!] = \mathscr{L}_{\mathbf{Z}_\beta}\alpha - \mathscr{L}_{\mathbf{Z}_\alpha}\beta - \mathrm{d}(\omega(\mathbf{Z}_\alpha,\mathbf{Z}_\beta))$;

(b) If α and β are closed, so is $[\![\alpha,\beta]\!]$.

(c) $\mathrm{d}\{f,g\}_\omega = [\![\mathrm{d}f,\mathrm{d}g]\!]$.

(d) $[\![\cdot,\cdot]\!]$ is a Lie bracket on $\Omega^1(M)$.

If Q is a smooth manifold and $B \in \Omega^2(Q)$ is closed, consider (T^*Q,ω_B).

(e) Show that for each $H \in \mathscr{C}^\infty(T^*Q)$, $\mathbf{X}_H^B = \mathbf{X}_H - \mathbf{Z}_{\iota_{\mathbf{X}_H}(\pi^*B)}$.

(f) Conclude that for $f,g \in \mathscr{C}^\infty(T^*Q)$, $\{f,g\}_{\omega_B} = \{f,g\}_{\omega_{\mathrm{can}}} - (\pi^*B)(\mathbf{X}_f,\mathbf{X}_g)$.

Hint: Expect the condition $\mathrm{d}\omega = 0$ to appear somewhere between (a) and (d).

Remark. If $(M,-\mathrm{d}\lambda)$ is an exact symplectic manifold, $\mathbf{E} \doteq \mathbf{Z}_\lambda$ is called the **Euler field**. Its flow (compute it in Darboux coordinates and you'll see the reason for the name "Euler" field) is an useful tool in the proof of Theorem 6, mentioned in p. 21.

It will be useful later in this chapter to have an abstraction of all we discussed so far. The inspiration comes from Theorem 13 (p. 42).

Definition 16 (Poisson geometry)

A **Poisson manifold** is a pair $(P,\{\cdot,\cdot\})$, where P is a smooth manifold and $\{\cdot,\cdot\}$ is a Lie bracket (to be called a **Poisson bracket**) on $\mathscr{C}^\infty(P)$ for which $\{f,\cdot\}$ is a derivation for all $f \in \mathscr{C}^\infty(P)$. Then:

(i) The vector field $\mathbf{X}_f \doteq -\{f,\cdot\}$ is called the **Hamiltonian field** of f.

(ii) The functions $f \in \mathscr{C}^\infty(P)$ for which $\mathbf{X}_f = 0$ are called **Casimir functions**.

(iii) The **Poisson bivector** is the twice-contravariant skew-symmetric tensor field Π defined by linearly extending the prescription $\Pi(\mathrm{d}f,\mathrm{d}g) \doteq \{f,g\}$, for all $f,g \in \mathscr{C}^\infty(P)$.

Remark. The negative sign in (i) is again an attempt to do some damage control: it makes $\mathbf{X}_f$ so defined coincide with the Hamiltonian field in a symplectic manifold. In any case, discussing Hamiltonian flows and coisotropic submanifolds now makes sense in any Poisson manifold (the latter in view of Proposition 7, p. 42). Writing (P,Π) instead of $(P,\{\cdot,\cdot\})$ is also usual.

Exercise 28

Check that in the above definition Π is indeed well-defined, e.g., by using coordinates for P. Namely, show that if we take coordinates $(x^1, \ldots, x^m)$ for P and write $\alpha, \beta \in \Omega^1(P)$ locally as

$$\alpha = \sum_{i=1}^m \alpha_i \, \mathrm{d}x^i \quad \text{and} \quad \beta = \sum_{j=1}^m \beta_j \, \mathrm{d}x^j,$$

then $\Pi(\alpha, \beta) = \sum_{i,j=1}^n \alpha_i \beta_j \{x^i, x^j\}$, and that the result is coordinate-independent. One usually writes $\Pi^{ij} \doteq \{x^i, x^j\}$ for the components of Π, so that

$$\Pi = \sum_{1 \leq i < j \leq m} \Pi^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}.$$

So, if (M, ω) is a symplectic manifold, then $(M, \{\cdot,\cdot\}_\omega)$ is a Poisson manifold, whose Casimir functions are precisely the (locally) constant ones. But not every Poisson manifold arises in this way.

Example 24 (Canonical Poisson prototype)

Consider $\mathbb{R}^{2n+r}$, with coordinates $(x^1, \ldots, x^n, y_1, \ldots, y_n, z^1, \ldots, z^r)$ and just mimic the usual formula seen in Example 22 (p. 40), ignoring the z-coordinates. Namely, define $\{\cdot,\cdot\}\colon \mathscr{C}^\infty(\mathbb{R}^{2n+r}) \times \mathscr{C}^\infty(\mathbb{R}^{2n+r}) \to \mathscr{C}^\infty(\mathbb{R}^{2n+r})$ by

$$\{f, g\} \doteq \sum_{k=1}^n \left(\frac{\partial f}{\partial x^k} \frac{\partial g}{\partial y_k} - \frac{\partial g}{\partial x^k} \frac{\partial f}{\partial y_k} \right).$$

Note that the Casimir functions for this bracket are precisely the ones depending only on the z-variables. There's a Darboux-type theorem (actually due to Lie) stating that every Poisson bracket on $\mathbb{R}^{2n+r}$ whose rank is equal to r (in general, it doesn't need to be constant) is locally equivalent to this standard one.

As the above example shows, there is no dimension restriction (dissapointingly, note that one may also always choose $\{\cdot,\cdot\} = 0$ on any manifold) and one has little to no control over the Casimir functions of an arbitrary Poisson bracket. Still, there are very interesting examples of non-trivial Poisson brackets coming from Mechanics.

Example 25 (Nambu mechanics)

Define $\{\cdot,\cdot\}_{\mathsf{N}}\colon \mathscr{C}^\infty(\mathbb{R}^3) \times \mathscr{C}^\infty(\mathbb{R}^3) \to \mathscr{C}^\infty(\mathbb{R}^3)$ by

$$\{f, g\}_{\mathsf{N}}(\boldsymbol{x}) \doteq \langle \boldsymbol{x}, \nabla f(\boldsymbol{x}) \times \nabla g(\boldsymbol{x}) \rangle,$$

where ∇ stands for the gradient operator, while $\langle \cdot,\cdot \rangle$ and $\times$ denote the standard Euclidean inner and cross products. This $\{\cdot,\cdot\}_{\mathsf{N}}$ is a Poisson bracket on $\mathbb{R}^3$, called the **Nambu bracket** (the only non-trivial thing to verify is the Jacobi identity). Here's some intuition: if $x, y, z\colon \mathbb{R}^3 \to \mathbb{R}$ denote the coordinate functions on $\mathbb{R}^3$, as a consequence of the cross product relations between vectors in the standard basis of $\mathbb{R}^3$, we have that

$$\{x,y\}_{\mathsf{N}} = z, \quad \{y,z\}_{\mathsf{N}} = x, \quad \text{and} \quad \{z,x\}_{\mathsf{N}} = y,$$

and thus the **Nambu bivector** is given by

$$\Pi_{\mathsf{N}} = x\,\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y\,\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + z\,\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

So, under the canonical isomorphism $(\mathbb{R}^3)^{\wedge 2} \cong \mathbb{R}^3$, Π_{N} corresponds to the position vector field on $\mathbb{R}^3$. We now claim that a function $f \in \mathscr{C}^\infty(\mathbb{R}^3)$ is Casimir if and only if it is radial. Assume that f is radial, so that $f(x) = \rho(\|\boldsymbol{x}\|^2)$ for some smooth $\rho\colon \mathbb{R} \to \mathbb{R}$. Then we have that $\nabla f(\boldsymbol{x}) = 2\rho'(\|\boldsymbol{x}\|^2)\boldsymbol{x}$, so

$$\{f,g\}_{\mathsf{N}}(\boldsymbol{x}) = 2\rho'(\|\boldsymbol{x}\|^2)\langle \boldsymbol{x}, \boldsymbol{x} \times \nabla g(\boldsymbol{x})\rangle = 0$$

for all $g \in \mathscr{C}^\infty(\mathbb{R}^3)$. Conversely, assume that f is Casimir. Let's show that for each $r > 0$, the restriction $f|_{r\mathbb{S}^2}$ is constant, by showing that its differential vanishes (here, $r\mathbb{S}^2$ denotes the sphere centered at the origin with radius r). The condition $\{f,\cdot\}_{\mathsf{N}} = 0$ implies that for all $\boldsymbol{x} \in \mathbb{R}^3$, we have $\boldsymbol{x} \times \nabla f(\boldsymbol{x}) = \boldsymbol{0}$, since the values $\nabla g(\boldsymbol{x})$ fill up $T_{\boldsymbol{x}}(\mathbb{R}^3)$ as g ranges over $\mathscr{C}^\infty(\mathbb{R}^3)$ (or, more simply, just take $g = x$, $g = y$ and $g = z$). Thus, write $\nabla f(\boldsymbol{x}) = \alpha(\boldsymbol{x})\boldsymbol{x}$ for some smooth $\alpha\colon \mathbb{R}^3 \to \mathbb{R}$. Now assume that $\gamma\colon I \to r\mathbb{S}^2$ is any smooth curve, and compute

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)) = \langle \nabla f(\gamma(t)), \dot{\gamma}(t)\rangle = \alpha(\gamma(t))\langle \gamma(t), \dot{\gamma}(t)\rangle = 0,$$

as desired (differentiate $\langle \gamma, \gamma\rangle = r^2$ to obtain $\langle \gamma, \dot{\gamma}\rangle = 0$). In higher dimensions, we have $(\mathbb{R}^n)^{\wedge(n-1)} \cong \mathbb{R}^n$, so multivectors instead of bivectors are required to study Nambu mechanics and Poisson brackets are no longer enough. See [21] for more details.

Remark. More generally, once can fix any smooth function $C\colon \mathbb{R}^3 \to \mathbb{R}$ and define a Nambu bracket from C, by $\{f,g\}_{\mathsf{N},C}(\boldsymbol{x}) \doteq \langle \nabla C(\boldsymbol{x}), \nabla f(\boldsymbol{x}) \times \nabla g(\boldsymbol{x})\rangle$. Then C itself is a Casimir function and, as you may now expect, the associated bivector $\Pi_{\mathsf{N},C}$ corresponds under $(\mathbb{R}^3)^{\wedge 2} \cong \mathbb{R}^3$ to ∇C.

Exercise 29

Verify that $\{\cdot,\cdot\}_{\mathsf{N}}$ satisfies the Jacobi identity. **Hint:** Vector calculus identities might help.

Exercise 30

Let $\mathfrak{g}$ be a Lie algebra. Define $\{\cdot,\cdot\}_{\mathfrak{g}^*}\colon \mathscr{C}^\infty(\mathfrak{g}^*)\times\mathscr{C}^\infty(\mathfrak{g}^*)\to\mathscr{C}^\infty(\mathfrak{g}^*)$ by setting

$$\{f,g\}_{\mathfrak{g}^*}(p) = p([\mathrm{d}f_p, \mathrm{d}f_p]),$$

where $\mathrm{d}f_p, \mathrm{d}g_p \in \mathfrak{g}^{**}\cong\mathfrak{g}$, so that taking their Lie bracket makes sense. This is called the **standard Lie-Poisson structure** on $\mathfrak{g}^*$. Show that $\{\cdot,\cdot\}_{\mathfrak{g}^*}$ is a Poisson bracket. **Hint:** You'll need the Jacobi identity for $[\cdot,\cdot]$ somewhere.

Unfortunately, the dimension and the presence of non-constant Casimir functions are not the only obstacle for a Poisson bracket to have been induced by a symplectic form. In $\mathbb{R}^2$, define

$$\{f,g\}(x,y) = y\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right).$$

This is clearly a Poisson bracket on $\mathbb{R}^2$. If f is Casimir, then f is constant on $\mathbb{R}\times\mathbb{R}_{>0}$ and $\mathbb{R}\times\mathbb{R}_{<0}$, hence constant on the entire $\mathbb{R}^2$ by continuity. But $\{\cdot,\cdot\}$ is not symplectic as

$$\det\begin{pmatrix} 0 & y \\ -y & 0\end{pmatrix} = y^2$$

vanishes along the x-axis. A bit more is required.

Theorem 14 (Pauli-Jost)

Let $(P,\{\cdot,\cdot\})$ be a Poisson manifold and assume that the induced map $T^*P\to TP$ characterized by $\mathrm{d}f_x \mapsto \mathbf{X}_f\big|_x$, for all $x\in P$ and $f\in\mathscr{C}^\infty(P)$, is an isomorphism. Then there is a symplectic form $\omega\in\Omega^2(P)$ for which $\{\cdot,\cdot\}_\omega = \{\cdot,\cdot\}$.

Here's a sketch of the proof: use the isomorphism $T^*P\cong TP$ to convert the bivector Π to a 2-form ω — it is non-degenerate by assumption. And it is closed by the Jacobi identity for $\{\cdot,\cdot\}$ (reverse the proof of Theorem 13, p. 42). It satisfies $\{\cdot,\cdot\}_\omega = \{\cdot,\cdot\}$ by construction.

For more on Poisson manifolds, see the delightful article [44], or the book [42] for a more thorough presentation.

3.2 Group actions on manifolds

Throughout the rest of this text, we will explore symmetries of symplectic manifolds. Symmetries are naturally encoded on group actions. We are naturally led to consider actions of Lie groups on manifolds. Thus, we'll start briefly reviewing group actions. Let's fix some notation. Here, G will denote a (connected) Lie group, e its identity element[4] and, for each $g\in G$, $\mathrm{L}_g, \mathrm{R}_g, \mathrm{C}_g, \mathrm{inv}\colon G\to G$ will denote, respectively, left

[4] The letter e comes from German, *einselement*.

translation by g, right translation by g, conjugation by g, and the inversion map. The Lie algebra $\mathfrak{g}$ (also denoted by $\mathrm{Lie}(G)$) of G is the tangent space T_eG, and we'll identify it with the algebra $\mathfrak{X}^L(G)$ of left-invariant vector fields on G, via $X \mapsto X^L$, where X^L is the unique left-invariant extension of $X \in \mathfrak{g}$, namely, $(X^L)_g \doteq \mathrm{d}(\mathrm{L}_g)_e(X)$. Similar comments hold for $\mathfrak{X}^R(G)$ and X^R.

Definition 17

Let G be a Lie group and M be a smooth manifold. A (smooth) **left action** of G on M, denoted $G \circlearrowright M$, is a smooth map $G \times M \ni (g, x) \mapsto g \cdot x \in M$, satisfying:

(i) $e \cdot x = e$ for all $x \in M$

(ii) $g \cdot (h \cdot x) = (gh) \cdot x$, for all $g, h \in G$, $x \in M$.

In other words, $G \circlearrowright M$ is equivalent to a Lie group homomorphism $G \to \mathrm{Diff}(M)$. We will identify $g \in G$ with the diffeomorphism $g\colon M \to M$ given by $x \mapsto g \cdot x$, and given $x \in M$, we'll write $\mathcal{O}_x\colon G \to M$ for the **orbit map** of x, $\mathcal{O}_x(g) \doteq g \cdot x$. The **orbit space** of $G \circlearrowright M$, denoted by M/G, is the quotient of M under the equivalence relation "$x \sim y \iff$ there is $g \in G$ such that $y = g \cdot x$".

Exercise 31

In the above definition, check that $\sim$ is indeed an equivalence relation.

Here are some frequent and useful concepts.

Definition 18

Let $G \circlearrowright M$ be a Lie group action on a manifold.

(i) Given $x \in M$, $G \cdot x \doteq \{g \cdot x \mid g \in G\}$ is the **orbit of** x;

(ii) Given $x \in M$, $G_x \doteq \{g \in G \mid g \cdot x = x\}$ is the **stabilizer of** x;

(iii) The action is **transitive** if there is $x \in M$ such that $G \cdot x = M$.

(iv) The action is **free** if $G_x = \{e\}$ for all $x \in M$.

(v) The action is **proper** if the **enriched action** $G \times M \to M \times M$ (given by $(g, x) \mapsto (x, g \cdot x)$) is a proper map (i.e., whose inverse images of compact sets are compact).

Remark. Given a smooth Lie group action $G \circlearrowright M$, freeness and properness of the action are enough to ensure that M/G has a unique smooth manifold structure for which the quotient projection $M \to M/G$ is a surjective submersion. In particular, if G is compact, properness is automatic. For more precise conditions ensuring that the quotient $M/_\sim$ of a manifold under an equivalence relation $\sim$ has a smooth manifold structure making $M \to M/_\sim$ a surjective submersion, see **Godement's Theorem**, for example, in [20].

Example 26

(1) A **real linear Lie group** is a Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$ (for example, $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{O}(n,\mathbb{R})$, $\mathrm{SO}(n,\mathbb{R})$, etc.). Each linear Lie group acts on $\mathbb{R}^n$ via evaluation. In particular, restricting we obtain actions such as $\mathrm{O}(n,\mathbb{R}) \circlearrowright \mathbb{S}^{n-1}$.

(2) $\mathbb{S}^1 \circlearrowright \mathbb{S}^2$ via rotations about the z-axis, where elements of $\mathbb{S}^1$ are regarded as angles.

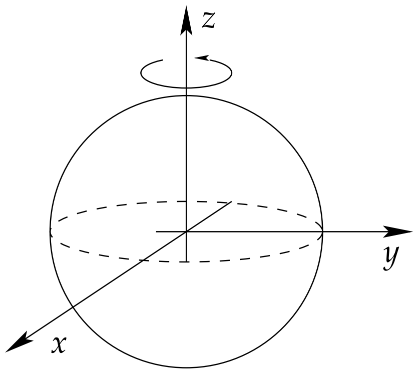


Figure 7: Action of the circle on the sphere.

(3) Any Lie group G acts on itself, in several ways:

- Via left translations: $g \cdot x \doteq \mathrm{L}_g(x) = gx$.
- Via right translations: $g \cdot x \doteq \mathrm{R}_{g^{-1}} = xg^{-1}$.
- Via conjugation: $g \cdot x \doteq \mathrm{C}_g(x) = gxg^{-1}$.

(4) Fixed $\alpha \in \mathbb{R}$, we have an action $\mathbb{R} \circlearrowright \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ via

$$t \cdot ((x,y) \bmod 1) \doteq (x+t, y+\alpha t) \bmod 1.$$

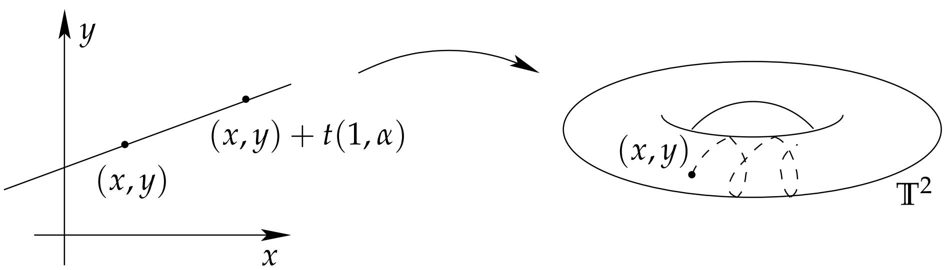


Figure 8: A line action on the torus.

It is possible to show that if $\alpha \in \mathbb{Q}$, the orbits of the action are embedded closed curves in $\mathbb{T}^2$, while if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, they're dense in $\mathbb{T}^2$ (this is called **Kronecker's flow theorem**).

(5) Let Q be a smooth manifold, and let $G \circlearrowright Q$ be a smooth action. This action induces actions:

- $G \circlearrowright TQ$, via $g \cdot (x, v) \doteq (g \cdot x, \mathrm{d}g_x(v))$.
- $G \circlearrowright T^*Q$, via $g \cdot (x, p) \doteq (g \cdot x, p \circ (\mathrm{d}g_x)^{-1})$.

Combining the above, G will also act on every tensor bundle over Q.

To complement the examples above, consider the following exercise:

Exercise 32

(a) Which of the actions given in the above example are transitive? Free? Try to describe the orbit spaces. **Hint:** The orbit-stabilizer theorem may come in handy.

(b) Show that if the action $G \circlearrowright Q$ is free and proper, so are $G \circlearrowright TQ$ and $G \circlearrowright T^*Q$. **Hint:** Recall that $G \circlearrowright Q$ is proper if and only if given any sequences $(g_n)_{n\geq 1}$ in G and $(x_n)_{n\geq 1}$ in Q such that both $(g_n \cdot x_n)_{n\geq 1}$ and $(x_n)_{n\geq 1}$ converge in Q, then $(g_n)_{n\geq 1}$ has a convergent subsequence in G.

And while we're talking about the orbit-stabilizer theorem, note that each stabilizer G_x is a closed subgroup of G (hence a Lie subgroup — not normal, in general), so the quotient G/G_x is a smooth manifold as well. Then $G/G_x \to G \cdot x \subseteq M$ says that each orbit $G \cdot x$ is an immersed submanifold of G. If G is compact, the orbits are embedded (simply because a proper injective immersion must be an embedding). To further exploit the fact we're dealing with Lie groups, derivatives must come into play.

Definition 19

Let $G \circlearrowright M$ be a smooth Lie group action. For any $X \in \mathfrak{g}$, the **action field** of X is $X^\# \in \mathfrak{X}(M)$ defined by $(X^\#)_x \doteq \mathrm{d}(\mathcal{O}_x)_e(X)$.

Remark. When emphasizing the manifold being acted on is needed, one might write $X^\#_M$ instead. Many texts directly define the action field by explictly writing

$$X^\#_x = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \exp(tX) \cdot x,$$

which automatically makes the flow relation $\Phi_{t,X^\#}(x) = \exp(tX) \cdot x$ true.

Exercise 33

Let $G \circlearrowright M$ be a smooth Lie group action. Show that:

(a) the map $\mathfrak{g} \ni X \mapsto X^\# \in \mathfrak{X}(M)$ is an anti-homomorphism of Lie algebras. For right actions, it's a homomorphism.

(b) given $x \in M$, $(X^\#)_x = \mathbf{0}$ if and only if $X \in \mathfrak{g}_x \doteq \mathrm{Lie}(G_x)$ (it is called the **stabilizer algebra at** x).

Here are quick examples, based on the previous ones.

Example 27

(1) For the rotation action $\mathbb{S}^1 \circlearrowright \mathbb{S}^2$, since $\mathrm{Lie}(\mathbb{S}^1) = \mathrm{i}\mathbb{R} \cong \mathbb{R}$ (the obvious isomorphism, delete i), we have that

$$1^\#_{(x,y,z)} = \frac{\mathrm{d}}{\mathrm{d}\theta}\bigg|_{t=0} (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z) = -y\frac{\partial}{\partial x}\bigg|_{(x,y,z)} + x\frac{\partial}{\partial y}\bigg|_{(x,y,z)}.$$

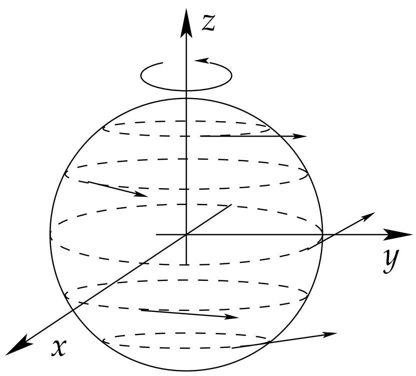


Figure 9: Infinitesimal generator for $\mathbb{S}^1 \circlearrowright \mathbb{S}^2$.

(2) Take a Lie group G acting on itself, and $X \in \mathfrak{g}$. For the action given by:

- left translations, we have $X^\# = X^R$.
- right translations, we have $X^\# = -X^L$ (here $\mathrm{d}(\mathrm{inv})_e = -\mathrm{Id}_{\mathfrak{g}}$ plays a role).
- conjugation, we have $X^\# = X^R - X^L$.

(3) For the action $\mathbb{R} \circlearrowright \mathbb{T}^2$ coming from a parameter $\alpha \in \mathbb{R}$, we have that

$$1^\#_{(x,y)} = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (x+t, y+\alpha t) \bmod 1 = \frac{\partial}{\partial\theta_1}\bigg|_{(x,y)} + \alpha\frac{\partial}{\partial\theta_2}\bigg|_{(x,y)}.$$

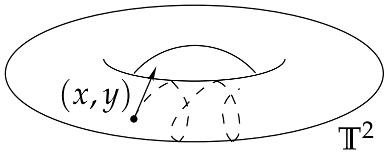


Figure 10: Infinitesimal generator for $\mathbb{R} \circlearrowright \mathbb{T}^2$.

Exercise 34

(a) Let $G_1 \circlearrowright M_1$ and $G_2 \circlearrowright M_2$ be two smooth Lie group actions, $\psi\colon G_1 \to G_2$ be a Lie group homomorphism, and $F\colon M_1 \to M_2$ be **ψ-equivariant**, that is, it satisfies $F(g \cdot x) = \psi(g) \cdot F(x)$, for all $g \in G_1$ and $x \in M_1$. Show that for each $X \in \mathfrak{g}_1$ and $x \in M_1$, we have $\mathrm{d}F_x(X^\#_{M_1}|_x) = \mathrm{d}\psi_e(X)^\#_{M_2}|_{F(x)}$.

(b) Conclude that if $G \circlearrowright Q$ is a smooth Lie group action, then for each $X \in \mathfrak{g}$, the fields $X^\#_{TQ}$ and $X^\#_{T^*Q}$ project to $X^\#_Q$ under the natural bundle projections.

Now, let's get back to the symplectic category.

Definition 20

Let (M, ω) be a symplectic manifold and G be a Lie group. A smooth action $G \circlearrowright M$ is called a **symplectic action** if for each $g \in G$, the action map $g\colon M \to M$ is a symplectomorphism. We'll denote a symplectic action by $G \circlearrowright (M, \omega)$.

Remark. This is equivalent to requiring the homomorphism $G \to \mathrm{Diff}(M)$ to be actually valued in $\mathrm{Sp}(M, \omega)$.

As an immediate consequence of the definition, since the flows of the action fields are given by $\Phi_{t,X^\#}(x) = \exp(tX) \cdot x$ (which in particular shows that $X^\#$ is complete), we have from Theorem 8 (p. 29) that all the action fields of a symplectic action are... symplectic.

Example 28

(1) If (M, ω, H) is a compact Hamiltonian system, we obtain $\mathbb{R} \circlearrowright (M, \omega)$ via $t \cdot x \doteq \Phi_{t,X_H}(x)$.

(2) More generally, if (M, ω) is a compact symplectic manifold and we have functions $f_1, \ldots, f_k \in \mathscr{C}^\infty(M)$ in involution, that is, $\{f_i, f_j\}_\omega = 0$ for all indices $i, j = 1, \ldots, k$, we obtain an action $\mathbb{R}^k \circlearrowright (M, \omega)$ by

$$(t_1, \cdots, t_k) \cdot x \doteq \Phi_{t_1, X_{f_1}} \circ \cdots \circ \Phi_{t_k, X_{f_k}}(x).$$

(3) Identify $\mathbb{R}^{2n} \cong \mathbb{C}^n$, so that we get an action $\mathbb{T}^n \circlearrowright (\mathbb{R}^{2n}, \omega_{2n})$ via

$$(\mathrm{e}^{\mathrm{i}\theta_1}, \cdots, \mathrm{e}^{\mathrm{i}\theta_n}) \cdot (z^1, \ldots, z^n) \doteq (\mathrm{e}^{\mathrm{i}\theta_1} z^1, \cdots, \mathrm{e}^{\mathrm{i}\theta_n} z^n).$$

By the way, you can check that in complex coordinates, one has

$$\omega_{2n} = \frac{\mathrm{i}}{2}\sum_{k=1}^{n} \mathrm{d}z^k \wedge \mathrm{d}\bar{z}^k.$$

(4) If $G \circlearrowright Q$, then $G \circlearrowright (T^*Q, \omega_{\mathrm{can}})$ by Proposition 4 (p. 20). And clearly, if $B \in \Omega^2(Q)$ is closed, we'll have $G \circlearrowright (T^*Q, \omega_B)$ if and only if the original action $G \circlearrowright Q$ preserves B.

With this in place, the next natural question is whether if given $G \circlearrowright (M, \omega)$, free and proper, ω induces a symplectic form in the orbit space M/G (which is a smooth manifold, under the given assumptions). The answer is an unfortunate "no", as $\dim(M/G) = \dim M - \dim G$ might not be even. Thus we are forced to move on to the next best thing that could survive in the quotient with good properties: the Poisson bracket $\{\cdot,\cdot\}_\omega$.

Theorem 15

Let $G \circlearrowright (M, \omega)$ be given. Then the **algebra $\mathscr{C}^\infty(M)^G$ of G-invariant functions** is closed under $\{\cdot,\cdot\}_\omega$. Hence, if $G \circlearrowright M$ is free and proper, there is a unique Poisson bracket $\{\{\cdot,\cdot\}\}$ on M/G for which the quotient projection $M \to M/G$ becomes a Poisson morphism.

Proof: By Exercise 24 (p. 41), for all $f_1, f_2 \in \mathscr{C}^\infty(M)^G$ and $g \in G$, we have that

$$\{f_1, f_2\}_\omega \circ g \overset{(*)}{=} \{f_1 \circ g, f_2 \circ g\}_\omega \circ g = \{f_1, f_2\}_{g^*\omega} = \{f_1, f_2\}_\omega,$$

where in $(*)$ we use invariance of f_1 and f_2. The rest is clear once we establish the correspondence $\mathscr{C}^\infty(M)^G \cong \mathscr{C}^\infty(M/G)$. But this is just the characteristic property of surjective submersions. Namely, each $f \in \mathscr{C}^\infty(M)^G$ passes to the quotient giving a function $\widetilde{f} \in \mathscr{C}^\infty(M/G)$, while composing any given $\widetilde{f} \in \mathscr{C}^\infty(M/G)$ with the quotient projection $M \to M/G$ gives $f \in \mathscr{C}^\infty(M)^G$ — these two processes are inverses to each other. With this in place, there's nothing else to do: define

$$\{\{\widetilde{f}_1, \widetilde{f}_2\}\}(G \cdot x) \doteq \{f_1, f_2\}_\omega(x),$$

where the representative x of $G \cdot x$ is chosen at will. □

Remark. The above result should be regarded as a little factory of Poisson manifolds from symplectic manifolds, on its own right: just find the good Lie groups. In fact, nothing changes if one replaces (M, ω) with some Poisson manifold $(P, \{\cdot,\cdot\})$ instead, assuming that the action $G \circlearrowright P$ itself is a **Poisson action**, in the sense that every action map $g\colon P \to P$ preserves $\{\cdot,\cdot\}$.

Example 29

Let G be a Lie group acting on itself by left translations (which is clearly free and proper), and also consider the induced action $G \circlearrowright T^*G$ via cotangent lifts. Let's fully understand the quotient T^*G/G. For that, fix the global trivialization $\Phi\colon G \times \mathfrak{g}^* \to T^*G$, given by $\Phi(g,p) \doteq (g, p \circ \mathrm{d}(\mathrm{L}_g)_e^{-1})$. Using Φ to pull-back the action on T^*G to $G \times \mathfrak{g}^*$, we compute

$$\begin{aligned} g\cdot(h,p) &= \Phi^{-1}(g\cdot\Phi(h,p)) = \Phi^{-1}(g\cdot(h, p\circ \mathrm{d}(\mathrm{L}_h)_e^{-1})) \\ &= \Phi^{-1}(gh, p\circ \mathrm{d}(\mathrm{L}_h)_e^{-1}\circ \mathrm{d}(\mathrm{L}_g)_h^{-1}) = \Phi^{-1}(gh, p\circ \mathrm{d}(\mathrm{L}_{gh})_e^{-1}) \\ &= \Phi^{-1}\Phi(gh,p) = (gh,p). \end{aligned}$$

That is, Φ undoes the twisting of the cotangent spaces. Since the action only happens on the first factor, it follows that

$$T^*G/G \xrightarrow{\ \Phi^{-1}\ } (G\times\mathfrak{g}^*)/G \xrightarrow{\ \mathrm{pr}_2\ } \mathfrak{g}^*,$$

but here's the catch: this diffeomorphism is Poisson, when we equip $\mathfrak{g}^*$ with its standard Lie-Poisson structure (of course, by Φ^{-1} and pr_2 above we understand the induced maps on the quotients). Note that

$$\mathscr{C}^\infty((G\times\mathfrak{g}^*)/G) \cong \mathscr{C}^\infty(G\times\mathfrak{g}^*)^G \ni f \mapsto f(e,\cdot) \in \mathscr{C}^\infty(\mathfrak{g}^*)$$

is an isomorphism. With this, it would suffice to check that for all $X, Y \in \mathfrak{g}$, one has $\{\hat{\hat{X}}, \hat{\hat{Y}}\}_{\Phi^*\omega}(g,p) = p([X,Y])$, where $\hat{\hat{X}}\colon G\times\mathfrak{g}^* \to \mathbb{R}$ is given by $\hat{\hat{X}}(g,p) = p(X)$, and this can be done in several ways (think about what is the most efficient one).

Exercise 35

In the setting of the above example, with the same notation, show that the "tautological form" $\Phi^*\lambda \in \Omega^1(G\times\mathfrak{g}^*)$ is given by $(\Phi^*\lambda)_{(g,p)} = p\circ(\mathrm{d}g_{(e,p)})^{-1}$, where by g we mean, as usual, the action $g\colon G\times\mathfrak{g}^* \to G\times\mathfrak{g}^*$ of the element $g\in G$.

3.3 Moment maps and Noether's Theorem

Recall that if a symplectic manifold (M,ω) has $H^1_{\mathrm{dR}}(M) = 0$, then every symplectic field is Hamiltonian. If a Lie group G acts on such (M,ω), then every action field $X^\#$ is Hamiltonian. It seems natural to try and collect all the Hamiltonian functions realizing these fields into a single map. So, assume that there is a map $\mu_c\colon \mathfrak{g} \to \mathscr{C}^\infty(M)$ for which $\boldsymbol{X}_{\mu_c^X} = X^\#$ for all $X \in \mathfrak{g}$, where we write μ_c^X for what would be $\mu_c(X)$ — let's say that if such map μ_c exists, the action is **weakly Hamiltonian**. This implies that for each $X, Y \in \mathfrak{g}$, the relation

$$\boldsymbol{X}_{\mu_c^{[X,Y]}} = [X,Y]^\# = -[X^\#, Y^\#] = -[\boldsymbol{X}_{\mu_c^X}, \boldsymbol{X}_{\mu_c^Y}] = \boldsymbol{X}_{\{\mu_c^X, \mu_c^Y\}_\omega}$$

holds, by exercises 25 (p. 41) and 33 (p. 51). In a perfect world, this would imply that $\mu_c^{[X,Y]} = \{\mu_c^X, \mu_c^Y\}_\omega$, but we can only say that those functions differ by a constant, which depends on X and Y themselves. This cannot be helped.

Definition 21

A symplectic action $G \circlearrowright (M,\omega)$ is called a **Hamiltonian action** if there exists a linear map $\mu_c\colon \mathfrak{g} \to \mathscr{C}^\infty(M)$, to be called **comoment map**, such that the diagram

$$\begin{array}{ccc} (\mathfrak{g}, [\cdot,\cdot]) & \xrightarrow{\ \mu_c\ } & (\mathscr{C}^\infty(M), \{\cdot,\cdot\}_\omega) \\ \Big\downarrow \# & & \Big\downarrow \boldsymbol{X}_\bullet \\ (\mathcal{S}(M,\omega), [\cdot,\cdot]) & \hookleftarrow & (\mathscr{H}(M,\omega), [\cdot,\cdot]) \end{array}$$

commutes, where $\mathcal{S}(M,\omega)$ and $\mathscr{H}(M,\omega)$ stand (as before), respectively, for the algebras of symplectic and Hamiltonian vector fields on (M,ω), and the vertical maps are anti-homomorphisms. The associated **moment map** is the function $\mu\colon M \to \mathfrak{g}^*$ given by $\mu(x)(X) \doteq \mu_c^X(x)$, and μ and μ_c carry the same information. We say that the data (M,ω,G,μ) forms a **Hamiltonian G-space**.

Remark. Digression. Moment/comoment maps need not exist for a given symplectic action and, when they do, they might not be unique. One possible tool to study this question is the so-called **Chevalley-Eilenberg cohomology** of a Lie algebra. Here's the quick gist of it: let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ in some vector space V. For each $0 \le k \le \dim \mathfrak{g}$, let $\mathscr{A}_\rho^k(\mathfrak{g}; V)$ denote the space of V-valued alternating k-linear maps on $\mathfrak{g}$, and define the **Chevalley-Eilenberg derivative** $\mathrm{d}\colon \mathscr{A}_\rho^k(\mathfrak{g};V) \to \mathscr{A}_\rho^{k+1}(\mathfrak{g};V)$ by mimicking the classical Palais formula:

$$\begin{aligned} \mathrm{d}\omega(X_0,\ldots,X_k) &= \sum_{i=0}^k (-1)^i \rho(X_i)\omega(X_0,\ldots,\widehat{X}_i,\ldots,X_k) \\ &\quad + \sum_{0\le i<j\le k} (-1)^{i+j}\omega([X_i,X_j], X_0,\ldots,\widehat{X}_i,\ldots,\widehat{X}_j,\ldots,X_k). \end{aligned}$$

Then we have the complex

$$\cdots \xrightarrow{\ \mathrm{d}\ } \mathscr{A}_\rho^{k-1}(\mathfrak{g};V) \xrightarrow{\ \mathrm{d}\ } \mathscr{A}_\rho^k(\mathfrak{g};V) \xrightarrow{\ \mathrm{d}\ } \mathscr{A}_\rho^{k+1}(\mathfrak{g};V) \xrightarrow{\ \mathrm{d}\ } \cdots$$

and $\mathrm{d}^2 = 0$, meaning that if we set $Z_\rho^k(\mathfrak{g};V)$ and $B_\rho^k(\mathfrak{g};V)$ as the kernel and range of the correct d's, we can form the cohomology $H_\rho^k(\mathfrak{g};V) \doteq Z_\rho^k(\mathfrak{g};V)/B_\rho^k(\mathfrak{g};V)$.

Exercise 36

Fix a Lie algebra $\mathfrak{g}$ and its adjoint representation $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. We have that $[\cdot,\cdot] \in \mathscr{A}^2_{\mathrm{ad}}(\mathfrak{g};\mathfrak{g})$. Show that the bracket $[\cdot,\cdot]$ is closed and defines the trivial class in $H^2_{\mathrm{ad}}(\mathfrak{g};\mathfrak{g})$. **Hint:** there is a (somewhat) obvious $f \in \mathscr{A}^1_{\mathrm{ad}}(\mathfrak{g};\mathfrak{g})$ for which $\mathrm{d}f = [\cdot,\cdot]$.

For example, $H^1_\rho(\mathfrak{g};V)$ controls the existence of ρ-invariant complements for ρ-invariant subspaces of V, and $H^2_\rho(\mathfrak{g};V)$ (for a certain ρ) controls whether $\mathfrak{g}$ admits abelian extensions. **Whitehead's Lemmas** state that if $\mathfrak{g}$ is semi-simple (that is, its Killing form is non-degenerate) then both $H^1_\rho(\mathfrak{g};V)$ and $H^2_\rho(\mathfrak{g};V)$ are trivial (in particular, the **Weyl's reducibility theorem** for semi-simple Lie algebras follows from this). As far as moment maps go, assume again that we have $G \circlearrowright (M,\omega)$ and a linear map $\mu_{\mathrm{c}}\colon \mathfrak{g} \to \mathscr{C}^\infty(M)$ such that for each $X \in \mathfrak{g}$, μ_{c}^X is a Hamiltonian function for $X^\#$, but no more, and indeed write $\mu_{\mathrm{c}}^{[X,Y]} = \{\mu_{\mathrm{c}}^X, \mu_{\mathrm{c}}^Y\}_\omega + \zeta_\mu(X,Y)$, for some $\zeta_\mu(X,Y) \in \mathbb{R}$.

Exercise 37

Show that the map $\zeta_\mu\colon \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ satisfies:

(a) ζ_μ is bilinear;

(b) $\zeta_\mu(Y,X) = -\zeta_\mu(X,Y)$ for all $X,Y \in \mathfrak{g}$;

(c) $\zeta_\mu(X,[Y,Z]) + \zeta_\mu(Y,[Z,X]) + \zeta_\mu(Z,[X,Y]) = 0$ for all $X,Y,Z \in \mathfrak{g}$.

Hence $\zeta_\mu \in Z^2_{\rho_0}(\mathfrak{g};\mathbb{R})$, where $\rho_0\colon \mathfrak{g} \to \mathfrak{gl}(\mathbb{R})$ is the trivial representation.
Note: item (c) says precisely that $\mathrm{d}(\zeta_\mu) = 0$.

It turns out that μ_{c} can be "corrected" to a legitimate comoment map if and only if the cohomology class $[\zeta_\mu]$ is trivial in $H^2_{\rho_0}(\mathfrak{g};\mathbb{R})$. Namely, we just add a primitive for ζ_μ to μ. If $[\zeta_\mu]$ is non-trivial, there is a way to make it trivial by passing to a certain central extension of $\mathfrak{g}$ (in other words, one must enlarge $\mathfrak{g}$) — this is related to phenomena of group quantization, see [23] and the other references in [33] for more details. One useful conclusion of all of this is the:

Theorem 16

A symplectic action $G \circlearrowright (M,\omega)$, where $H^1_{\mathrm{dR}}(M) = 0$ and G is semi-simple, is Hamiltonian.

As for the uniqueness, we'll just mention that two "comoment maps" for a weakly Hamiltonian action differ by an element of the annihilator of the commutant ideal $[\mathfrak{g},\mathfrak{g}]$. End of digression.

With this in place, let's revisit a few of the previous examples of symplectic actions.

Example 30

(1) Consider a symplectic manifold (M, ω) and a smooth function $H\colon M \to \mathbb{R}$ for which the Hamiltonian field $\boldsymbol{X}_H$ is complete, and define the action $\mathbb{R} \circlearrowleft (M, \omega)$ via Hamiltonian flow: $t \cdot x = \Phi_{t,\boldsymbol{X}_H}(x)$. We know that $\mathrm{Lie}(\mathbb{R}) = \mathbb{R}$, so given $a \in \mathbb{R}$, we can note that $\Phi_{ta,\boldsymbol{X}_H} = \Phi_{t,a\boldsymbol{X}_H} = \Phi_{t,\boldsymbol{X}_{aH}}$ and compute

$$(a^\#)_x = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}(ta)\cdot x = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\Phi_{ta,\boldsymbol{X}_H}(x) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\Phi_{t,\boldsymbol{X}_{aH}}(x) = \boldsymbol{X}_{aH}\big|_x,$$

so that $\mu_{\mathrm{c}}\colon \mathbb{R} \to \mathscr{C}^\infty(M)$ given by $\mu_{\mathrm{c}}^a = aH$ works. It is a homomorphism since $\{\mu_{\mathrm{c}}^a, \mu_{\mathrm{c}}^b\}_\omega = ab\{H, H\}_\omega = 0$ for all $a, b \in \mathbb{R}$ and $\mathbb{R}$ is abelian. The corresponding moment map $\mu\colon M \to \mathbb{R}^*$ takes a point $x \in M$ to the map $\mu(x)\colon \mathbb{R} \to \mathbb{R}$ given by multiplication by $H(x)$ (so in this sense, $\mu = H$).

(2) Consider the circle action $\mathbb{S}^1 \circlearrowleft (\mathbb{C}^n, \omega_{2n})$ given just by scalar multiplication: $\mathrm{e}^{\mathrm{i}\theta}\cdot(z^1, \ldots, z^n) = (\mathrm{e}^{\mathrm{i}\theta}z^1, \ldots, \mathrm{e}^{\mathrm{i}\theta}z^n)$. In other words, compose the diagonal inclusion $\mathbb{S}^1 \hookrightarrow \mathbb{T}^n$ with the action seen in Example 28 (p. 52). Note that $\mathrm{Lie}(\mathbb{S}^1) = \mathrm{i}\mathbb{R}$:

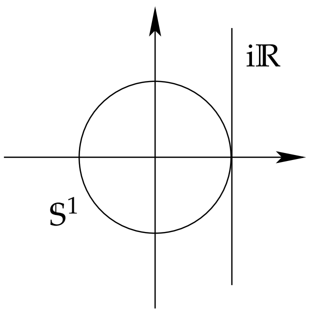


Figure 11: Lie algebra of the circle group.

A direct computation (being careful with the identification $\mathbb{R}^{2n} \cong \mathbb{C}^n$) shows that if $\mathrm{i}a \in \mathrm{i}\mathbb{R}$, then

$$(\mathrm{i}a)^\# = \sum_{k=1}^n \left(\mathrm{i}az^k\frac{\partial}{\partial z^k} - \mathrm{i}a\bar{z}^k\frac{\partial}{\partial \bar{z}^k}\right),$$

so that

$$\begin{aligned}\iota_{(\mathrm{i}a)^\#}\omega_{2n} &= \omega_{2n}((\mathrm{i}a)^\#,\cdot) = \frac{\mathrm{i}}{2}\sum_{k=1}^n (\mathrm{d}z^k\wedge \mathrm{d}\overline{z}^k)((\mathrm{i}a)^\#,\cdot)\\ &= \frac{\mathrm{i}}{2}\sum_{k=1}^n(\mathrm{i}az^k\,\mathrm{d}\overline{z}^k - (-\mathrm{i}a\overline{z}^k\,\mathrm{d}z^k)) = -\frac{a}{2}\sum_{k=1}^n(z^k\,\mathrm{d}\overline{z}^k+\overline{z}^k\,\mathrm{d}z^k)\\ &= -\frac{a}{2}\sum_{k=1}^n \mathrm{d}(|z^k|^2) = \mathrm{d}\left(-\frac{a}{2}\sum_{k=1}^n|z^k|^2\right).\end{aligned}$$

This suggests that this circle action is indeed Hamiltonian with comoment map $\mu_{\mathrm{c}}\colon \mathrm{i}\mathbb{R}\to\mathscr{C}^\infty(\mathbb{C}^n)$ given by

$$\mu_{\mathrm{c}}^{\mathrm{i}a}(\boldsymbol{z}) = -\frac{a\|\boldsymbol{z}\|^2}{2}.$$

You can directly check that $\{\mu_{\mathrm{c}}^{\mathrm{i}a},\mu_{\mathrm{c}}^{\mathrm{i}b}\}_{\omega_{2n}} = 0$ for all $a,b\in\mathbb{R}$.

(3) Let $(M,-\mathrm{d}\lambda)$ be an exact symplectic manifold and assume that $G \circlearrowright (M,-\mathrm{d}\lambda)$ is an action not only preserving $\omega=-\mathrm{d}\lambda$, but the primitive λ as well (clearly the case of interest is when λ is indeed the tautological form on some cotangent bundle $M=T^*Q$). We claim that this already ensures that the action is Hamiltonian. Namely, given $X\in\mathfrak{g}$, Cartan's homotopy formula comes into play again:

$$0=\mathscr{L}_{X^\#}\lambda = \iota_{X^\#}\mathrm{d}\lambda + \mathrm{d}\iota_{X^\#}\lambda \implies \iota_{X^\#}\omega = \mathrm{d}(\lambda(X^\#)).$$

This leads us to consider $\mu_{\mathrm{c}}\colon\mathfrak{g}\to\mathscr{C}^\infty(M)$ given by $\mu_{\mathrm{c}}^X = \lambda(X^\#)$. Let's verify that this is a comoment map. For that, start noting that for any $X,Y\in\mathfrak{g}$, we have

$$X^\#(\lambda(Y^\#)) = X^\#(\mu_{\mathrm{c}}^Y) = \mathrm{d}\mu_{\mathrm{c}}^Y(X^\#) = \omega(X_{\mu_{\mathrm{c}}^Y},X^\#) = -\omega(X^\#,Y^\#).$$

Now, we use the definition of exterior derivative to compute

$$\begin{aligned}\{\mu_{\mathrm{c}}^X,\mu_{\mathrm{c}}^Y\}_\omega &= \omega(X^\#,Y^\#)\\ &= Y^\#(\lambda(X^\#)) - X^\#(\lambda(Y^\#)) + \lambda([X^\#,Y^\#])\\ &= \omega(X^\#,Y^\#) - \omega(Y^\#,X^\#) - \lambda([X,Y]^\#),\end{aligned}$$

so that cancelling one $\omega(X^\#,Y^\#)$ on both sides and reusing the very first line of the computation, it follows that $\{\mu_{\mathrm{c}}^X,\mu_{\mathrm{c}}^Y\}_\omega = \mu_{\mathrm{c}}^{[X,Y]}$, as required. Lastly, note that we may write the moment map $\mu\colon M\to\mathfrak{g}^*$ as $\mu(x) = (\mathcal{O}_x^*\lambda)_e$, where $\mathcal{O}_x\colon G\to M$ is the orbit map of x.

Remark. Note that if a smooth action $G\circlearrowright Q$ preserves some $A\in\Omega^1(Q)$, then the action $G\circlearrowright (T^*Q,\omega_B)$ induced via cotangent lifts, where $B=\mathrm{d}A$, is Hamiltonian with comoment map $\mu_{\mathrm{c}}\colon\mathfrak{g}\to\mathscr{C}^\infty(T^*Q)$ given by $\mu_{\mathrm{c}}^X(x,p) = (p-A_x)(X_Q^\#\big|_x)$.

Exercise 38

Let (M,ω) be a compact symplectic manifold, $f_1,\dots,f_k \in \mathscr{C}^\infty(M)$ be in involution, and consider the action $\mathbb{R}^k \circlearrowright (M,\omega)$ given by composition of Hamiltonian flows, as in Example 28 (p. 52). Namely, set

$$(t_1,\dots,t_k)\cdot x = \Phi_{t_1,X_{f_1}} \circ \cdots \circ \Phi_{t_k,X_{f_k}}(x).$$

Show that this action is Hamiltonian with comoment map $\mu_c\colon \mathbb{R}^k \to \mathscr{C}^\infty(M)$ given by

$$\mu_c^{(a_1,\dots,a_k)} = \sum_{i=1}^k a_i f_i$$

and moment map $\mu\colon M \to (\mathbb{R}^k)^*$ given by $\mu(x) = \langle \boldsymbol{f}(x),\cdot\rangle$, where $\boldsymbol{f} = (f_1,\dots,f_k)$ and $\langle\cdot,\cdot\rangle$ is the standard inner product in $\mathbb{R}^k$. **Hint:** to show that μ_c is a Poisson map, the involution assumption is crucial.

Let's look at one last example, of mechanical flavor:

Example 31 (The reason for the name "co/moment" map)

Consider $\mathbb{R}^3$ equipped with its standard Euclidean inner product $\langle\cdot,\cdot\rangle$, and consider the diagonal derivative action $\mathrm{SO}(3) \circlearrowright T\mathbb{R}^3 \cong \mathbb{R}^6$, given by

$$R\cdot(\boldsymbol{x},\boldsymbol{y}) \doteq (R\boldsymbol{x}, R\boldsymbol{y}).$$

Here, $\mathbb{R}^6$ is equipped with the standard symplectic form ω_6. Recall that we have a natural isomorphism $(\mathfrak{so}(3),[\cdot,\cdot]) \cong (\mathbb{R}^3,\times)$, via the map

$$\mathbb{R}^3 \ni (a,b,c) \mapsto \mathsf{A}_{(a,b,c)} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \in \mathfrak{so}(3)$$

That is, $\mathsf{A}_{(a,b,c)}$ is the matrix of $(a,b,c)\times\_$ relative to the standard basis of $\mathbb{R}^3$. The inner product on $\mathbb{R}^3$ then induces an isomorphism $\mathfrak{so}(3) \cong \mathfrak{so}(3)^*$. We claim that this action is Hamiltonian with comoment map $\mu_c\colon \mathbb{R}^3 \to \mathscr{C}^\infty(\mathbb{R}^6)$ given by

$$\mu_c^{\boldsymbol{v}}(\boldsymbol{x},\boldsymbol{y}) = \langle \boldsymbol{v}, \boldsymbol{x}\times\boldsymbol{y}\rangle.$$

A quick way to see this is to compute, for each $\boldsymbol{v}\in\mathbb{R}^3$, the action field $\boldsymbol{v}^\#$ as $\boldsymbol{v}^\#_{(\boldsymbol{x},\boldsymbol{y})} = (\boldsymbol{v}\times\boldsymbol{x}, \boldsymbol{v}\times\boldsymbol{y})$ (the total derivative of a linear map is itself), and to note that for any function $f\in\mathscr{C}^\infty(\mathbb{R}^6)$, if $\nabla f = (\nabla_{\boldsymbol{x}} f, \nabla_{\boldsymbol{y}} f)$, then $\boldsymbol{X}_f = (\nabla_{\boldsymbol{y}} f, -\nabla_{\boldsymbol{x}} f)$, so that $(\nabla \mu_c^{v})_{(x,y)} = (y \times v, v \times x)$ leads to $X_{\mu_c^v} = v^\#$ as required. So, with all due isomorphisms, the moment map $\mu\colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ appears as $\mu(x,v) = x \times y$, which is the actual **angular momentum**.

Here's one last exercise describing a simple construction providing more examples:

Exercise 39

Let $(M_1, \omega_1, G, \mu_1)$ and $(M_2, \omega_2, G, \mu_2)$ be two Hamiltonian G-spaces. Show that $(M_1 \times M_2, \omega_1 \oplus \omega_2, G, \mu_1 \oplus \mu_2)$ is also a Hamiltonian G-space, where the action of G on $M_1 \times M_2$ is diagonal and the new moment $\mu_1 \oplus \mu_2\colon M_1 \times M_2 \to \mathfrak{g}^*$ is given by $(\mu_1 \oplus \mu_2)(x,y) \doteq \mu_1(x) + \mu_2(y)$.

We have seen that comoment maps are Poisson maps. It turns out that moment maps are also Poisson maps.

Exercise 40

For a Hamiltonian G-space (M, ω, G, μ), show that the moment map $\mu\colon M \to \mathfrak{g}^*$ is a Poisson map, that is,

$$\{f \circ \mu, g \circ \mu\}_\omega = \{f, g\}_{\mathfrak{g}^*} \circ \mu$$

holds for all $f, g \in \mathscr{C}^\infty(\mathfrak{g}^*)$. Here, $\{\cdot,\cdot\}_{\mathfrak{g}^*}$ is the standard Lie-Poisson structure on $\mathfrak{g}^*$, as seen in Exercise 30 (p. 47). **Hint:** It suffices to show it holds for $f = \widehat{X}$ and $g = \widehat{Y}$, where $X, Y \in \mathfrak{g}$ and $\widehat{X}$ and $\widehat{Y}$ are the natural images of X and Y in $\mathfrak{g}^{**}$ (hence maps $\mathfrak{g}^* \to \mathbb{R}$). Use that $\mu_c^X = \widehat{X} \circ \mu$.

To further explore the properties of moment maps, let's introduce a useful little device: if (M, ω, G, μ) is a Hamiltonian G-space, for each $x \in M$ let $\theta_x\colon \mathfrak{g} \to T_x^*M$ be given by $\theta_x(X) = \omega_x(X_x^\#, \cdot)$. In other words, we have that

$$\theta_x(X)(v) = \omega_x(X_x^\#, v) = \mathrm{d}(\mu_c^X)_x(v) = \mathrm{d}\mu_x(v)X,$$

so θ_x is the dual of the derivative $\mathrm{d}\mu_x\colon T_xM \to \mathfrak{g}^*$. Thus, simple linear algebra gives us the:

Proposition 8

Let (M, ω, G, μ) be a Hamiltonian G-space. Then $\mu\colon M \to \mathfrak{g}^*$ is a submersion at $x \in M$ if and only if the stabilizer G_x is discrete.

Proof: Just note that $\mathrm{Ran}(\mathrm{d}\mu_x) = \mathrm{Ann}(\ker \theta_x) = \mathrm{Ann}(\mathfrak{g}_x)$, since ω_x is non-degenerate and we use item (b) of Exercise 33 (p. 51). So $\mathrm{d}\mu_x$ is surjective if and only if $\mathfrak{g}_x = 0$. The conclusion follows because we assume that G is connected. □

The next natural thing to do would be to analyze the kernel of $\mathrm{d}\mu_x$.

Proposition 9

Let (M,ω,G,μ) be a Hamiltonian G-space. Then for each $x \in M$, we have that $\ker \mathrm{d}\mu_x$ and $T_x(G\cdot x)$ are ω_x-orthogonal.

Proof: Recall that $T_x(G\cdot x) = \{X^\#_x \mid X \in \mathfrak{g}\}$ (this is a general fact about Lie group actions on smooth manifolds), so it suffices to note that if $X \in \mathfrak{g}$ and $v \in \ker \mathrm{d}\mu_x$, we have $\omega_x(X^\#_x, v) = \mathrm{d}\mu_x(v)X = 0$, because $\mathrm{d}\mu_x(v)$ is the zero linear functional. □

Let's leave no stone unturned:

Proposition 10

Let (M,ω,G,μ) be a Hamiltonian G-space. Then the moment map $\mu\colon M \to \mathfrak{g}^*$ is G-equivariant, where in $\mathfrak{g}^*$ we consider the **coadjoint action** $G \circlearrowright \mathfrak{g}^*$ given by $g\cdot p \doteq p\circ \mathrm{Ad}(g^{-1})$. Namely, we have that

$$\mu(g\cdot x) = \mu(x)\circ \mathrm{Ad}(g^{-1})$$

for all $x\in M$ and $g \in G$.

Proof: Since G is connected, we may proceed infinitesimally, and just prove that

$$\mathrm{d}\mu_x(X^\#_M|_x) = X^\#_{\mathfrak{g}^*}|_{\mu(x)},$$

by showing that both sides yield the same result when tested against an arbitrary $Y \in \mathfrak{g}$. On one hand, we have that

$$\begin{aligned}\mathrm{d}\mu_x(X^\#_M|_x)Y &= \theta_x(Y)(X^\#_M|_x) = \omega_x(Y^\#_M|_x, X^\#_M|_x)\\ &= \{\mu_c^Y,\mu_c^X\}_\omega(x) = \mu_c^{[Y,X]}(x)\\ &= \mu(x)([Y,X]).\end{aligned}$$

On the other hand, we may take an arbitrary element $p \in \mathfrak{g}^*$, the 1-parameter subgroup $\gamma_X\colon \mathbb{R}\to G$ given by $\gamma_X(t) = \exp(tX)$, and compute

$$\begin{aligned}X^\#_{\mathfrak{g}^*}|_p &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\gamma_X(t)\cdot p = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} p\circ \mathrm{Ad}(\gamma_X(t)^{-1})\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} p\circ \mathrm{Ad}(\gamma_{-X}(t)) = p\circ \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathrm{Ad}(\gamma_{-X}(t))\\ &= p\circ \mathrm{ad}(-X) = p([\cdot, X]),\end{aligned}$$

concluding the proof. □

Exercise 41

Let G be a Lie group. Show that the coadjoint action $G \circlearrowright \mathfrak{g}^*$ is a Poisson action, when we equip $\mathfrak{g}^*$ with its standard Lie-Poisson structure. Namely, show that for all $f_1, f_2 \in \mathscr{C}^\infty(\mathfrak{g}^*)$ and $g\in G$, we have $\{f_1\circ g, f_2\circ g\}_{\mathfrak{g}^*} = \{f_1,f_2\}_{\mathfrak{g}^*}\circ g$.

Hint: Again it suffices to show that the conclusion holds for $f = \widehat{X}$ and $g = \widehat{Y}$, where $X, Y \in \mathfrak{g}$. Verify that $\widehat{X} \circ g = \widehat{\mathrm{Ad}(g^{-1})X}$.

Remark. For dimension reasons, a Lie algebra need not carry a symplectic form. It turns out that the **coadjoint orbits** carry natural symplectic forms — they're called the **Kirillov-Kostant-Souriau forms** (KKS forms, for short). The coadjoint action restricted to each orbit is Hamiltonian, and the moment map is given by the inclusion into $\mathfrak{g}^*$.

Let's conclude this section with one of the most powerful theorems in Mathematical Physics.

Theorem 17 (Noether)

Let (M, ω, G, μ) be a Hamiltonian G-space, and $H \in \mathscr{C}^\infty(M)$ be G-invariant. Then μ is constant along the integral curves of the Hamiltonian field $\boldsymbol{X}_H$.

Proof: Since H is G-invariant, its derivative kills all action fields. So, let $\gamma\colon I \to M$ be an integral curve of $\boldsymbol{X}_H$. For any $X \in \mathfrak{g}$, compute:

$$\begin{aligned}(\mu \circ \gamma)'(t)X &= \mathrm{d}\mu_{\gamma(t)}(\gamma'(t))X = \theta_{\gamma(t)}(X)(\gamma'(t)) \\ &= \omega_{\gamma(t)}(X^\#_{\gamma(t)}, \gamma'(t)) = \omega_{\gamma(t)}(X^\#_{\gamma(t)}, \boldsymbol{X}_H|_{\gamma(t)}) \\ &= -\mathrm{d}H_{\gamma(t)}(X^\#_{\gamma(t)}) = 0.\end{aligned}$$

□

The fact that a powerful and celebrated theorem like this has such an easy proof should not be cause for concern. This Hamiltonian formulation essentially sweeps the dirt under the rug, as applying this theorem to some mechanical system requires not only recognizing the underlying symplectic structure, but also recognizing a convenient group G of symmetries acting on the configuration or phase space. It goes by the slogan **"symmetries generate conservation laws"**.

As our first example of how to apply this, let's put together everything we have learned so far.

Example 32 (Killing fields)

Let (Q, g) be a Riemannian manifold, and $\boldsymbol{X} \in \mathfrak{X}(Q)$ be a **Killing field** (that is, each stage of the flow of $\boldsymbol{X}$ is an isometry of Q or, equivalently, the covariant differential $\nabla \boldsymbol{X}$ is skew-adjoint, where ∇ is the Levi-Civita connection of (Q, g)). Let's assume, for simplicity, that $\boldsymbol{X}$ is complete[a], so we obtain an isometric action $\mathbb{R} \circlearrowright (Q, \mathrm{g})$ given by its flow. This action lifts to an action $\mathbb{R} \circlearrowright (T^*Q, \omega_{\mathrm{can}})$, which is Hamiltonian by item (3) of Example 30 (p. 57), with moment map $\mu\colon T^*Q \to \mathbb{R}^*$ given by

$$\mu(x, p)a = \lambda_{(x,p)}(a^\#_{T^*Q}|_{(x,p)}) = p(a^\#_Q|_x) = p(a\boldsymbol{X}_x) = ap(\boldsymbol{X}_x).$$

The Hamiltonian $H\colon T^*Q \to \mathbb{R}$ given by $H(x, p) = \mathrm{g}_x(p, p)/2$ is $\mathbb{R}$-invariant, and we know that if $(x,p)\colon I \to T^*Q$ is an integral curve of $\boldsymbol{X}_H$, then $x\colon I \to Q$ is a geodesic, by Example 19 (p. 27). Noether's conclusion is that if $\boldsymbol{X} \in \mathfrak{X}(Q)$ is a Killing field and $x\colon I \to Q$ is a geodesic, then the function $t \mapsto \mathrm{g}_{x(t)}(\boldsymbol{X}_{x(t)}, \dot{x}(t))$ is constant, since if $p \in T^*_xQ$ corresponds to $\boldsymbol{v} \in T_xQ$, then $p(\boldsymbol{X}_x)$ reads $\mathrm{g}_x(\boldsymbol{X}_x, \boldsymbol{v})$.

[a] This assumption can be dropped.

Exercise 42

Let (Q,g) be a Riemannian manifold, $A \in \Omega^1(Q)$ be a magnetic potential, and $\boldsymbol{X} \in \mathfrak{X}(Q)$ be a Killing field such that $\mathscr{L}_{\boldsymbol{X}} A = 0$. Show that if $\gamma\colon I \to Q$ is a magnetic geodesic (in the sense of Exercise 19, p. 28), then there is $c \in \mathbb{R}$ such that

$$\mathrm{g}_{\gamma(t)}(\dot{\gamma}(t), \boldsymbol{X}_{\gamma(t)}) = A_{\gamma(t)}(\boldsymbol{X}_{\gamma(t)}) + c, \quad \text{for all } t \in I,$$

and conclude that

$$B_{\gamma(t)}(\dot{\gamma}(t), \boldsymbol{X}_{\gamma(t)}) = (\nabla_{\dot{\gamma}(t)} A)(\boldsymbol{X}_{\gamma(t)}) + A_{\gamma(t)}\left(\frac{\mathrm{D}\boldsymbol{X}}{\mathrm{d}t}(t)\right), \quad \text{for all } t \in I.$$

To what extent is this constant c gauge invariant? **Hint:** You can assume that $\boldsymbol{X}$ is complete, for simplicity (which allows you to mimic what we just did on the previous example). See the remark after Example 30 (p. 57), and justify it.

Example 33

Consider again the Hamiltonian $\mathrm{SO}(3)$-space $(\mathbb{R}^6, \omega_6, \mathrm{SO}(3), \times)$ seen in Example 31 (p. 59), and the Hamiltonian $H\colon \mathbb{R}^6 \to \mathbb{R}$ given by

$$H(\boldsymbol{x},\boldsymbol{y}) = \frac{\|\boldsymbol{y}\|^2}{2} + V(\boldsymbol{x}),$$

where $V\colon \mathbb{R}^3 \to \mathbb{R}$ is a radial function. Then H is clearly $\mathrm{SO}(3)$-invariant. But integral curves of $\boldsymbol{X}_H$ have the form $(\gamma,\dot{\gamma})\colon I \to \mathbb{R}^6$, with $\gamma\colon I \to \mathbb{R}^3$ a solution to Newton's force equation $\ddot{\gamma}(t) = \boldsymbol{F}(\gamma(t))$, where $\boldsymbol{F}(\boldsymbol{x}) \doteq -\nabla V(\boldsymbol{x})$ is the gradient of V. Noether's theorem now says that if $\gamma\colon I \to \mathbb{R}^3$ is a solution to Newton's force equation, the angular momentum $t \mapsto \gamma(t) \times \dot{\gamma}(t)$ is constant.

Exercise 43 (Challenge #4 - The Lagrangian World)

Let Q be a smooth manifold and $L\colon TQ \to \mathbb{R}$ be a **Lagrangian function**[a]. For each $(x,\boldsymbol{v}) \in T^*Q$, define the **fiber derivative** $\mathbb{F}L\colon TQ \to T^*Q$ by

$$\mathbb{F}L(x,\boldsymbol{v})\boldsymbol{w} \doteq \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} L(x, \boldsymbol{v} + t\boldsymbol{w}).$$

Say that L is **hyperregular** if $\mathbb{F}L$ is a diffeomorphism. If $G \circlearrowright Q$ is an action and L is G-invariant, show that for each $X \in \mathfrak{g}$, the **Noether charge** $\mathcal{J}^X : TQ \to \mathbb{R}$ given by $\mathcal{J}^X(x,v) \doteq \mathbb{F}L(x,v)X^\#_x$ is constant along critical points of the action functional

$$\mathcal{A}_L[x] = \int_a^b L(x(t), \dot{x}(t))\,\mathrm{d}t.$$

Hints: The lifted action $G \circlearrowright (TQ, (\mathbb{F}L)^*\omega_{\mathrm{can}})$ is Hamiltonian, the moment map is the Noether charge, and integral curves of the Hamiltonian field of L computed with $(\mathbb{F}L)^*\omega_{\mathrm{can}}$ project to critical points of the action functional of L. Alternatively, use that L is hyperregular and send everything to $(T^*Q, \omega_{\mathrm{can}})$ with $\mathbb{F}L$ and that Hamilton's equations for the Legendre transform of L are equivalent to the Euler-Lagrange equations for L. Note that the conclusion still holds if L is not assumed to be hyperregular. See [38] for more details.

[a] The name only means that the domain is TQ, just like when we say that a Hamiltonian on Q is a function defined on T^*Q.

3.4 Marsden-Weinstein reduction

Let's revisit the issue with quotients: quotients of symplectic manifolds under symplectic actions do not need to be symplectic, for dimension reasons. But, again, what exactly is the problem caused by the dimension of the space being odd? Since derivatives comute with pull-backs, and the pull-back map induced by a surjective submersion is injective, whatever 2-form survives in the quotient will be closed. But in odd dimensions, it will be degenerate. Think of the linear setting:

Exercise 44

Let (V, Ω) be a symplectic vector space, and $S \subseteq V$ be a subspace.

(a) If S is isotropic, Ω survives in the quotient S^Ω/S as a symplectic form.

(b) If S is coisotropic, Ω survives in the quotient S/S^Ω as a symplectic form.

(c) If S is any subspace, Ω survives in the quotient $S/(S \cap S^\Omega)$ as a symplectic form.

On the above exercise, obviously items (a) and (b) follow from (c), but it still instructive to do them. The point here is that $S \cap S^\Omega$ is the **radical** of $\Omega|_{S\times S}$, namely, the kernel of the map $S \to S^*$ induced by Ω, and quotienting out the radical leads to non-degeneracy in the quotient. Related to dimensions of kernels, are dimensions of ranges.

Definition 22

Let M be a smooth manifold and $\omega \in \Omega^2(M)$. The **rank** of ω at $x \in M$ is the dimension of the range of the map $T_xM \to T_x^*M$ induced by ω_x.

Note that the rank function is lower semicontinuous and, in particular, we see that symplectic forms have full rank. So, consider the situation where ω has constant rank, even if not full.

Definition 23

Let M be a smooth manifold and $\omega \in \Omega^2(M)$ have constant rank. The **radical distribution** of ω is the distribution $\mathrm{rad}(\omega) \hookrightarrow TM$ given by

$$\mathrm{rad}(\omega)_x \doteq \mathrm{rad}(\omega_x) = \{\boldsymbol{v} \in T_xM \mid \omega_x(\boldsymbol{v}, \cdot) = 0\}.$$

The first thing (probably) anyone does when they come across a distribution of subspaces, is to ask themselves when is it integrable. So, one checks for involutivity directly: if the manifold M has a constant rank $\omega \in \Omega^2(M)$ and $\boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{X}(M)$ are tangent to $\mathrm{rad}(\omega)$, we may use Cartan's homotopy formula together with the identity mentioned in the hint for Exercise 25 (p. 41), and $\iota_{\boldsymbol{X}}\omega = \iota_{\boldsymbol{Y}}\omega = 0$, to compute:

$$\begin{aligned}\iota_{[\boldsymbol{X},\boldsymbol{Y}]}\omega &= \mathscr{L}_{\boldsymbol{X}}(\iota_{\boldsymbol{Y}}\omega) - \iota_{\boldsymbol{X}}(\mathscr{L}_{\boldsymbol{Y}}\omega) \\ &= \mathscr{L}_{\boldsymbol{X}}(\iota_{\boldsymbol{Y}}\omega) - \iota_{\boldsymbol{X}}(\iota_{\boldsymbol{Y}}(\mathrm{d}\omega) + \mathrm{d}(\iota_{\boldsymbol{Y}}\omega)) \\ &= -\mathrm{d}\omega(\boldsymbol{X}, \boldsymbol{Y}).\end{aligned}$$

Proposition 11

Let M be a smooth manifold and $\omega \in \Omega^2(M)$ have constant rank. Then $\mathrm{rad}(\omega)$ is integrable if and only if $\mathrm{d}\omega$ annihilates $\mathrm{rad}(\omega)$ — which in particular happens when ω is closed.

This motivatives the following definition:

Definition 24

Let M be a smooth manifold. A 2-form $\omega \in \Omega^2(M)$ is called **pre-symplectic** if it closed and has constant rank. We'll say that the pair (M, ω) is a **pre-symplectic manifold**. The **radical foliation** $\mathscr{F}_\omega$ of (M, ω) consists of the leaves of $\mathrm{rad}(\omega)$ and, we say that $\mathscr{F}_\omega$ is **simple** if the quotient leaf space $M/\mathscr{F}_\omega$ has the structure of a smooth manifold for which the projection $\pi_\omega \colon M \to M/\mathscr{F}_\omega$ is a surjective submersion.

Example 34

(1) Let (M,ω) be a symplectic manifold. Then it is obviously pre-symplectic, $\mathrm{rad}(\omega)$ is trivial, and $\mathscr{F}_\omega$ consists of points, so $M/\mathscr{F}_\omega$ is just M itself.

(2) Consider the vector space $\mathbb{R}^{2n+r} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^r$ and define $\omega_{2n,r} \in \Omega^2(\mathbb{R}^{2n+r})$ by assigning to each tangent space to $\mathbb{R}^{2n+r}$ the skew-symmetric bilinear map $\Omega_{2n,r}\colon \mathbb{R}^{2n+r} \times \mathbb{R}^{2n+r} \to \mathbb{R}$ given by

$$\Omega_{2n,r}\big((\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}),(\boldsymbol{x}',\boldsymbol{y}',\boldsymbol{z}')\big) \doteq \langle \boldsymbol{x},\boldsymbol{y}'\rangle - \langle \boldsymbol{x}',\boldsymbol{y}\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the standard Euclidean inner product in $\mathbb{R}^n$. Then $\omega_{2n,r}$ is pre-symplectic, we have that $\mathrm{rad}(\omega_{2n,r})_{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})} \cong \mathbb{R}^r$, and the leaf of $\mathscr{F}_{\omega_{2n,r}}$ passing through $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ is $\{(\boldsymbol{x},\boldsymbol{y})\} \times \mathbb{R}^r$. The quotient $\mathbb{R}^{2n+r}/\mathscr{F}_{2n,r}$ is just $(\mathbb{R}^{2n},\omega_{2n})$.

Exercise 45

Let (V,Ω) be a pre-symplectic vector space. Show that V admits a basis

$$\mathscr{B} = (e_1,\ldots,e_n,f_1,\ldots,f_n,h_1,\ldots,h_r)$$

such that for all $i,j = 1,\ldots,n$ and $k,\ell = 1,\ldots,r$ we have

$$\Omega(e_i,e_j) = \Omega(f_i,f_j) = \Omega(e_i,h_k) = \Omega(f_j,h_k) = \Omega(h_k,h_\ell) = 0 \quad \text{and} \quad \Omega(e_i,f_j) = \delta_{ij}.$$

Or, in other words, the matrix of Ω relative to $\mathscr{B}$ is $-J_{2n} \oplus 0_r$. The number r is intrinsic to (V,Ω), identify it. You can call this $\mathscr{B}$ a **pre-Darboux basis**.

The fact that in the two previous examples the leaf spaces turned out to be symplectic was not a coincidence.

Proposition 12

Let (M,ω) be a pre-symplectic manifold with simple radical distribution. Then $M/\mathscr{F}_\omega$ has a unique symplectic structure $\widetilde{\omega} \in \Omega^2(M/\mathscr{F}_\omega)$ such that $(\pi_\omega)^*\widetilde{\omega} = \omega$.

Proof: Given any point $x \in M$, denote by $[x]$ the leaf of $\mathscr{F}_\omega$ passing through x. If $\pi_\omega\colon M \to M/\mathscr{F}_\omega$ is the quotient projection, as before, consider now the derivative $\mathrm{d}(\pi_\omega)_x\colon T_xM \to T_{[x]}(M/\mathscr{F}_\omega)$. By definition of the leaf space, we have that its kernel is $\ker \mathrm{d}(\pi_\omega)_x = T_x[x] = \mathrm{rad}(\omega_x)$. Thus, we get an isomorphism

$$T_{[x]}(M/\mathscr{F}_\omega) \cong T_xM/\mathrm{rad}(\omega_x),$$

and so ω_x passes to the quotient as a symplectic linear form $\widetilde{\omega_x}$ on $T_{[x]}(M/\mathscr{F}_\omega)$. This defines a smooth $\widetilde{\omega} \in \Omega^2(M/\mathscr{F}_\omega)$, which is non-degenerate and satisfies $(\pi_\omega)^*\widetilde{\omega} = \omega$ by construction, and is closed since

$$0 = \mathrm{d}\omega = \mathrm{d}((\pi_\omega)^*\widetilde{\omega}) = (\pi_\omega)^*(\mathrm{d}\widetilde{\omega}) \implies \mathrm{d}\widetilde{\omega} = 0,$$

as π_ω^* is injective. The condition $(\pi_\omega)^*\widetilde{\omega} = \omega$ forces such $\widetilde{\omega}$ to be unique. □

Another way to produce symplectic quotients, apart from dealing with pre-symplectic objects, is to explore Hamiltonian actions further. Let (M, ω, G, μ) be a Hamiltonian G-space, and take a regular value $p \in \mathfrak{g}^*$ of $\mu\colon M \to \mathfrak{g}^*$, so that the inverse image $\mu^{-1}(p)$ is an embedded submanifold of M. Since we're always assuming that G is connected, μ is G-equivariant by Proposition 10 (p. 61). Now, the stabilizer G_p acts on $\mu^{-1}(p)$, for if $x \in \mu^{-1}(p)$, we have

$$\mu(g \cdot x) = \mu(x) \circ \mathrm{Ad}(g^{-1}) = p \circ \mathrm{Ad}(g^{-1}) = p,$$

so $g \cdot x \in \mu^{-1}(p)$ as well. Write $\iota_p\colon \mu^{-1}(p) \hookrightarrow M$ for the inclusion. The pull-back $(\iota_p)^*\omega$ is surely closed, but as you might expect, not necessarily non-degenerate. All we have seen so far would come full circle if $(\iota_p)^*\omega$ were pre-symplectic. But it is.

Lemma 4

Let (M, ω, G, μ) be a Hamiltonian G-space and $p \in \mathfrak{g}^*$ be a regular value of μ. Then for all $x \in \mu^{-1}(p)$, we have that

$$\mathrm{rad}((\iota_p)^*\omega)_x = T_x(G_p \cdot x),$$

so that $\dim \mathrm{rad}((\iota_p)^*\omega)_x = \dim G_p$ is independent of x, and thus $(\mu^{-1}(p), (\iota_p)^*\omega)$ is pre-symplectic.

Proof: Let's use the definition of radical with the result of Proposition 9 (p. 61) to compute

$$\begin{aligned}\mathrm{rad}((\iota_p)^*\omega)_x &= T_x(\mu^{-1}(p)) \cap (T_x(\mu^{-1}(p)))^{\omega_x} \\ &= \ker \mathrm{d}\mu_x \cap \big((T_x(G \cdot x))^{\omega_x}\big)^{\omega_x} \\ &= \ker \mathrm{d}\mu_x \cap T_x(G \cdot x) \\ &= T_x(G_p \cdot x),\end{aligned}$$

as required. The very last step requires a quick explanation: if $\boldsymbol{v} \in T_x(G \cdot x)$, then $\boldsymbol{v} = (X_M^\#)_x$ for some $X \in \mathfrak{g}$ and, with this in place, $\boldsymbol{v} \in \ker \mathrm{d}\mu_x$ if and only if $(X_{\mathfrak{g}}^\#)_p = 0$ by Proposition 10 (p. 61) — which is equivalent to having $X \in \mathfrak{g}_p$ by item (b) of Exercise 33 (p. 51), meaning that $\boldsymbol{v} \in T_x(G_p \cdot x)$. □

Theorem 18 (Marsden-Weinstein, Meyer)

Let (M, ω, G, μ) be a Hamiltonian G-space and $p \in \mathfrak{g}^*$ be a regular value for μ. Assume that the action $G_p \circlearrowright \mu^{-1}(p)$ is free and proper. Then the quotient $\mu^{-1}(p)/G_p$ has a unique symplectic form $\omega_\circ$ characterized by the relation $\pi^*\omega_\circ = (\iota_p)^*\omega$, where $\pi\colon \mu^{-1}(p) \to \mu^{-1}(p)/G_p$ is the quotient projection. We call $(\mu^{-1}(p)/G_p, \omega_\circ)$ the **reduction of** (M, ω) **to level** p.

Proof: The space $\mu^{-1}(p)/G_p$ is a smooth manifold because the action is assumed to be free and proper. Now the 2-form $(i_p)^*\omega$ in $\mu^{-1}(p)$ is pre-symplectic and the leaves of radical foliation $\mathscr{F}_{(\iota_p)^*\omega}$ are precisely the G_p-orbits. We are done, by Proposition 12 (p. 66).

$$(\mu^{-1}(p), (\iota_p)^*\omega) \xhookrightarrow{\iota_p} (M, \omega)$$

$$(\mu^{-1}(p), (\iota_p)^*\omega) \xrightarrow{\pi} (\mu^{-1}(p)/G_p, \omega_\circ)$$

□

Remark. When the Lie group G is abelian, the adjoint representation is trivial, so we have that $G = G_p$ for all $p \in \mathfrak{g}^*$. And, for $p = 0$, we have that $G_0 = G$. In this case, $\mu^{-1}(0) \to \mu^{-1}(0)/G$ turns out to be a principal G-bundle.

Just like what happened with Noether's theorem, we again see a strong theorem with a very simple proof. In practice, one wants to explicitly describe the **reduced form** $\omega_\circ$. Let's see how one can usually try to do this in practice.

Example 35

(1) Let (M, ω) be a symplectic manifold, and $H\colon M \to \mathbb{R}$ be a smooth function for which the field $\boldsymbol{X}_H$ is complete. We have seen that $(M, \omega, \mathbb{R}, \mu)$ is a Hamiltonian $\mathbb{R}$-space, where $\mu = H$ and the action is given by Hamiltonian flows. So, for each $e \in \mathbb{R}$, each quotient $\Sigma_e/\mathbb{R}$ of an energy level which turns out to be a manifold automatically inherits a symplectic structure $\omega_\circ$. Explicitly, if $L \in \Sigma_e/\mathbb{R}$, and $\widetilde{\boldsymbol{v}}, \widetilde{\boldsymbol{w}} \in T_L(\Sigma_e/\mathbb{R})$, we have

$$(\omega_\circ)_L(\widetilde{\boldsymbol{v}}, \widetilde{\boldsymbol{w}}) = \omega_x(\boldsymbol{v}, \boldsymbol{w}),$$

where $x \in L$ and $\boldsymbol{v}, \boldsymbol{w} \in T_x\Sigma_e$ are such that $\mathrm{d}\pi_x(\boldsymbol{v}) = \widetilde{\boldsymbol{v}}$, and similarly for $\widetilde{\boldsymbol{w}}$ — where $\pi\colon \Sigma_e \to \Sigma_e/\mathbb{R}$ is the quotient projection. And as a quick sanity check, do note that $\dim(\Sigma_e/\mathbb{R}) = \dim M - 2$ is even.

(2) Consider the circular Hamiltonian action of $\mathbb{S}^1$ on $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$, equipped with ω_{2n+2} — we have seen that the moment map is $\mu\colon \mathbb{C}^{n+1} \to (\mathrm{i}\mathbb{R})^*$ given by

$$\mu(z) = -\frac{\|z\|^2}{2}.$$

In particular, $\mu^{-1}(-1/2) = \mathbb{S}^{2n+1}$. Hence the quotient $\mathbb{CP}^n = \mathbb{S}^{2n+1}/\mathbb{S}^1$ inherits a symplectic form $\omega_{\rm FS} \in \Omega^2(\mathbb{CP}^n)$ — called the **Fubini-Study form**. There is something slightly deeper going on here, in the following way: $\mathbb{S}^{2n+1}$ inherits a Riemannian metric from $\mathbb{R}^{2n+2}$ and the $\mathbb{S}^1$-action consists of isometries, so there is a unique Riemannian metric $\mathrm{g}_{\rm FS}$ (called the **Fubini-Study metric**) on the quotient $\mathbb{S}^{2n+1}/\mathbb{S}^1$ for which the quotient projection $\mathbb{S}^{2n+1} \to \mathbb{S}^{2n+1}/\mathbb{S}^1$ is a **Riemannian submersion** (see [32] for more on those). With the natural complex structure J that $\mathbb{C}P^n$ has, the relation $g_{FS} = \omega_{FS}(\cdot, J\cdot)$ holds, making $\mathbb{C}P^n$ a Kähler manifold.

(3) Let $(M, -\mathrm{d}\lambda)$ be an exact symplectic manifold, and $G \circlearrowright M$ be a Lie group action preserving λ – hence automatically Hamiltonian by item (3) of Example 30 (p. 57). Assume that $0 \in \mathfrak{g}^*$ is a regular value of the moment map μ and that we are under the conditions of Theorem 18 (p. 67). In this case, the reduction of $(M, -\mathrm{d}\lambda)$ to level 0 is called the **Marsden-Weinstein reduction of** $(M, -\mathrm{d}\lambda)$ and it is denoted by $M /\!\!/ G$. Let's show that the reduced symplectic structure $(-\mathrm{d}\lambda)_\circ$ on $M/\!\!/G$ is also exact. Write $\iota\colon \mu^{-1}(0) \hookrightarrow M$ for the inclusion, $\pi\colon \mu^{-1}(0) \to M/\!\!/G$ for the projection, and consider $\iota^*\lambda \in \Omega^1(\mu^{-1}(0))$. Clearly $\iota^*\lambda$ is G-invariant, but we also have that if $x \in \mu^{-1}(0)$ and $\boldsymbol{v} \in \ker \mathrm{d}\mu_x$ satisfies $\mathrm{d}\pi_x(\boldsymbol{v}) = \boldsymbol{0}$, then $(\iota^*\lambda)_x(\boldsymbol{v}) = 0$, since $x \in \mu^{-1}(0)$ says that λ_x annihilates $T_x(G\cdot x) = \ker \mathrm{d}\pi_x$. Hence there is $\lambda_\circ \in \Omega^1(M/\!\!/G)$ such that $\pi^*(\lambda_\circ) = \iota^*\lambda$. Once this is in place, we have that

$$\pi^*((\mathrm{d}\lambda)_\circ) = \iota^*(\mathrm{d}\lambda) = \mathrm{d}(\iota^*\lambda) = \mathrm{d}(\pi^*\lambda_\circ) = \pi^*(\mathrm{d}\lambda_\circ),$$

and since π being a surjective submersion implies that π^* is injective, it follows that $(-\mathrm{d}\lambda)_\circ = -\mathrm{d}\lambda_\circ$ is exact.

(4) Let (M, ω, G, μ) be a Hamiltonian G-space under the conditions of Theorem 18 (p. 67), $p \in \mathfrak{g}^*$ be a regular value of the moment map μ, and $H\colon M \to \mathbb{R}$ be a G-invariant Hamiltonian. By G-invariance of H, we know that the restriction $H|_{\mu^{-1}(p)}\colon \mu^{-1}(p) \to \mathbb{R}$ passes to the quotient as a function $H_p\colon \mu^{-1}(p)/G_p$, namely, satisfying $H_p \circ \pi = H|_{\mu^{-1}(p)}$, where $\pi\colon \mu^{-1}(p) \to \mu^{-1}(p)/G_p$ is the quotient projection. But there's more to it. First, we claim that the Hamiltonian field $\boldsymbol{X}_H$ is tangent to $\mu^{-1}(p)$ along its points. More precisely, if $x \in \mu^{-1}(p)$ and γ is the integral curve of H starting at x, we have that

$$\mathrm{d}\mu_x(\boldsymbol{X}_H|_x) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mu(\gamma(t)) = 0$$

by Noether's theorem, so $\boldsymbol{X}_H|_x \in T_x\mu^{-1}(p)$. Next, G-invariance of H and ω also gives us that $\mathrm{d}(\mathrm{L}_g)_x(\boldsymbol{X}_H|_x) = \boldsymbol{X}_H|_{g\cdot x}$, since for each $\boldsymbol{v} \in T_xM$ and $g \in G$ we know that $H \circ \mathrm{L}_g = H$ implies $\mathrm{d}H_{g\cdot x} \circ \mathrm{d}(\mathrm{L}_g)_x = \mathrm{d}H_x$, so evaluating at $\boldsymbol{v}$ gives

$$\omega_x(\boldsymbol{X}_H|_x, \boldsymbol{v}) = \omega_{g\cdot x}(\boldsymbol{X}_H|_{g\cdot x}, \mathrm{d}(\mathrm{L}_g)_x(\boldsymbol{v})) = \omega_x(\mathrm{d}(\mathrm{L}_{g^{-1}})_{g\cdot x}\boldsymbol{X}_H|_{g\cdot x}, \boldsymbol{v}).$$

Thus $\boldsymbol{X}_H|_x = \mathrm{d}(\mathrm{L}_{g^{-1}})_{g\cdot x}\boldsymbol{X}_H|_{g\cdot x}$ implies that $\mathrm{d}(\mathrm{L}_g)_x(\boldsymbol{X}_H|_x) = \boldsymbol{X}_H|_{g\cdot x}$, as claimed. Hence there is a vector field $\widetilde{\boldsymbol{X}} \in \mathfrak{X}(\mu^{-1}(p)/G_p)$ such that $\mathrm{d}\pi \circ \boldsymbol{X}_H = \widetilde{\boldsymbol{X}} \circ \pi$, and we claim that this field $\widetilde{\boldsymbol{X}}$ is nothing more than $\boldsymbol{X}_{H_p}$, computed using the reduced symplectic form $\omega_\circ$. Let's check this, identifying ω with its pull-back and H with its restriction, both to $\mu^{-1}(p)$. Again, we'll use that π^* is injective, so that

$$\begin{aligned}\pi^*\iota_{\widetilde{\boldsymbol{X}}}\omega_\circ &= \iota_{\boldsymbol{X}_H}(\pi^*\omega_\circ) = \iota_{\boldsymbol{X}_H}\omega \\ &= \mathrm{d}H = \mathrm{d}(H_p \circ \pi) \\ &= \mathrm{d}H_p \circ \mathrm{d}\pi = \pi^*(\mathrm{d}H_p) \\ &= \pi^*\iota_{\boldsymbol{X}_{H_p}}\omega_\circ,\end{aligned}$$

meaning that $\iota_{\widetilde{\boldsymbol{X}}}\omega_\circ = \iota_{\boldsymbol{X}_{H_p}}\omega_\circ$ finally yields $\widetilde{\boldsymbol{X}} = \boldsymbol{X}_{H_p}$ due to non-degeneracy of $\omega_\circ$.

Exercise 46 (Passing differential forms to action quotients)

Let's clarify one part of the argument used in item (3) of the previous example. Assume that M is a smooth manifold, $G \circlearrowright M$ is a free and proper Lie group action, with quotient projection $\pi\colon M \to M/G$, and $\alpha \in \Omega^k(M)$ is:

- G-invariant, and;
- G-**horizontal**, in the sense that for all $x \in M$ and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in T_xM$, we have that $\alpha_x(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k) = 0$ whenever there is some i such that $\boldsymbol{v}_i \in \ker \mathrm{d}\pi_x$.

Show that there is a unique $\widetilde{\alpha} \in \Omega^k(M/G)$ such that $\pi^*\widetilde{\alpha} = \alpha$.

Hint: there is only one choice of definition for $\widetilde{\alpha}$, but you need to check that the expression you're forced to consider is indeed well-defined; do the case $k = 1$ first to get intuition, and then telescope a certain difference for the general case.

Remark: note that requiring α to be G-horizontal is to be expected, as we have the isomorphism $T_{G\cdot x}(M/G) \cong T_xM/\ker \mathrm{d}\pi_x$ — G-invariance alone is not strong enough to ensure α survives in M/G.

In particular, reducing Hamiltonians on phase spaces is a useful technique for reducing the number of degrees of freedom of a mechanical system.

Example 36 (Mechanics on the plane)

Consider, in $\mathbb{R}^2$, the motion of a particle with mass $m > 0$ under the action of a potential function $V\colon \mathbb{R}^2 \to \mathbb{R}$. Such motion is described by the integral curves of the Hamiltonian field of the total energy $H\colon T^*\mathbb{R}^2 \cong \mathbb{R}^4 \to \mathbb{R}$ given by

$$H(x, y, p_x, p_y) = \frac{1}{2m}(p_x^2 + p_y^2) + V(x, y).$$

Assume that V has rotational symmetry, that is, that V is invariant under the action $\mathbb{S}^1 \circlearrowright \mathbb{R}^2$ given by rotation. The action $\mathbb{S}^1 \circlearrowright \mathbb{R}^4$ given by cotangent lifts is Hamiltonian with moment map $\mu\colon \mathbb{R}^4 \to \mathbb{R}$ given by

$$\mu(x,y,p_x,p_y) = -yp_x + xp_y.$$

Indeed, by item (3) of Example 30 (p. 57), and identifying $\mathrm{Lie}(\mathbb{S}^1) = \mathrm{i}\mathbb{R} \cong \mathbb{R} \cong \mathbb{R}^*$ by setting $\mu(x,y,p_x,q_y) \doteq \mu(x,y,p_x,p_y)(1)$, we just have that

$$\mu(x,y,p_x,p_y) = (p_x\,\mathrm{d}x + p_y\,\mathrm{d}y)|_{(x,y)}(-y\partial_x|_{(x,y)} + x\partial_y|_{(x,y)}) = -yp_x + xp_y,$$

as claimed. It's easy to see that every $a \in \mathbb{R}$, $a \neq 0$, is a regular value for μ. Using polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, a straightforward computation shows that

$$p_x = p_r\cos\theta - \frac{p_\theta\sin\theta}{r} \quad\text{and}\quad p_y = p_r\sin\theta + \frac{p_\theta\cos\theta}{r},$$

so that

$$H(r,\theta,p_r,p_\theta) = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + V(r) \quad\text{and}\quad \mu(r,\theta,p_r,p_\theta) = p_\theta.$$

By Noether's theorem, the angular momentum p_θ is constant along the path the particle follows. Given $a \in \mathbb{R}$, $a \neq 0$, the level set $\mu^{-1}(a)$ consists of all possible initial conditions for the particle for which the angular momentum is constant and equal to a. Only the coordinates r and p_r survive in the quotient $\mu^{-1}(a)/\mathbb{S}^1$, and the reduced Hamiltonian is $H_a\colon \mu^{-1}(a)/\mathbb{S}^1 \to \mathbb{R}$ given by

$$H_a(r,p_r) = \frac{p_r^2}{2m} + V_{\mathrm{eff}}(r),$$

where $V_{\mathrm{eff}}\colon \mu^{-1}(a)/\mathbb{S}^1 \to \mathbb{R}$ given by

$$V_{\mathrm{eff}}(r) = \frac{a^2}{2mr^2} + V(r)$$

is the **effective potential**.

Remark. Interesting dynamics are also induced by very complicated potentials. One example is the **Henon-Heiles potential** $V_{\mathrm{HH},\lambda}\colon \mathbb{R}^2 \to \mathbb{R}$, given by

$$V_{\mathrm{HH},\lambda}(x,y) = \frac{1}{2m}(x^2+y^2) + \lambda\left(x^2y - \frac{y^3}{3}\right).$$

Moving through energy levels, one even has chaotic behavior. It's a very frequent illustrative example in two-dimensional dynamics, usually studied with numerical techniques.

Exercise 47 (Mechanics on the sphere)

Consider, in the sphere $\mathbb{S}^2$, the motion of a particle with mass $m > 0$ under the action of a potential function $V\colon \mathbb{S}^2 \to \mathbb{R}$. Such motion is described by the integral curves of the Hamiltonian field of the total energy $H\colon T^*\mathbb{S}^2 \to \mathbb{R}$, given by

$$H(x,y,z,p_x,p_y,p_z) = \frac{1}{2m}(p_x^2+p_y^2+p_z^2) + V(x,y,z).$$

Of course, we have (by unwittingly identifying $T\mathbb{S}^2 \cong T^*\mathbb{S}^2$) that

$$T^*\mathbb{S}^2 = \{(x,y,z,p_x,p_y,p_z) \in T^*\mathbb{R}^3 \mid x^2+y^2+z^2 = 1 \text{ and } xp_x+yp_y+zp_z = 0\}.$$

Consider the circular action $\mathbb{S}^1 \circlearrowright \mathbb{S}^2$, as in item (2) of Example 26 (p. 49), and assume that V is $\mathbb{S}^1$-invariant — which is to say that V only depends on z. Take spherical coordinates

$$x = \cos\theta\cos\phi, \quad y = \sin\theta\cos\phi, \quad \text{and} \quad z = \sin\phi,$$

where θ ranges from 0 to 2π and ϕ from $-\pi/2$ to $\pi/2$.

(a) Show that

$$p_x = -p_\theta\frac{\sin\theta}{\cos\phi} - p_\phi\cos\theta\sin\phi, \quad p_y = p_\theta\frac{\cos\theta}{\cos\phi} - p_\phi\sin\theta\sin\phi,$$

and $p_z = p_\phi\cos\phi$.

(b) Conclude that

$$H(\theta,\phi,p_\theta,p_\phi) = \frac{1}{2m}\left(\frac{p_\theta^2}{\cos^2\phi} + p_\phi^2\right) + V(\phi) \quad \text{and} \quad \mu(\theta,\phi,p_\theta,p_\phi) = p_\theta.$$

By Noether, p_θ is again a conserved quantity.

(c) Show that for each regular value $a \in \mathbb{R}$ of μ, the reduced Hamiltonian and the effective potential $H_a, V_{\rm eff}\colon \mu^{-1}(a)/\mathbb{S}^1 \to \mathbb{R}$ are given by and related via

$$H_a(\phi,p_\phi) = \frac{p_\phi^2}{2m} + V_{\rm eff}(\phi) \quad \text{and} \quad V_{\rm eff}(\phi) = \frac{a^2}{2m\cos^2\phi} + V(\phi).$$

Exercise 48 (Mechanics on the cylinder)

Consider, in the sphere $\mathbb{S}^1 \times \mathbb{R}$, the motion of a particle with mass $m > 0$ under the action of a potential function $V\colon \mathbb{S}^1 \times \mathbb{R} \to \mathbb{R}$. Again, such motion will be described by the integral curves of the Hamiltonian field of the total energy

$H\colon T^*(\mathbb{S}^1\times\mathbb{R})\to\mathbb{R}$, given by

$$H(x,y,z,p_x,p_y,p_z) = \frac{1}{2m}(p_x^2+p_y^2+p_z^2)+V(x,y,z),$$

and we may write (again with the aid of the metric) that

$$T^*(\mathbb{S}^1\times\mathbb{R}) = \{(x,y,z,p_x,p_y,p_z)\in T^*\mathbb{R}^3 \mid x^2+y^2=1 \text{ and } xp_x+yp_y=0\}.$$

Take cylindrical coordinates $x=r\cos\theta$, $y=r\sin\theta$, $z=z$.

(a) Show that $p_\theta = -p_x\sin\theta+p_y\cos\theta$, and conclude that in such coordinates, the Hamiltonian is given by

$$H(\theta,z,p_\theta,p_z) = \frac{1}{2m}(p_\theta^2+p_z^2)+V(\theta,z).$$

Note we did not impose conditions on V, nor considered symmetries yet.

(b) Consider the action $\mathbb{S}^1 \circlearrowright \mathbb{S}^1\times\mathbb{R}$ given by rotation and assume that the potential V is $\mathbb{S}^1$-invariant, which is to say that it depends only on z. Show that the moment map $\mu\colon T^*(\mathbb{S}^1\times\mathbb{R})\to\mathbb{R}$ is given by $\mu(\theta,z,p_\theta,p_z)=p_\theta$. Conclude that for each regular value $a\in\mathbb{R}$ of μ, we have

$$H_a(z,p_z) = \frac{p_z^2}{2m}+V_{\rm eff}(z), \quad \text{with} \quad V_{\rm eff}(z) = \frac{a^2}{2m}+V(z)$$

on the quotient $\mu^{-1}(a)/\mathbb{S}^1$.

(c) Consider the action $\mathbb{R}\circlearrowright \mathbb{S}^1\times\mathbb{R}$ given by translations and assume that the potential V is $\mathbb{R}$-invariant, which is to say it depends only on θ. Show that the moment map $\mu\colon T^*(\mathbb{S}^1\times\mathbb{R})\to\mathbb{R}$ is given by $\mu(\theta,z,p_\theta,p_z)=p_z$. Conclude that for each regular value $a\in\mathbb{R}$ of μ, we have

$$H_a(\theta,p_\theta) = \frac{p_\theta^2}{2m}+V_{\rm eff}(\theta), \quad \text{with} \quad V_{\rm eff}(\theta) = \frac{a^2}{2m}+V(\theta)$$

on the quotient $\mu^{-1}(a)/\mathbb{R}$.

We'll conclude the discussion with one last "fake" example:

Example 37

Let's see that what would be a linear Marsden-Weinstein quotient is in fact trivial. Namely, let (V,Ω) be a symplectic vector space, and let ω be the constant symplectic form, which assigns to each tangent space $T_xV\cong V$, $\omega_x \doteq \Omega$ itself. We have the obvious evaluation action ${\rm Sp}(V,\Omega)\circlearrowright V$. Recall that the Lie algebra

$$\mathfrak{sp}(V,\Omega) = \{\psi\in\mathfrak{gl}(V) \mid \Omega(\psi(x),y)+\Omega(x,\psi(y))=0, \text{ for all } x,y\in V\}$$

consists of the so-called Hamiltonian operators. Important examples come as the velocities of curves of symplectic transvections, namely, $\dot{\tau}_v$ given by

$$\dot{\tau}_v(x) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{t=0} \tau_{\lambda,v}(x) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=0} (x + \lambda\Omega(x,v)v) = \Omega(x,v)v,$$

for any non-zero $v \in V$. Now, given $\psi \in \mathfrak{sp}(V,\Omega)$, we have $(\psi^\#)_x = \psi(x)$ because the original action is linear. Given any $(x,v) \in TV$, we have the relation $\omega_x((\psi^\#)_x, v) = \Omega(\psi(x), v)$, so that $\mu\colon V \to \mathfrak{sp}(V,\Omega)^*$ given by

$$\mu_{\mathrm{c}}^{\psi}(x) = \frac{1}{2}\Omega(\psi(x), x)$$

does satisfy $\mathrm{d}(\mu_{\mathrm{c}}^{\psi})_x = \omega_x((\psi^\#)_x, \cdot)$, using that ψ is Ω-skew. If $x \in \mu^{-1}(0)$, then $\mu_{\mathrm{c}}^{\dot{\tau}_v}(x) = 0$ leads to $\Omega(x,v)^2 = 0$ for all $v \in V$, so $x = 0$ from non-degeneracy of Ω. But here a disaster happens: μ_{c} is not a comoment map, but instead is an anti-Poisson map. On one hand we have

$$\{\mu_{\mathrm{c}}^{\psi_1}, \mu_{\mathrm{c}}^{\psi_2}\}_\omega(x) = \omega_x((\psi_1^\#)_x, (\psi_2^\#)_x) = \Omega(\psi_1(x), \psi_2(x))$$

and on the other hand

$$\begin{aligned}\mu_{\mathrm{c}}^{[\psi_1,\psi_2]}(x) &= \frac{1}{2}\Omega([\psi_1,\psi_2](x), x) \\ &= \frac{1}{2}(\Omega(\psi_1\psi_2(x), x) - \Omega(\psi_2\psi_1(x), x)) \\ &= -\Omega(\psi_1(x), \psi_2(x)).\end{aligned}$$

For more on symplectic reductions, see [13] and [26].

Where to go from here?

Now you have the basic Symplectic Geometry kit needed to move on with your mathematical life. Here are some topics you can try to explore next (perhaps in no particular order).

Symplectic Topology: We have seen that if an even-dimensional compact manifold is to admit a symplectic form, then all even degree de Rham cohomology spaces must be non-trivial, and the manifold has to be orientable — but those are not the only obstructions ($\mathbb{S}^2 \times \mathbb{S}^4$ is one possible witness). This raises the question of whether is there a complete obstruction. In dimension 2, the complete criterion is just orientability. In dimension 4, things already get much more complicated, and conditions must be imposed on the **Seiberg-Witten invariants** of the manifold (generally speaking, topology of 4-manifolds is a very complicated subject). In higher dimensions, some information is certainly encoded on the cohomology ring of the classifying space $\mathrm{BSp}_{2n}(\mathbb{R})$ (via characteristic classes), but it seems not much is known about it. On the other hand, every symplectic manifold carries an almost-complex structure, which is compatible with ω, but not integrable in general — and there are also complex and almost-complex manifolds which carry no symplectic structure. In fact, given any group described with finitely many generators and relations is the fundamental group of a four-dimensional compact symplectic manifold. More on these relations between symplectic and complex geometry leads us to the next topic.

Some references: [3], [18], [28], [37], [39].

Kähler Geometry: It is the intersection of Riemannian geometry, complex geometry, and symplectic geometry. In the linear setting, we can say that a **Kähler vector space** is a quadruple $(V, \Omega, J, \mathrm{g})$, where Ω is a symplectic structure, J is an almost complex structure which is also a symplectomorphism, and $\mathrm{g} = \Omega(\cdot, J\cdot)$ is an inner product on V. We'll say that (Ω, J, g) is a **compatible triple** for V. A **Kähler basis** is a basis $\mathscr{B} = (e_1, \ldots, e_n, Je_1, \ldots, Je_n)$ which is both Darboux and orthonormal. If $S \subseteq V$ is a subspace, then $J[S^\Omega] = S^\perp$ holds. On the smooth setting, we say that a **Kähler manifold** is a quadruple $(M, \omega, J, \mathrm{g})$ where (ω, J, g) is a field of compatible triples on the tangent spaces to M, with ω closed and J integrable. This turns out to be equivalent to requiring a M to be a complex manifold equipped with a **hermitian metric** (i.e., a Riemannian metric for which $J^*\mathrm{g} = \mathrm{g}$) such that $\nabla J = 0$, where ∇ is the Levi-Civita connection. One can talk about things such as Lagrangian submanifolds, minimal and totally geodesic submanifolds, complex submanifolds, holomorphic curvature, and much more. Being complex manifolds, one can talk about complex differential forms of some bidegree (k, ℓ) on any Kähler manifold, and talk about the **Dolbeaut cohomology** spaces $H^{k,\ell}_{\bar\partial}(M)$ — in particular, the symplectic form ω itself defines a Dolbeaut class $[\omega] \in H^{1,1}_{\bar\partial}(M)$. Kähler manifolds are also thoroughly studied in Algebraic Geometry, appearing very prominently as submanifolds of complex projective spaces.

Some references: [5], [19], [31], [45], etc.

Geometric Quantization: A **quantization procedure** consists in passing from a classical description of a physical system, to a quantum description. For example, if Q is the configuration space for a classical mechanical system and T^*Q is the phase space of positions and momenta, classical observables (by which we mean "physical quantities") are simply functions $f\colon T^*Q \to \mathbb{R}$, which take a state (x, p) and return the value $f(x, p)$ of the physical quantity represented by f when the system is at state (x, p) (for example, the energy of the system when it is at said state). In the quantum setting, measurements of physical quantities — now understood as quantum observables — only give us statistical distributions of possible values instead of a determined value. The spectrum of such values is encoded as the mathematical spectrum of a symmetric operator on a complex Hilbert space, which will now play the role of the quantum state space. Namely, symmetric operators have real eigenvalues, and this is perhaps the first motivation for the use of functional analysis tools in the study of quantum mechanics. However, is it possible that two distinct classical descriptions give rise to the same quantum description under quantization, and so each such classical description is called a **classical limit**. In any case, T^*Q carries its standard symplectic structure ω_{can}, and one might wonder what this becomes after quantization. Since symplectomorphisms should naturally correspond to unitary operators (because both preserve the structure of whatever space they'll act on) and the Poisson bracket of two classical observables should correspond to the commutator of the corresponding quantum observables, one can formulate the quantization problem in terms of category theory, by trying to find a functor from the core of symplectic category to the core of the category of Hilbert spaces. It turns out that imposing even weak assumptions on the spaces considered (motivated by the behavior of Schrödinger's equation) prohibits the existence of such a functor (this is called **Van Hove's theorem**). The next keyword, then, is **prequantization**. Again, this has more than one possible formulation. One can say that a prequantization of a symplectic manifold (M, ω) with integral class $[\omega]$ is a (complex) Hermitian line bundle $(\mathscr{L}, \langle \cdot, \cdot \rangle) \to M$ with a $\mathrm{U}(1)$-connection $\nabla\colon \mathfrak{X}(M) \times \Gamma(\mathscr{L}) \to \Gamma(\mathscr{L})$ parallelizing $\langle \cdot, \cdot \rangle$ and satisfying $R^\nabla = \omega$ (in the sense that for all $\boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{X}(M)$ and $\psi \in \Gamma(\mathscr{L})$, the relation $R^\nabla(\boldsymbol{X}, \boldsymbol{Y})\psi = \omega(\boldsymbol{X}, \boldsymbol{Y})\psi$ holds) — taking the associated Hilbert space to be the space of (suitably normalized) L^2-sections of $\mathscr{L}$ (to be thought of as wavefunctions), one obtains a (pre)quantum description satisfying most of the Dirac's axioms for quantization procedures.

Some references: [2], [16], [40], [43], etc.

Contact Geometry: A **contact manifold** is a pair (M, Σ), where $\Sigma \hookrightarrow TM$ is a non-integrable distribution of hyperplanes — called a **contact distribution**. Up to a non-zero functional multiple, choosing Σ amounts to choosing a non-vanishing 1-form $\alpha \in \Omega^1(M)$ such that $\Sigma = \ker \alpha$ and $\mathrm{d}\alpha_x$ is non-degenerate on Σ_x, for all $x \in M$ — then α is called a **contact form**, and one often writes (M, α) instead of (M, Σ). Darboux's theorem implies that the dimension of a contact manifold must be odd, and then non-degenerability can be rephrased as $\alpha \wedge (\mathrm{d}\alpha)^{\wedge n} \neq 0$. The **Reeb field** of (M, α) is the unique vector field $\boldsymbol{\xi} \in \mathfrak{X}(M)$ satisfying $\alpha(\boldsymbol{\xi}) = 1$

and $\mathrm{d}\alpha(\boldsymbol{\xi}, \cdot) = 0$ (the latter condition does not contradict non-degenerability of $\mathrm{d}\alpha$ on Σ because $\boldsymbol{\xi}$ is transverse to Σ, not tangent). This field helps control the geometry of (M, α), as $\mathscr{L}_{\boldsymbol{\xi}}\alpha = 0$ by Cartan's homotopy formula. A diffeomorphism between contact manifolds preserving the contact forms is called a **contactomorphism**. One factory of examples comes from symplectic geometry: if $(M, -\mathrm{d}\lambda)$ is an exact symplectic manifold, then $(M \times \mathbb{R}, \pi^*\lambda + \mathrm{d}t)$ is a contact manifold, where $\pi\colon M \times \mathbb{R} \to M$ and $t\colon M \times \mathbb{R} \to \mathbb{R}$ are the projections. On the other hand, if (M, α) is a contact manifold, then $(M \times \mathbb{R}, \mathrm{d}(\mathrm{e}^t\alpha))$ is a symplectic manifold. The contact analogues of Lagrangian submanifolds are called **Legendrian submanifolds**. In summary, contact geometry is the natural odd-dimensional analogue of symplectic geometry, with applications in differential equations, control theory, geometric quantization, among others.

Some references: [10], [26], [29], etc.

Symplectic Connections: Given a symplectic manifold (M, ω), a **symplectic connection** is a torsionfree connection ∇ which parallelizes ω. If S is any tensor field of type $(1, 2)$, and we define a connection ∇' by $\nabla'_{\boldsymbol{X}}\boldsymbol{Y} = \nabla_{\boldsymbol{X}}\boldsymbol{Y} + S_{\boldsymbol{X}}\boldsymbol{Y}$, then a very short computation shows that ∇' is symplectic if and only if the trilinear map $(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}) \mapsto \omega(S_{\boldsymbol{X}}\boldsymbol{Y}, \boldsymbol{Z})$ is fully symmetric; since the difference of any two connections is a tensor, we conclude that the space of symplectic connections on (M, ω) is an affine space whose translation space is $\Gamma(T^*M^{\odot 3})$. This is very different from what happens in Riemannian geometry, where a metric uniquely determines its Levi-Civita connection. It is not possible to make a "canonical" choice of symplectic connection without having extra structure on the manifold (e.g., being Kähler, pseudo-Kähler, or a symmetric symplectic space). Generally, the Ricci endomorphism r^∇ of a symplectic connection (characterized by $\mathrm{Ric}^\nabla(\boldsymbol{X}, \boldsymbol{Y}) = \omega(\boldsymbol{X}, \mathrm{r}^\nabla(\boldsymbol{Y})))$ carries relevant geometric information (even though what would be the "scalar curvature" vanishes as ω is skew), and one usually looks for the so-called **preferred connections**, satisfying the cyclic identity $(\nabla_{\boldsymbol{X}}\mathrm{r}^\nabla)(\boldsymbol{Y}, \boldsymbol{Z}) + (\nabla_{\boldsymbol{Y}}\mathrm{r}^\nabla)(\boldsymbol{Z}, \boldsymbol{X}) + (\nabla_{\boldsymbol{Z}}\mathrm{r}^\nabla)(\boldsymbol{X}, \boldsymbol{Y}) = 0$. Such connections also have a variational characterization, and this entire theory has some links with geometric quantization. Symplectic connections are also useful in the study of **symplectic Lie groups**, that is, Lie groups equipped with left-invariant symplectic forms.

Some references: [6], [7], [9], [12], etc.

Mirror Symmetry: Briefly, a **Calabi-Yau manifold** is a compact and Ricci-flat Kähler manifold with vanishing first Chern class. Such manifolds are considered in **String Theory**, where point-like particles are treated as one-dimensional strings, and spacetimes have extra "hidden" dimensions. **Mirror symmetry** essentially discusses when two given Calabi-Yau manifolds yield the "same" physical theory, even though they might have different geometries. This is a relatively recent area of research in mathematical physics (from roughly around the 90's). Important tools and questions indeed come from symplectic geometry. Part of the challenge seems to be making the aforementioned equivalence mathematically precise. One such attempt is called **Homological mirror symmetry** (also known as **Kontsevich's conjecture**) — it discusses the so-called **Floer homology**: one defines the Floer cochain groups as certain modules generated by transversal intersections of Lagrangian submanifolds of a given symplectic manifold, dualizes, and computes the homology of the resulting chain complex. The **Fukaya category** of a symplectic manifold has Lagrangian submanifolds as objects, and chain groups as morphisms. Understanding all of this requires a deep knowledge not only of symplectic geometry, but also of algebraic geometry, algebraic topology, category theory and, to some extent, actual Physics.

Some references: [3], [4], [22], etc.

Toric manifolds and Delzant polytopes: A **toric manifold** is simply a Hamiltonian $\mathbb{T}^k$-space for which the action is effective. The **Atiyah-Guillemin-Sternberg convexity theorem** says that the set of fixed points of the action on a compact and connected toric manifold is a finite union of connected symplectic submanifolds, the moment map is constant on each submanifold, and the image of the moment map equals the convex hull of those values (inside the the dual Lie algebra $\mathfrak{t}^* \cong \mathbb{R}^k$ of $\mathbb{T}^k$). A **Delzant polytope** is a polytope in $\mathbb{R}^k$ such that for which each vertex is the intersection of exactly n edges, and for each vertex we can find vectors $v_1, \ldots, v_k \in \mathbb{Z}^k$ along those edges which form a basis for $\mathbb{Z}^k$. **Delzant's theorem** says that there is a bijective correspondence between toric manifolds and Delzant polytopes — namely, the correspondence takes a toric manifold to the image of its moment map. So, given a toric manifold, the associated Delzant polytope actually encodes all of its geometry.

Some references: [13], [28].

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