PHYSICAL MEASUREMENTS VIA INTEGRATION

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ABSTRACT. Many physical quantities can be expressed using integrals, thereby reducing many questions about the world to questions about math. In this note we'll focus on three frequently occurring examples of this: area, volume, and length.

1. AREA

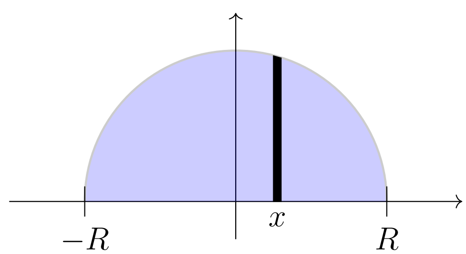
Recall that, by definition, $\int_a^b f(x)\,dx$ is the area between $f(x)$, the x-axis, and the vertical lines a and b (with the convention that area below the x-axis are counted negatively). But this isn't the end of the story: given the same region, there are often several different ways to represent its area as an integral. Let's look at a nice example: the circle.

1.1. **Area of a circle.** Say we wanted to find the area of a circle of radius R. There are at least three ways to use integration to do so.

Method 1: The circle itself isn't a function (it fails the 'vertical line test'), but a semicircle is: it's the function

$$f(x) = \sqrt{R^2 - x^2}.$$

Here's a picture:



A typical narrow vertical strip drawn at x has width dx and height $\sqrt{R^2 - x^2}$, hence has area $\sqrt{R^2 - x^2}\,dx$. Summing all the different strips yields the total area:

$$\text{Area of semicircle of radius } R = \int_{-R}^{R} \sqrt{R^2 - x^2}\,dx.$$

How do we actually compute this integral? A nice trick is to make a substitution that simplifies the integrand: writing

$$x = R\sin\theta,$$

we find that

$$dx = R\cos\theta\,d\theta.$$

EXERCISE 1. Using the substitution above, show that

$$\int \sqrt{R^2 - x^2}\,dx = R^2 \int \cos^2\theta\,d\theta.$$

SOLUTION. Plugging in the trigonometric substitution for x given above, we find

$$\begin{aligned}\int \sqrt{R^2 - x^2}\,dx &= \int \sqrt{R^2 - (R\sin\theta)^2} \cdot (R\cos\theta\,d\theta) = \int \sqrt{R^2 - R^2\sin^2\theta} \cdot R\cos\theta\,d\theta \\ &= \int \sqrt{R^2(1-\sin^2\theta)} \cdot R\cos\theta\,d\theta \\ &= \int \sqrt{R^2} \cdot \sqrt{1-\sin^2\theta} \cdot R\cos\theta\,d\theta \\ &= R^2 \int \sqrt{\cos^2\theta} \cdot \cos\theta\,d\theta = R^2 \int \cos^2\theta\,d\theta\end{aligned}$$

EXERCISE 2. Use integration by parts to show that

$$\int \cos^2\theta\,d\theta = \frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta.$$

[*Hint:* $\sin^2 x = 1 - \cos^2 x$.]

SOLUTION. Taking $u = \cos\theta$ and $dv = \cos\theta\,d\theta$, we find $du = -\sin\theta\,d\theta$ and $v = \sin\theta$, so the integration by parts formula yields

$$\begin{aligned}\int \cos^2\theta\,d\theta &= \sin\theta\cos\theta + \int \sin^2\theta\,d\theta \\ &= \sin\theta\cos\theta + \int (1-\cos^2\theta)\,d\theta \\ &= \sin\theta\cos\theta + \int d\theta - \int \cos^2\theta\,d\theta \\ &= \sin\theta\cos\theta + \theta - \int \cos^2\theta\,d\theta\end{aligned}$$

Note that we have $\int \cos^2\theta\,d\theta$ on both sides of the equation, and we're trying to solve for $\int \cos^2\theta\,d\theta$! Adding this quantity to both sides and simplifying gives

$$\int \cos^2\theta\,d\theta = \frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta.$$

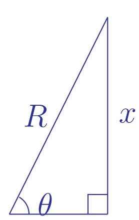
Plugging this in above, we find

$$\int \sqrt{R^2 - x^2}\,dx = R^2 \int \cos^2\theta\,d\theta = \frac{1}{2}R^2\sin\theta\cos\theta + \frac{1}{2}R^2\theta.$$

But of course, the antiderivative of $\sqrt{R^2 - x^2}$ shouldn't be in terms of θ – it should be in terms of x!

EXERCISE 3. Use the right triangle below to deduce that

$$\int \sqrt{R^2 - x^2}\,dx = \frac{1}{2}x\sqrt{R^2 - x^2} + \frac{1}{2}R^2\sin^{-1}\left(\frac{x}{R}\right).$$



SOLUTION. Recall that we set $x = R\sin\theta$; rewriting this, we have $\sin\theta = x/R$, which is where the picture of the right triangle above comes from. By the Pythagorean theorem, the third side of the triangle must be $\sqrt{R^2 - x^2}$. Thus $\cos\theta = \frac{\sqrt{R^2-x^2}}{R}$. Plugging this in, we obtain

$$\begin{aligned}\int \sqrt{R^2 - x^2}\,dx &= \frac{1}{2}R^2 \sin\theta\cos\theta + \frac{1}{2}R^2\theta \\ &= \frac{1}{2}R^2 \cdot \frac{x}{R} \cdot \frac{\sqrt{R^2 - x^2}}{R} + \frac{1}{2}R^2 \sin^{-1}\frac{x}{R} \\ &= \frac{1}{2}x\sqrt{R^2 - x^2} + \frac{1}{2}R^2 \sin^{-1}\frac{x}{R}\end{aligned}$$

Finally we can evaluate the definite integral we started with:

EXERCISE 4. Use the above to prove that

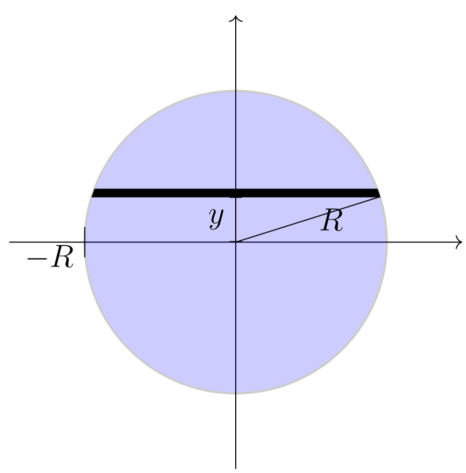
$$\int_{-R}^{R} \sqrt{R^2 - x^2}\,dx = \frac{\pi R^2}{2}.$$

SOLUTION.

$$\begin{aligned}\int_{-R}^{R} \sqrt{R^2 - x^2}\,dx &= \frac{1}{2}x\sqrt{R^2 - x^2} + \frac{1}{2}R^2 \sin^{-1}\frac{x}{R}\Bigg|_{-R}^{R} \\ &= \frac{1}{2}R^2\sin^{-1}(1) - \frac{1}{2}R^2\sin^{-1}(-1) = \frac{1}{2}R^2\frac{\pi}{2} - \frac{1}{2}R^2\left(-\frac{\pi}{2}\right) \\ &= \frac{\pi R^2}{2}\end{aligned}$$

Since this represents the area of the semicircle, we deduce the famous formula πR^2 for the area of the circle of radius R.

<u>Method 2</u>: The circle isn't a function, but we can still draw a picture of it in the plane:



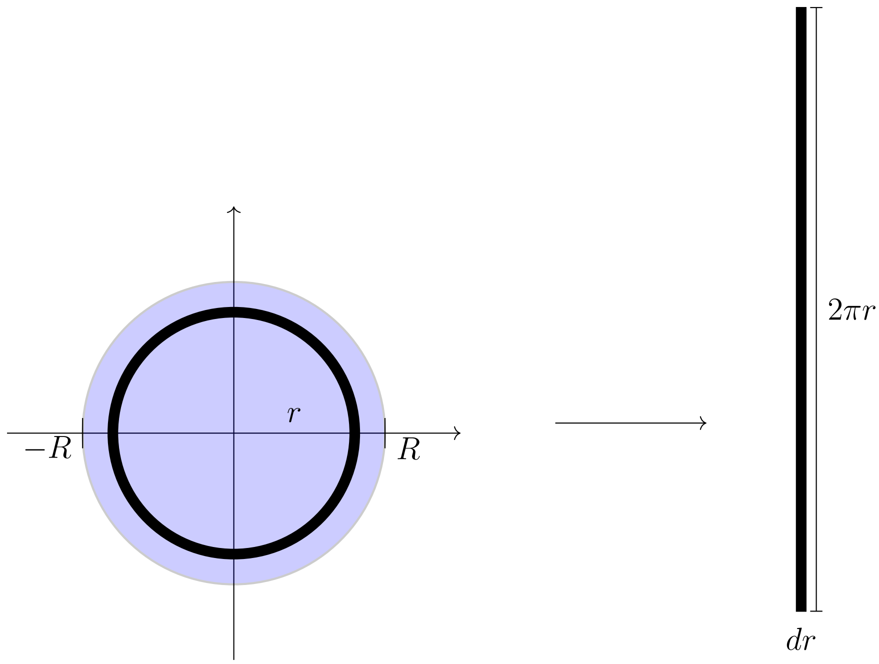
Consider a typical horizontal slice located at height y with tiny thickness dy. What is the length of this slice? From the Pythagorean theorem we see that the right endpoint is $(\sqrt{R^2 - y^2}, y)$; similarly, the left endpoint is $(-\sqrt{R^2 - y^2}, y)$. Thus the length of the slice is $2\sqrt{R^2 - y^2}$, so its area is $2\sqrt{R^2 - y^2}\,dy$. Summing together the areas of all such slices yields the area of the entire circle:

$$\text{Area of circle of radius } R = \int_{-R}^{R} 2\sqrt{R^2 - y^2}\,dy.$$

Now that we've set up the area as an integral, we can forget about what the letters represent geometrically and evaluate the integral. But notice that this is precisely the same integral we've already evaluated:

$$\text{Area of circle of radius } R = \int_{-R}^{R} 2\sqrt{R^2 - y^2}\, dy = 2\int_{-R}^{R} \sqrt{R^2 - y^2}\, dy = 2\int_{-R}^{R} \sqrt{R^2 - x^2}\, dx = \pi R^2.$$

Method 3: Rather than partitioning the circle into straight-line strips, we can split it up into thin circular strips:



Taking a typical strip of radius r around the origin and thickness dr, we can cut and unfold it into a straight line of thickness dr and length $2\pi r$; thus, this strip would have area $2\pi r\, dr$. Summing the areas of all the strips together produces the area of the circle:

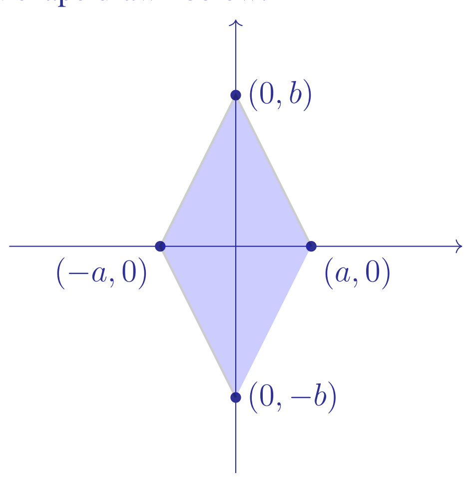
$$\text{Area of circle of radius } R = \int_0^R 2\pi r\, dr.$$

This integral is much more straightforward than the ones from Methods 1 and 2:

$$\int_0^R 2\pi r\, dr = 2\pi \int_0^R r\, dr = 2\pi \cdot \frac{r^2}{2}\bigg|_0^R = 2\pi \frac{R^2}{2} - 0 = \pi R^2.$$

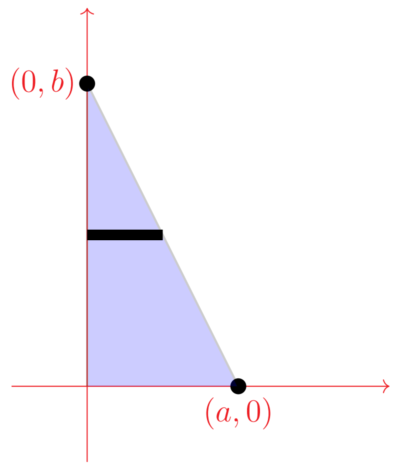
1.2. **Exercises.**

EXERCISE 5. Consider the diamond shape drawn below:

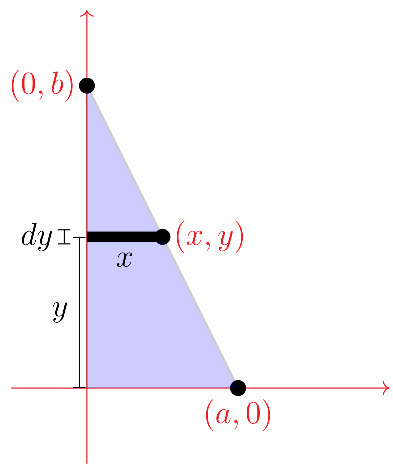


(a) Write down an integral that represents the area of the diamond as a sum of horizontal slices. Then evaluate the integral.

Solution. Let's draw a picture of the upper right quarter of the diamond; if we figure out the area of this, we can simply multiply it by 4 to get the area of the entire diamond. We will slice this up into many thin horizontal slices. A typical slice is pictured below:



Next, we'll find the area of our typical slice, and then sum up over all the areas of all the slices. To find the area of the slice, we need to add some labels to our picture from above:



Note that we don't *have* to call these x and y; we only did so because it's traditional to label the coordinates of a point (x, y), and that forces the labels we chose. Once we settled on y for the height, however, this forces us to call the thickness of the slice dy, since this represents a tiny change in the height.

Having labelled the diagram, we can now carry out our strategy. The area of the slice at height y is $x\,dy$. Summing all these slices together yields

$$\text{Area of triangle} = \int_0^b x\,dy.$$

Two important comments on this integral:

- Both x and y are non-constant variables – the higher up we take the slice, the larger y is and the smaller x is. By contrast, both a and b are constants – they're not changing throughout the problem.
- As things stand, we cannot evaluate this integral because it's in terms of two different variables. So we have to find some way to relate the two variable x and y to one another.
- The fact that the integral is with respect to dy tells you that you want to rewrite the entire integrand in terms of y. In other words, we need to solve for x in terms of y.

There are (at least) two ways to solve for x in terms of y. One approach is to recognize that the triangle we're looking at is similar to the smaller right triangle whose base is our slice. This implies that the ratios of the sides are equal:

$$\frac{x}{a} = \frac{b-y}{b}.$$

A different approach is to notice that the right endpoint of the slice has coordinates (x, y), and that point lies on the line connecting $(a, 0)$ to $(0, b)$. This line has equation $y = -\frac{b}{a}x + b$; in other words, every point on the line satisfies this equation. In particular, the right endpoint of our slice satisfies this equation. Using either of these methods, we're led to discover

$$x = \frac{a}{b}(b-y).$$

Plugging this back into our integral produces

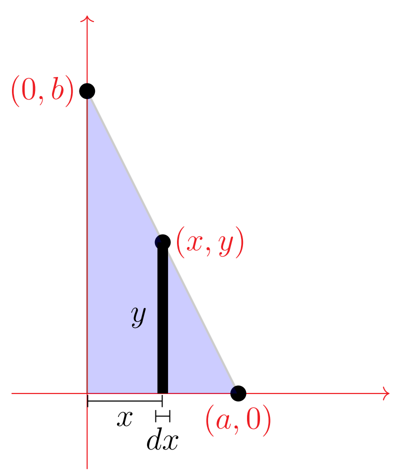
$$\begin{aligned}
\text{Area of triangle} &= \int_0^b x\,dy \\
&= \int_0^b \frac{a}{b}(b-y)\,dy \\
&= \frac{a}{b}\int_0^b (b-y)\,dy \\
&= \frac{a}{b}\left(by - \frac{1}{2}y^2\right)\Big|_0^b \\
&= \frac{a}{b}\left(b^2 - \frac{1}{2}b^2\right) = \frac{1}{2}ab.
\end{aligned}$$

Since this is $1/4$ of the diamond, the area of the diamond must be $4 \times \frac{1}{2}ab = 2ab$.

Note that we just used calculus to figure out the formula for the area of a right triangle. This is silly – there are much easier ways to discover the formula! – but it's a good illustration of the ideas involved in finding the area of a general region where simpler approaches are unavailable.

(b) Write down an integral that represents the area of the diamond as a sum of vertical slices. Then evaluate the integral.

Solution. This is extremely similar to the previous problem, so we just set up the integral and leave the rest to the reader. First things first: draw a picture of a typical slice and label both the location and the size of the slice:



Since the area of the slice is $y\,dx$, we have

$$\text{Area of triangle} = \int_0^a y\,dx.$$

Since (x, y) lies on the line $y = -\frac{b}{a}x + b$, we deduce

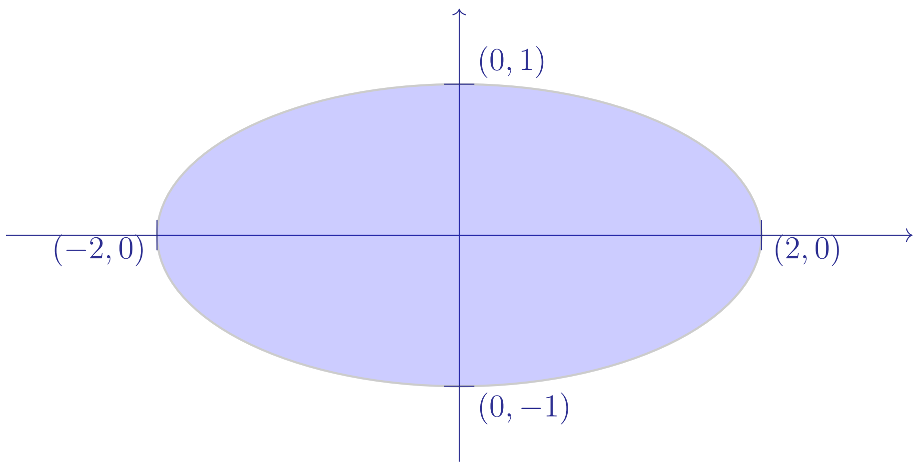
$$\text{Area of triangle} = \int_0^a \left(-\frac{b}{a}x + b\right)\,dx = -\frac{b}{a}\int_0^a x\,dx + b\int_0^a dx = \frac{1}{2}ab$$

(c) (Optional) Can you find another way to set up an integral representing the area of the diamond?

Exercise 6. A standard ellipse is a circle that's stretched out by some amount horizontally and by some other amount vertically. Any standard ellipse that's centered at the origin can be described as the set of points (x, y) satisfying

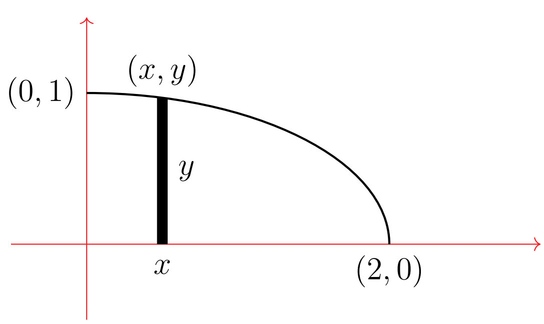
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$$

Here's what the graph of this looks like when $a = 2$ and $b = 1$:



Write down an integral for the area of the illustrated ellipse, and evaluate it. You must explain how the integral is related to the drawing.

Solution. We figure out the area of $1/4$ of the ellipse. We start by drawing a vertical slice at x, of width dx. Let's say it has height y.



Since the area of the slice is $y\,dx$, we find that the area of the whole region is the sum of all these slice areas:

$$\text{Area of quarter-ellipse} = \int_0^2 y\,dx.$$

Now we must find a relationship between x and y. Since (x, y) are the coordinates of the top of this slice, which is located on the ellipse, we deduce that x and y must satisfy the equation of the ellipse:

$$\left(\frac{x}{2}\right)^2 + y^2 = 1.$$

Since the integral is with respect to dx, we need to solve for y:

$$y = \sqrt{1 - \frac{x^2}{4}}.$$

Plugging this in, we have now expressed the area of the entire ellipse as an integral:

$$\text{Area of ellipse} = 4\int_0^2 \sqrt{1 - \frac{x^2}{4}}\, dx = 2\int_0^2 \sqrt{4 - x^2}\, dx.$$

We recognize this integral from above: $\int_0^2 \sqrt{4 - x^2}\, dx$ is the area of a quarter circle of radius 2. Thus,

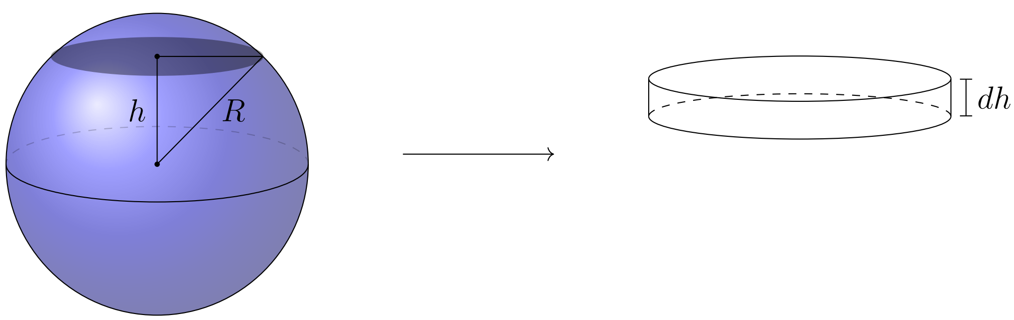
$$\text{Area of ellipse} = 2\int_0^2 \sqrt{4 - x^2}\, dx = 2 \times \frac{1}{4}\pi(2^2) = 2\pi.$$

It's a good exercise to carry out the same argument for a general ellipse with x-radius a and y-radius b; the area you should get is πab.

2. Volume

It turns out that the ideas above generalize beautifully to finding volumes of three-dimensional solids. We consider an example.

2.1. **Volume of a sphere.** Given a solid sphere of radius R. What is its volume? Consider the following picture:



Consider a thin horizontal slice of the sphere at some height h above the center of the sphere. The slice is essentially cylindrical, with thickness dh.

Exercise 7. Explain why the volume of the slice is $\pi(R^2 - h^2)\, dh$.

Solution. The slice looks like a cylinder with thickness dh and radius $\sqrt{R^2 - h^2}$ (by the Pythagorean theorem). Thus the volume of the cylinder is $\pi(\sqrt{R^2 - h^2})^2\, dh = \pi(R^2 - h^2)\, dh$.

Summing up over all the slices, we obtain the volume of the sphere:

$$\text{Volume of sphere of radius } R = \int_{-R}^{R} \pi(R^2 - h^2)\, dh$$

Exercise 8. Conclude that the volume of a sphere of radius R is $\frac{4}{3}\pi R^3$.

Solution. We have

$$\text{Volume of sphere of radius } R = \int_{-R}^{R} \pi(R^2 - h^2)\, dh = \pi\left(R^2 h - \frac{1}{3}h^3\right)\Bigg|_{-R}^{R} = \frac{4}{3}\pi R^3.$$

Unexpectedly, we can use this formula to derive a formula for the surface area of a sphere. Let $S(r)$ represent the surface area of the sphere of radius r. Break a solid sphere of radius R up into very thin spherical shells of thickness dr. Since the shell of radius r has volume $S(r)\,dr$, we can find the total volume of the sphere of radius R:

$$\text{Volume of sphere of radius } R = \int_0^R S(r)\,dr.$$

But we just figured out above a formula for the volume. Plugging this in, we get

$$\frac{4}{3}\pi R^3 = \int_0^R S(r)\,dr.$$

Now we do something crazy: we've been thinking of R as a specific number, but since the above formula is true for any choice of R, we can think of R as a variable. Differentiating with respect to R, we find

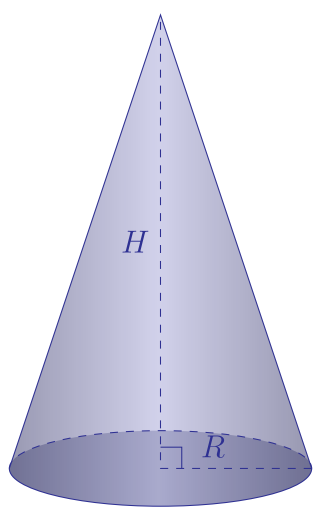
$$4\pi R^2 = \frac{d}{dR}\int_0^R S(r)\,dr.$$

By the Fundamental Theorem of Calculus, the right hand side is simply $S(R)$. In other words, we've figured out that

$$\text{Surface area of a sphere of radius } R = S(R) = 4\pi R^2.$$

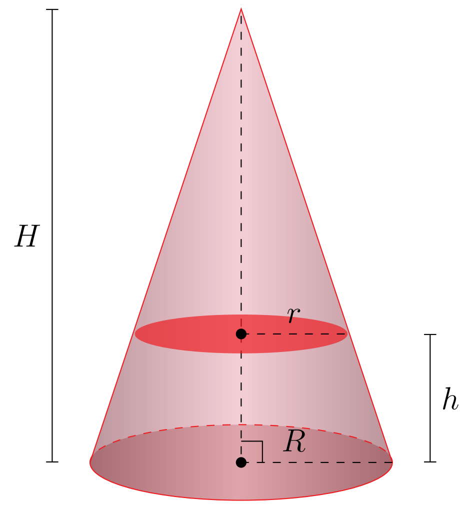
2.2. Exercises.

Exercise 9. Consider a right circular cone, i.e. a cone whose base is a circle and whose top vertex lies directly above the center of the base. Here's a picture:



By considering a thin horizontal slice at height h above the base and thickness dh, represent the volume of the cone as an integral. Then evaluate the integral to find a simple formula for the volume of the cone.

Solution. We start by drawing a typical slice, say, a distance h above the base of the cone (and therefore of thickness dh) and of radius r:



This slice is essentially a cylinder; its volume is $\pi r^2\,dh$. Thus, the volume of the cone is

$$\text{Volume of cone} = \int_0^H \pi r^2\,dh.$$

All that remains is to find the relationship between r and h.

Note that in the illustration, the dashed lines outline two similar right triangles: a small one inside of a bigger one. Using similarity, we have

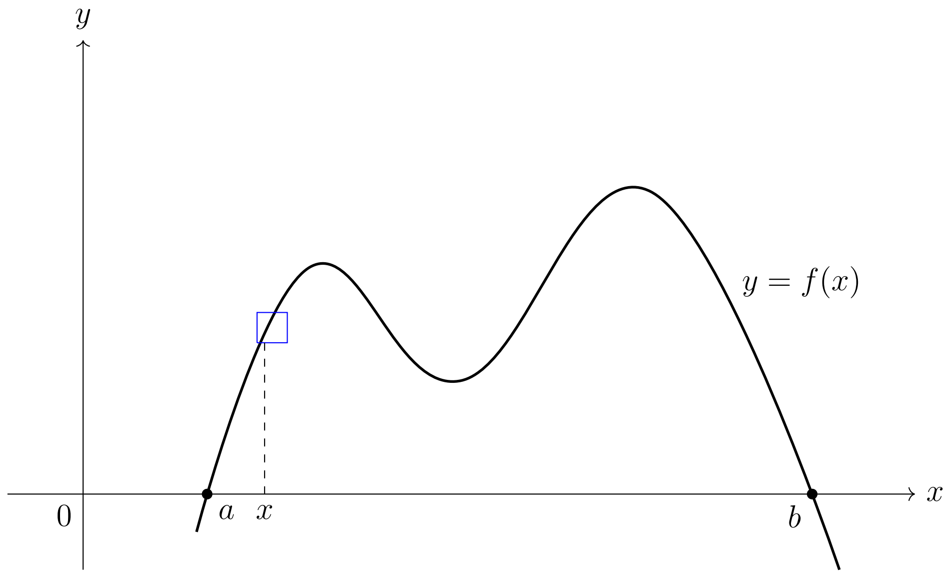
$$\frac{r}{R} = \frac{H-h}{H}.$$

We deduce $r = \frac{R}{H}(H-h)$, whence

$$\begin{aligned}
\text{Volume of cone} &= \int_0^H \pi r^2\,dh \\
&= \pi \int_0^H \left(\frac{R}{H}(H-h)\right)^2 dh \\
&= \frac{\pi R^2}{H^2}\int_0^H (H-h)^2\,dh \\
&= \frac{\pi R^2}{H^2} \times \frac{-1}{3}(H-h)^3\Big|_0^H \\
&= \frac{\pi R^2}{H^2} \times \frac{1}{3}H^3 = \frac{1}{3}\pi R^2 H.
\end{aligned}$$

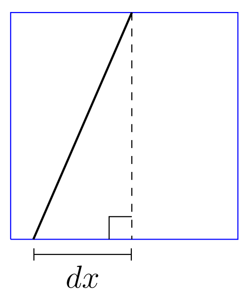
3. Length

Surprisingly, the same slicing approach can be employed to obtain a formula for the length of a curve. For example, consider the following picture:



Question: If you walk along the curve $y = f(x)$, starting at a and ending at b, how far have you walked?

Slice the curve into many very short pieces, and zoom in on a typical piece (say, the one I've indicated in the blue box, located at x). If the piece is short enough, it looks like a straight line segment:



Assuming the total horizontal change during this part of the walk is a tiny quantity dx, how long is this piece of the walk? In other words, how long is the hypotenuse of the right triangle drawn above? If we can figure out the height, we can use the Pythagorean theorem to find out!

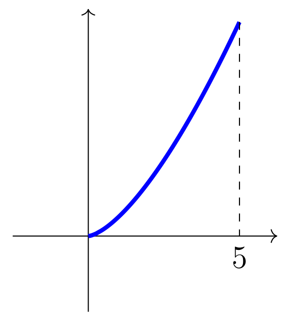
This whole picture should look familiar to you – it's precisely the same set-up we used to define the derivative of f at x! Recall that $f'(x)$ is the slope of the hypotenuse. When the slope of a straight line is m, that means that if you go over 1, you go up m; if you go over 2, you go up $2m$; if you go over $\frac{1}{3}$, you go up $\frac{1}{3}m$. Thus, since the slope of the line drawn above is $f'(x)$ and we're going over dx, we must go up $f'(x)\,dx$. By the Pythagorean theorem, we find that the length of the hypotenuse is

$$\sqrt{(dx)^2 + \Big(f'(x)\,dx\Big)^2} = \sqrt{(dx)^2 + f'(x)^2\,(dx)^2} = \sqrt{\Big(1 + f'(x)^2\Big)(dx)^2} = \sqrt{1 + f'(x)^2}\;dx$$

Summing up the lengths of all these tiny pieces together yields the total length of the walk:

$$\text{Total length of walk along } f \text{ from } a \text{ to } b = \int_a^b \sqrt{1 + f'(x)^2}\;dx.$$

3.1. **Example.** Suppose a piece of string happens to look precisely like the curve $f(x) = x^{3/2}$ between $x = 0$ and $x = 5$:



How long is the piece of string? By our formula, the total length is

$$\int_0^5 \sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}\,dx = \int_0^5 \left(1+\frac{9}{4}x\right)^{1/2} dx = \frac{2}{3}\cdot\frac{4}{9}\left(1+\frac{9}{4}x\right)^{3/2}\Bigg|_0^5 = \frac{335}{27}$$

3.2. **Exercises.**

Exercise 10. Find the length along the curve $y = x^{2/3}$ from $x = 0$ to $x = 13\sqrt{13}$.

Solution. The most natural approach is to simply employ the given formula for length: we have

$$\text{Length} = \int_0^{13\sqrt{13}} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^{13\sqrt{13}} \sqrt{1+\frac{4}{9}x^{-2/3}}\,dx$$

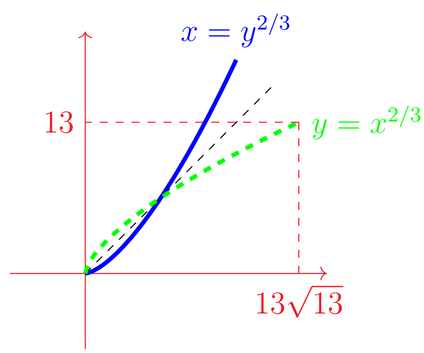
The integral is complicated, so guessing seems out of the question. The next line of attack, as always, is u-substitution. Take $u = x^{2/3}$, so that $x = u^{3/2}$ and $dx = \frac{3}{2}\sqrt{u}\,du$. Plugging these into the integral and simplifying yields

$$\int \sqrt{1+\frac{4}{9}x^{-2/3}}\,dx = \frac{3}{2}\int \sqrt{1+\frac{4}{9u}}\cdot\sqrt{u}\,du = \frac{3}{2}\int \left(u+\frac{4}{9}\right)^{1/2} du = \left(u+\frac{4}{9}\right)^{3/2} = \left(x^{2/3}+\frac{4}{9}\right)^{3/2}.$$

This allows us to evaluate our definite integral:

$$\text{Length} = \int_0^{13\sqrt{13}} \sqrt{1+\frac{4}{9}x^{-2/3}}\,dx = \left(x^{2/3}+\frac{4}{9}\right)^{3/2}\Bigg|_0^{13\sqrt{13}} = \left(13+\frac{4}{9}\right)^{3/2} - \left(\frac{4}{9}\right)^{3/2} = 49$$

Alternative (tricky) solution. Note that if we take the curve and move it around without changing its shape, the length will remain the same. In particular, we can reflect (aka flip) the curve over the diagonal line $y = x$ without changing the length:



The new (flipped) curve is the inverse function of the original; everything about it is given by exchanging the roles of x and y. In particular, its equation is $x = y^{2/3}$, and we are considering the portion of this where y is between 0 and $13\sqrt{13}$. In other words, the length we're looking for is the same as the length of the curve

$y = x^{3/2}$ between $x = 0$ and $x = 13$. This is precisely the same problem we solved in the example, except with a different endpoint! Thus,

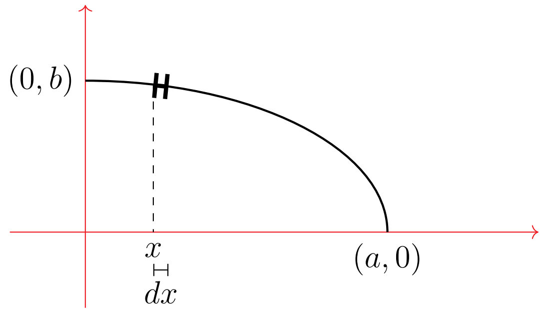
$$\text{Length} = \int_0^{13} \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\, dx = \frac{2}{3} \cdot \frac{4}{9} \left(1 + \frac{9}{4}x\right)^{3/2} \Bigg|_0^{13} = 49.$$

EXERCISE 11. Recall that the equation of a standard ellipse centered at the origin with x-radius a and y-radius b is

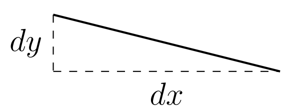
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$$

Figure out an integral representing the perimeter (circumference) of this ellipse. [*Don't worry about evaluating this integral – no one knows how to do that.*]

SOLUTION. As usual, we simplify the problem by looking at a quarter-ellipse (and then multiplying by 4). Taking a very short slice of the curve, located at x and having a horizontal width of dx:



The length of the segment, however, is *not* dx, because the segment is tilted. Here's a picture (having zoomed in on the segment):



By the Pythagorean theorem, the length of the segment is

$$\text{Length of segment} = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(1 + \frac{(dy)^2}{(dx)^2}\right)(dx)^2}$$

$$= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Summing up over all the different segments, we find

$$\text{Length of quarter-ellipse} = \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Note that this is precisely the same formula we obtained earlier, using a slightly different approach.

In any event, all that remains is to determine $\frac{dy}{dx}$. Rearranging the equation of the ellipse yields

$$y = \frac{b}{a}\sqrt{a^2 - x^2},$$

and differentiating produces

$$\frac{dy}{dx} = \frac{b}{a} \times \frac{-2x}{2\sqrt{a^2 - x^2}} = -\frac{bx}{a\sqrt{a^2 - x^2}}$$

Plugging this back into our length formula gives

$$\begin{aligned}\text{Circumference of ellipse} &= 4\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx\\ &= 4\int_0^a \sqrt{1+\frac{b^2}{a^2}\cdot\frac{x^2}{a^2-x^2}}\,dx\\ &= \frac{4b}{a}\int_0^a \sqrt{\frac{a^2}{b^2}-1+\frac{a^2}{a^2-x^2}}\,dx\end{aligned}$$

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