THE "EINSTEIN MANIFOLD"

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Fix once and for all a real pseudo-Euclidean vector space (V, g) with indefinite signature, i.e., g is not positive-definite nor negative-negative. This means that if the signature of g is (i_+, i_-), we have $i_+, i_- \geq 1$. We'll also write $\mathrm{g} = \langle \cdot, \cdot \rangle$ whenever convenient. Let $\mathscr{C} = \{x \in V \setminus \{0\} \mid \langle x, x \rangle = 0\}$ denote the lightcone of (V, g). On $\mathscr{C}$, define a equivalence relation $\sim$ by saying that $u \sim v$ if $v = \lambda u$ for some non-zero $\lambda \in \mathbb{R}$, and consider the quotient $\mathbb{E} = \mathscr{C}/_\sim$. Equivalently, $\mathbb{E}$ is the quotient of $\mathscr{C}$ under the linear action $\mathbb{R}^\times \circlearrowright \mathscr{C}$ given by multiplication. Geometrically, $\mathbb{E}$ is the set of all lightrays in (V, g), and it is called the *Einstein manifold*. This name is a historical accident, and is unrelated to the notion of an Einstein manifold, where the Ricci tensor is a constant multiple of the metric. The quotient projection $\pi \colon \mathscr{C} \to \mathbb{E}$ defines a principal $\mathbb{R}^\times$-bundle. Indeed, the action $\mathbb{R}^\times \circlearrowright \mathscr{C}$ is free and the (enriched) action map $\mathscr{C} \times \mathbb{R}^\times \to \mathscr{C} \times \mathscr{C}$ is closed. Now, given $L \in \mathbb{E}$, we may choose a non-zero $u \in L$ and consider the derivative $\mathrm{d}\pi_u \colon T_u\mathscr{C} = u^\perp \to T_L\mathbb{E}$. The vertical spaces are the kernels $\ker \mathrm{d}\pi_u = \mathbb{R}u$, which establishes that $T_L\mathbb{E} \cong u^\perp/\mathbb{R}u = L^\perp/L$.

These identifications allow us to try and transfer geometric structures from (V, g) or $\mathscr{C}$ to $\mathbb{E}$, but the problem is that the isomorphism $T_L\mathbb{E} \cong L^\perp/L$ is not natural, and depends on a choice of non-zero vector $u \in L$. Since the vector u is lightlike, the scalar product g passes to the quotient $T_L\mathbb{E} \cong u^\perp/\mathbb{R}u$ as a scalar product g_u, and has signature (p, q), where $i_+ = p + 1$ and $i_- = q + 1$ (indeed, the degenerate metric signature of the lightlike hyperplane $u^\perp$ is $(i_+ - 1, i_- - 1, 1)$, and modding out $\mathbb{R}u$ eliminates the degenerate dimension). However, this does not mean we have defined a pseudo-Riemannian metric on $\mathbb{E}$, as we'll see later that it is impossible to make consistent choice of u's for each L unless g has Lorentzian (or anti-Lorentzian) signature. To understand better what happens, consider a non-zero $\lambda \in \mathbb{R}$, and let's show that $\mathrm{g}_{\lambda u} = \lambda^2 \mathrm{g}_u$ on $T_L\mathbb{E}$. To wit: g_u is the only scalar product in $T_L\mathbb{E}$ for which $(\mathrm{d}\pi_u)^*(\mathrm{g}_u) = \mathrm{g}$, but if $m_\lambda \colon V \to V$ is the multiplication by λ, we have that $\pi \circ m_\lambda = \pi$, leading to $\mathrm{d}\pi_{\lambda u} \circ m_\lambda = \mathrm{d}\pi_u$, so that

$$\mathrm{g} = (\mathrm{d}\pi_u)^*(\mathrm{g}_u) = (\mathrm{d}\pi_{\lambda u} \circ m_\lambda)^*(\mathrm{g}_u) = (m_\lambda)^*(\mathrm{d}\pi_{\lambda u})^*(\mathrm{g}_u) = (\mathrm{d}\pi_{\lambda u})^*(\lambda^2 \mathrm{g}_u).$$

But $\mathrm{g} = (\mathrm{d}\pi_{\lambda u})^*(\mathrm{g}_{\lambda u})$ as well, and $(\mathrm{d}\pi_{\lambda u})^*$ is injective (because π is a submersion), and thus $\mathrm{g}_{\lambda u} = \lambda^2 \mathrm{g}_u$, as required. This means that we have defined a field of pointwise conformal structures (i.e., inner products up to a positive scalar factor in each $T_L\mathbb{E}$) on $\mathbb{E}$, and "smoothness" follows from the fact that $\pi \colon \mathscr{C} \to \mathbb{E}$ admits smooth local sections: if ψ_1 and ψ_2 are two smooth local sections of π, then we may write $\psi_2 = \lambda \psi_1$ with some local smooth function λ, and the above says that on their common domain, the conformal factor λ^2 between the local pseudo-Riemannian metrics induced by ψ_2

and ψ_2 is smooth. Of course, one may glue local representatives via partitions of unity to obtain a global representative of the smooth conformal structure $\mathfrak{C}$ so defined.

Moving on, instead of considering the equivalence relation $\sim$ previously defined, one can define on $\mathscr{C}$ a second relation $\approx$ by saying that $x \approx y$ if $y = \lambda x$ for some positive $\lambda \in \mathbb{R}_{>0}$. The quotient $\widetilde{\mathbb{E}} = \mathscr{C}/_{\approx}$ is the set of all lightlike half-lines in (V, g). Equivalently, $\widetilde{\mathbb{E}}$ is the quotient of $\mathscr{C}$ under the linear action $\mathbb{R}_{>0} \circlearrowright \mathscr{C}$ given by multiplication. The projection $\widetilde{\pi} \colon \mathscr{C} \to \widetilde{\mathbb{E}}$ defines a principal $\mathbb{R}_{>0}$-bundle and everything claimed for $\mathbb{E}$ remains true for $\widetilde{\mathbb{E}}$ as well. In particular, the identity map $\mathscr{C} \to \mathscr{C}$ induces a two-fold covering map $\widetilde{\mathbb{E}} \to \mathbb{E}$, which takes a lightlike half-line to the lightray it spans. With this in place, to understand $\mathbb{E}$ it suffices to understand $\widetilde{\mathbb{E}}$, and for this we'll take an orthogonal decomposition $V = V_+ \oplus V_-$, where the restriction of g to V_+ is positive-definite and to V_- is negative-definite (and hence $\dim V_\pm = i_\pm$). The obvious map

$$\widetilde{\mathbb{E}} \ni \mathbb{R}_{>0}x \mapsto \left(\frac{x_+}{\|x_+\|}, \frac{x_-}{\|x_-\|} \right) \in \mathbb{S}^p \times \mathbb{S}^q$$

is a diffeomorphism, with inverse

$$\mathbb{S}^p \times \mathbb{S}^q \ni (u_+, u_-) \mapsto \mathbb{R}_{>0}(u_+ + u_-) \in \widetilde{\mathbb{E}},$$

where $\mathbb{S}^p$ is the unit sphere of V_+ and $\mathbb{S}^q$ is the unit sphere of V_- (it is equipped with the induced negative-definite round metric). Note that here we cannot replace $\widetilde{\mathbb{E}}$ with $\mathbb{E}$, as the tentative map $\mathbb{E} \to \mathbb{S}^p \times \mathbb{S}^q$ would not be well-defined. However, on the other direction, the sum map $\mathbb{S}^p \times \mathbb{S}^q \to V$ is an isometric immersion, which happens to take values in $\mathscr{C}$ — and composing with π, we obtain a two-fold covering map $\mathbb{S}^p \times \mathbb{S}^q \to \mathbb{E}$. The non-trivial deck transformation $(u_+, u_-) \mapsto (-u_+, -u_-)$ is an isometry, so the metric in $\mathbb{S}^p \times \mathbb{S}^q$ passes to the quotient, giving a global representative of $\mathfrak{C}$. Since the sectional curvatures of $\mathbb{S}^p$ and $\mathbb{S}^q$ are constant and opposites, it follows that $\mathbb{S}^p \times \mathbb{S}^q$ is conformally flat (and hence the conformal structure $\mathfrak{C}$ on $\mathbb{E}$ is flat). Now, let's conclude the discussion with two remarks regarding the Lorentz case.

- The bundle $\pi \colon \mathscr{C} \to \mathbb{E}$ is trivial if and only if g is Lorentzian or anti-Lorentzian. Say that g is Lorentzian and write, with the above notation, $V_- = \mathbb{R}w$ for a unit timelike vector $w \in V$, so that $\mathscr{C} \ni x \mapsto (\mathbb{R}x, \langle x, w \rangle) \in \mathbb{E} \times \mathbb{R}^\times$ is a global trivialization of π. Conversely, assume that g is not Lorentzian or anti-Lorentzian, but that π defines a trivial bundle: since it is a line bundle, it is orientable, meaning that if we choose a null plane $\Pi \subseteq V$, the orientation of $\mathscr{C}$ induces an orientation of the tautological line bundle of the projective line $P\Pi$, which is a contradiction (the total space of such tautological bundle is a Möbius strip).

- The bundle $\mathbb{S}^p \times \mathbb{S}^q \to \mathbb{E}$ is trivial if and only if g is Lorentzian or anti-Lorentzian. This happens since the total space of two-fold covering map over a connected base space is disconnected if and only if the covering map is a trivial $\mathbb{Z}_2$-bundle, and $\mathbb{S}^p \times \mathbb{S}^q$ is disconnected if and only if $p = 0$ (g is anti-Lorentzian) or $q = 0$ (g is Lorentzian).