PHYSICAL MEASUREMENTS VIA INTEGRATION

LEO GOLDMAKHER

ABSTRACT. Many physical quantities can be expressed using integrals, thereby reducing many questions about the world to questions about math. In this note we'll focus on three frequently occurring examples of this: area, volume, and length.

1. AREA

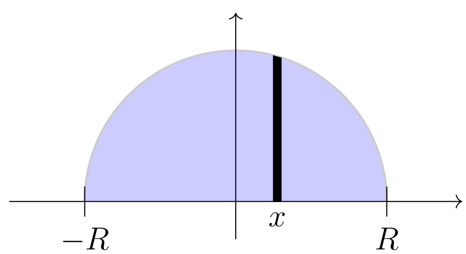
Recall that, by definition, $\int_a^b f(x)\,dx$ is the area between $f(x)$, the x-axis, and the vertical lines a and b (with the convention that area below the x-axis are counted negatively). But this isn't the end of the story: given the same region, there are often several different ways to represent its area as an integral. Let's look at a nice example: the circle.

1.1. **Area of a circle.** Say we wanted to find the area of a circle of radius R. There are at least three ways to use integration to do so.

Method 1: The circle itself isn't a function (it fails the 'vertical line test'), but a semicircle is: it's the function

$$f(x) = \sqrt{R^2 - x^2}.$$

Here's a picture:



A typical narrow vertical strip drawn at x has width dx and height $\sqrt{R^2 - x^2}$, hence has area $\sqrt{R^2 - x^2}\,dx$. Summing all the different strips yields the total area:

$$\text{Area of semicircle of radius } R = \int_{-R}^{R} \sqrt{R^2 - x^2}\,dx.$$

How do we actually compute this integral? A nice trick is to make a substitution that simplifies the integrand: writing

$$x = R\sin\theta,$$

we find that

$$dx = R\cos\theta\,d\theta.$$

Exercise 1. Using the substitution above, show that

$$\int \sqrt{R^2 - x^2}\,dx = R^2 \int \cos^2\theta\,d\theta.$$

Exercise 2. Use integration by parts to show that

$$\int \cos^2\theta \, d\theta = \frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta.$$

[*Hint:* $\sin^2 x = 1 - \cos^2 x$.]

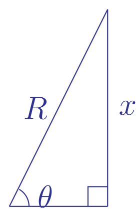
Plugging this in above, we find

$$\int \sqrt{R^2 - x^2}\, dx = R^2 \int \cos^2\theta \, d\theta = \frac{1}{2}R^2 \sin\theta\cos\theta + \frac{1}{2}R^2\theta.$$

But of course, the antiderivative of $\sqrt{R^2 - x^2}$ shouldn't be in terms of θ – it should be in terms of x!

Exercise 3. Use the right triangle below to deduce that

$$\int \sqrt{R^2 - x^2}\, dx = \frac{1}{2}x\sqrt{R^2 - x^2} + \frac{1}{2}R^2 \sin^{-1}\left(\frac{x}{R}\right).$$



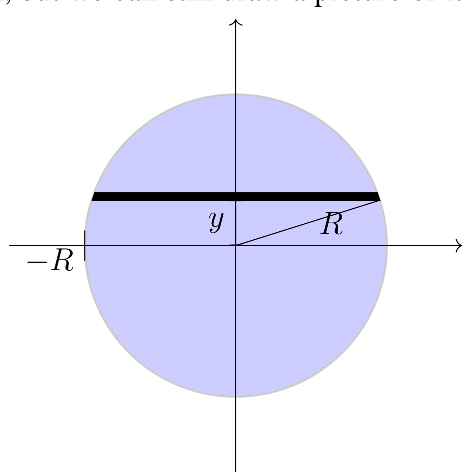
Finally we can evaluate the definite integral we started with:

Exercise 4. Use the above to prove that

$$\int_{-R}^{R} \sqrt{R^2 - x^2}\, dx = \frac{\pi R^2}{2}.$$

Since this represents the area of the semicircle, we deduce the famous formula πR^2 for the area of the circle of radius R.

Method 2: The circle isn't a function, but we can still draw a picture of it in the plane:



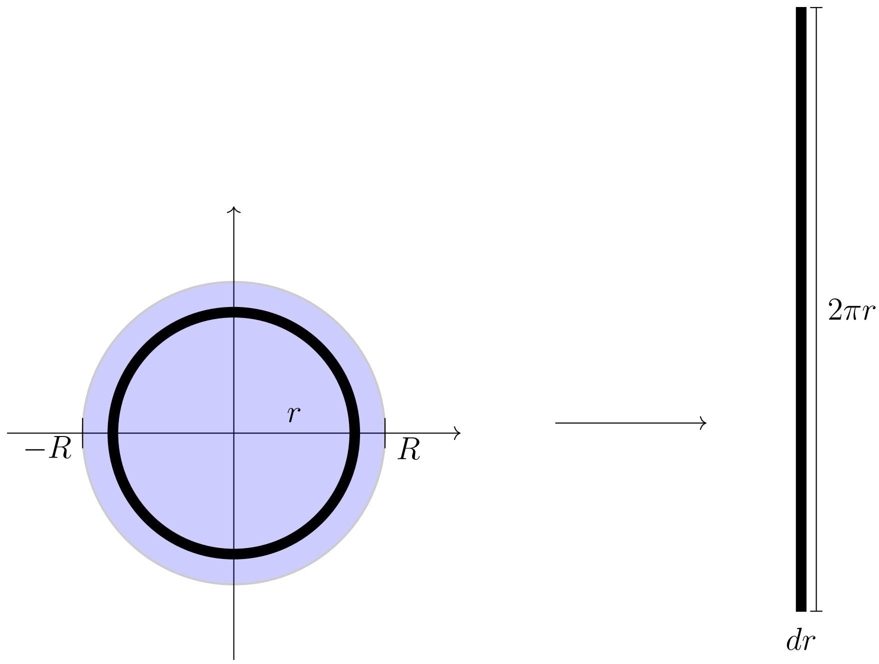
Consider a typical horizontal slice located at height y with tiny thickness dy. What is the length of this slice? From the Pythagorean theorem we see that the right endpoint is $(\sqrt{R^2 - y^2}, y)$; similarly, the left endpoint is $(-\sqrt{R^2 - y^2}, y)$. Thus the length of the slice is $2\sqrt{R^2 - y^2}$, so its area is $2\sqrt{R^2 - y^2}\, dy$. Summing together the areas of all such slices yields the area of the entire circle:

$$\text{Area of circle of radius } R = \int_{-R}^{R} 2\sqrt{R^2 - y^2}\, dy.$$

Now that we've set up the area as an integral, we can forget about what the letters represent geometrically and evaluate the integral. But notice that this is precisely the same integral we've already evaluated:

$$\text{Area of circle of radius } R = \int_{-R}^{R} 2\sqrt{R^2 - y^2}\,dy = 2\int_{-R}^{R} \sqrt{R^2 - y^2}\,dy = 2\int_{-R}^{R} \sqrt{R^2 - x^2}\,dx = \pi R^2.$$

<u>Method 3</u>: Rather than partitioning the circle into straight-line strips, we can split it up into thin circular strips:



Taking a typical strip of radius r around the origin and thickness dr, we can cut and unfold it into a straight line of thickness dr and length $2\pi r$; thus, this strip would have area $2\pi r\,dr$. Summing the areas of all the strips together produces the area of the circle:

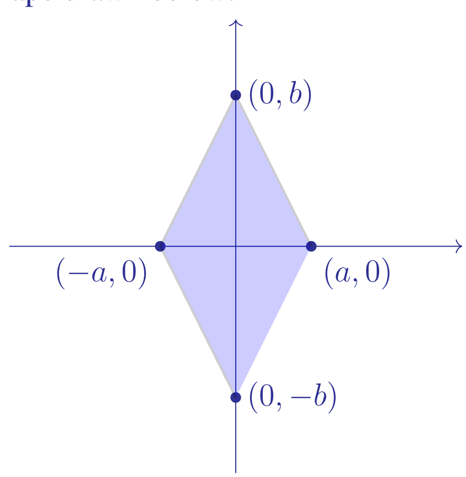
$$\text{Area of circle of radius } R = \int_0^R 2\pi r\,dr.$$

This integral is much more straightforward than the ones from Methods 1 and 2:

$$\int_0^R 2\pi r\,dr = 2\pi\int_0^R r\,dr = 2\pi \cdot \frac{r^2}{2}\bigg|_0^R = 2\pi\frac{R^2}{2} - 0 = \pi R^2.$$

1.2. **Exercises.**

Exercise 5. Consider the diamond shape drawn below:

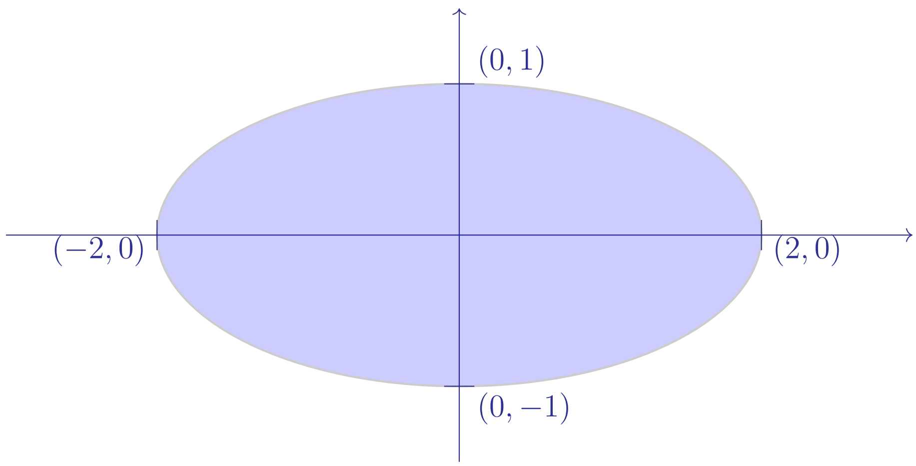


(a) Write down an integral that represents the area of the diamond as a sum of horizontal slices. Then evaluate the integral.
(b) Write down an integral that represents the area of the diamond as a sum of vertical slices. Then evaluate the integral.
(c) (Optional) Can you find another way to set up an integral representing the area of the diamond?

Exercise 6. A standard ellipse is a circle that's stretched out by some amount horizontally and by some other amount vertically. Any standard ellipse that's centered at the origin can be described as the set of points (x, y) satisfying

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$$

Here's what the graph of this looks like when $a = 2$ and $b = 1$:

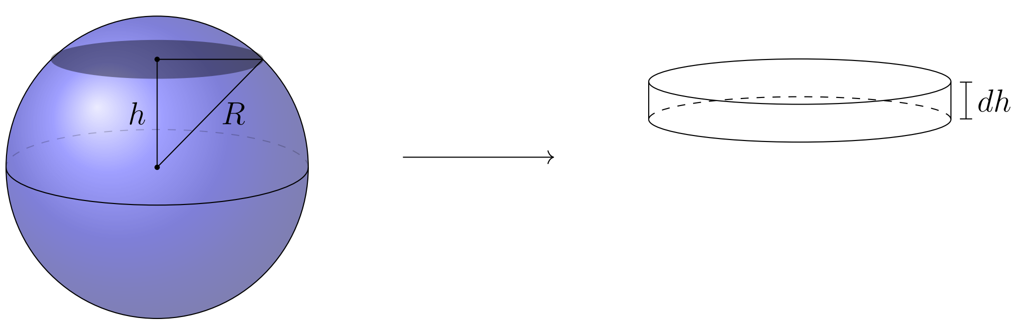


Write down an integral for the area of the illustrated ellipse, and evaluate it. You must explain how the integral is related to the drawing.

2. Volume

It turns out that the ideas above generalize beautifully to finding volumes of three-dimensional solids. We consider an example.

2.1. **Volume of a sphere.** Given a solid sphere of radius R. What is its volume? Consider the following picture:



Consider a thin horizontal slice of the sphere at some height h above the center of the sphere. The slice is essentially cylindrical, with thickness dh.

Exercise 7. Explain why the volume of the slice is $\pi(R^2 - h^2)\,dh$.

Summing up over all the slices, we obtain the volume of the sphere:

$$\text{Volume of sphere of radius } R = \int_{-R}^{R} \pi(R^2 - h^2)\,dh$$

Exercise 8. Conclude that the volume of a sphere of radius R is $\frac{4}{3}\pi R^3$.

Unexpectedly, we can use this formula to derive a formula for the surface area of a sphere. Let $S(r)$ represent the surface area of the sphere of radius r. Break a solid sphere of radius R up into very thin spherical shells of thickness dr. Since the shell of radius r has volume $S(r)\,dr$, we can find the total volume of the sphere of radius R:

$$\text{Volume of sphere of radius } R = \int_0^R S(r)\,dr.$$

But we just figured out above a formula for the volume. Plugging this in, we get

$$\frac{4}{3}\pi R^3 = \int_0^R S(r)\,dr.$$

Now we do something crazy: we've been thinking of R as a specific number, but since the above formula is true for any choice of R, we can think of R as a variable. Differentiating with respect to R, we find

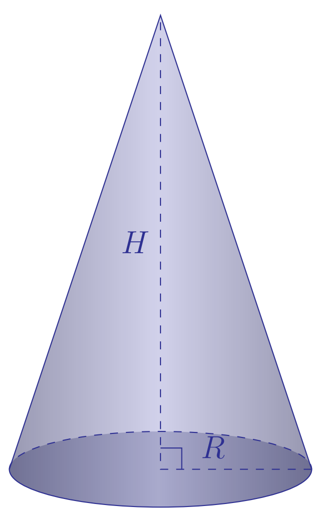
$$4\pi R^2 = \frac{d}{dR}\int_0^R S(r)\,dr.$$

By the Fundamental Theorem of Calculus, the right hand side is simply $S(R)$. In other words, we've figured out that

$$\text{Surface area of a sphere of radius } R = S(R) = 4\pi R^2.$$

2.2. **Exercises.**

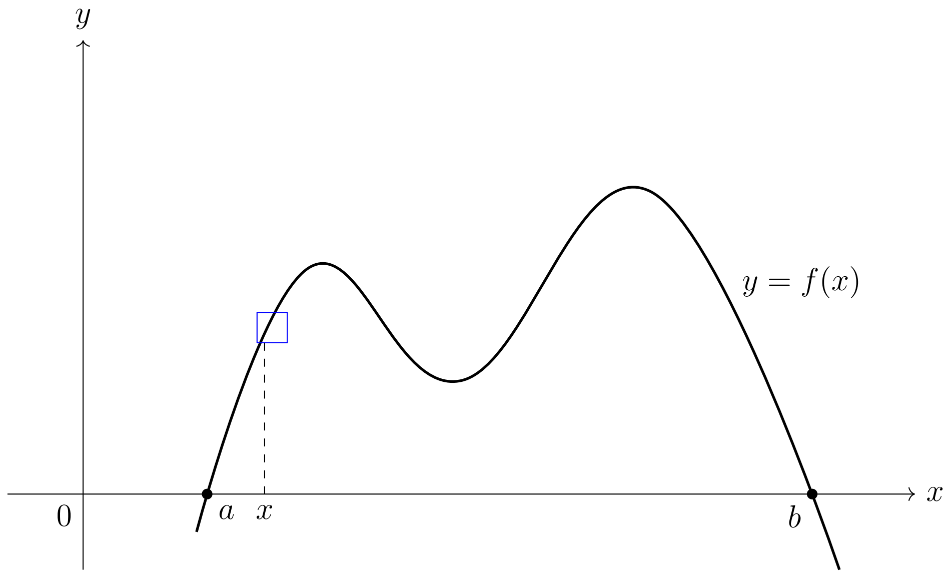
Exercise 9. Consider a right circular cone, i.e. a cone whose base is a circle and whose top vertex lies directly above the center of the base. Here's a picture:



By considering a thin horizontal slice at height h above the base and thickness dh, represent the volume of the cone as an integral. Then evaluate the integral to find a simple formula for the volume of the cone.

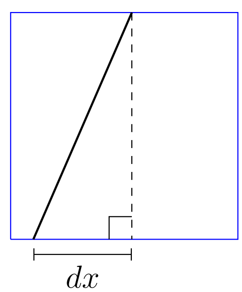
3. Length

Surprisingly, the same slicing approach can be employed to obtain a formula for the length of a curve. For example, consider the following picture:



Question: If you walk along the curve $y = f(x)$, starting at a and ending at b, how far have you walked?

Slice the curve into many very short pieces, and zoom in on a typical piece (say, the one I've indicated in the blue box, located at x). If the piece is short enough, it looks like a straight line segment:



Assuming the total horizontal change during this part of the walk is a tiny quantity dx, how long is this piece of the walk? In other words, how long is the hypotenuse of the right triangle drawn above? If we can figure out the height, we can use the Pythagorean theorem to find out!

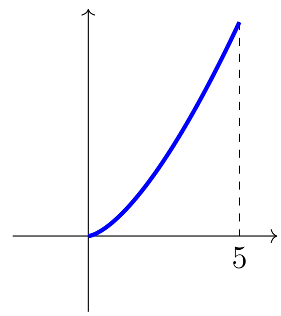
This whole picture should look familiar to you – it's precisely the same set-up we used to define the derivative of f at x! Recall that $f'(x)$ is the slope of the hypotenuse. When the slope of a straight line is m, that means that if you go over 1, you go up m; if you go over 2, you go up $2m$; if you go over $\frac{1}{3}$, you go up $\frac{1}{3}m$. Thus, since the slope of the line drawn above is $f'(x)$ and we're going over dx, we must go up $f'(x)\,dx$. By the Pythagorean theorem, we find that the length of the hypotenuse is

$$\sqrt{(dx)^2 + \Big(f'(x)\,dx\Big)^2} = \sqrt{(dx)^2 + f'(x)^2\,(dx)^2} = \sqrt{\Big(1 + f'(x)^2\Big)(dx)^2} = \sqrt{1 + f'(x)^2}\;dx$$

Summing up the lengths of all these tiny pieces together yields the total length of the walk:

$$\text{Total length of walk along } f \text{ from } a \text{ to } b = \int_a^b \sqrt{1 + f'(x)^2}\;dx.$$

3.1. **Example.** Suppose a piece of string happens to look precisely like the curve $f(x) = x^{3/2}$ between $x = 0$ and $x = 5$:



How long is the piece of string? By our formula, the total length is

$$\int_0^5 \sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}\,dx = \int_0^5 \left(1+\frac{9}{4}x\right)^{1/2} dx = \frac{2}{3}\cdot\frac{4}{9}\left(1+\frac{9}{4}x\right)^{3/2}\bigg|_0^5 = \frac{335}{27}$$

3.2. **Exercises.**

Exercise 10. Find the length along the curve $y = x^{2/3}$ from $x = 0$ to $x = 13\sqrt{13}$.

Exercise 11. Recall that the equation of a standard ellipse centered at the origin with x-radius a and y-radius b is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$$

Figure out an integral representing the perimeter (circumference) of this ellipse. [*Don't worry about evaluating this integral – no one knows how to do that.*]

DEPT OF MATHEMATICS AND STATISTICS, WILLIAMS COLLEGE, WILLIAMSTOWN, MA, USA 01267
Email address: `Leo.Goldmakher@williams.edu`