INTEGRATING RATIONAL FUNCTIONS WITH LINEAR DENOMINATOR

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ABSTRACT. A rational function is a function of the form $\frac{f(x)}{g(x)}$, where f and g are polynomials. In this note I'll describe a method for finding the anti-derivative of any rational function with a linear denominator.

1. FIRST EXAMPLES

Recall that

$$\int \frac{1}{x}\, dx = \ln|x| + C.$$

What if we're trying to integrate a more complicated expression, like

$$\int \frac{2x^2 - 5x + 3}{x}\, dx?$$

It's not at all obvious (to me, at least) how to guess an anti-derivative of the integrand. A second approach might be u-substitution; taking $u = 2x^2 - 5x + 3$, we find that $du = (4x - 5)dx$, but plugging this into our integral only makes it more complicated, not less.

But there's a nice trick that simplifies the integral considerably – we separate the fraction into a sum of three simpler fractions. Here's how this looks:

$$\begin{aligned}\int \frac{2x^2 - 5x + 3}{x}\, dx &= \int \frac{2x^2}{x} - \frac{5x}{x} + \frac{3}{x}\, dx \\ &= \int 2x - 5 + \frac{3}{x}\, dx \\ &= x^2 - 5x + 3\ln|x| + C.\end{aligned}$$

This idea applies to a much more general situation – in which we have any linear polynomial in the denominator of the fraction – with a few small twists.

2. A MORE COMPLICATED EXAMPLE

Consider the anti-derivative

$$\int \frac{2x^2 - 5x + 3}{x - 2}\, dx.$$

How do we deal with this integral? One way is to attempt to imitate the example above and split the fraction into three simpler fractions:

$$\frac{2x^2 - 5x + 3}{x - 2} = \frac{2x^2}{x - 2} - \frac{5x}{x - 2} + \frac{3}{x - 2}$$

Unfortunately, this doesn't simplify the problem. For example, how would we find the anti-derivative of $\frac{2x^2}{x-2}$? There are two standard approaches:

2.1. **Polynomial long division.** This is a process that works well but is a bit complicated, so I generally don't use it. Thus, rather than explain this approach in detail, I'll merely illustrate how it works on the above example. We have

$$\begin{aligned}\frac{2x^2-5x+3}{x-2} &= \frac{2x(x-2)+4x}{x-2} - \frac{5x}{x-2} + \frac{3}{x-2} \\ &= 2x - \frac{x}{x-2} + \frac{3}{x-2} \\ &= 2x - \frac{(x-2)+2}{x-2} + \frac{3}{x-2} \\ &= 2x - 1 + \frac{5}{x-2}.\end{aligned}$$

As a consequence, we deduce

$$\begin{aligned}\int \frac{2x^2-5x+3}{x-2}\,dx &= \int 2x - 1 + \frac{5}{x-2}\,dx \\ &= x^2 - x + 5\ln|x-2| + C.\end{aligned}$$

Having demonstrated how polynomial long division works, I'll now turn to a method I prefer.

2.2. **u-substitution.** In our introductory examples we had just a single x in the denominator, and this made the process of finding the anti-derivative relatively simple. Our next example

$$\int \frac{2x^2-5x+3}{x-2}\,dx$$

is made more complicated because the denominator is $x-2$, rather than a single variable.

The trick is to force the denominator to just be a single variable by using u-substitution. Setting $u = x-2$, we find $du = dx$, so

$$\int \frac{2x^2-5x+3}{x-2}\,dx = \int \frac{2x^2-5x+3}{u}\,du.$$

The expression on the right hand side is not a nice integral because it's in terms of two different variables, x and u. To make it all in terms of u, we solve for x: since $u = x-2$, we have $x = u+2$. Plugging this into our integral we find

$$\begin{aligned}\int \frac{2x^2-5x+3}{x-2}\,dx &= \int \frac{2x^2-5x+3}{u}\,du \\ &= \int \frac{2(u+2)^2-5(u+2)+3}{u}\,du \\ &= \int \frac{2(u^2+4u+4)-5(u+2)+3}{u}\,du \\ &= \int \frac{2u^2+3u+1}{u}\,du \\ &= \int \frac{2u^2}{u} + \frac{3u}{u} + \frac{1}{u}\,du \\ &= \int 2u + 3 + \frac{1}{u}\,du \\ &= u^2 + 3u + \ln|u| + C \\ &= (x-2)^2 + 3(x-2) + \ln|x-2| + C.\end{aligned}$$

Although this looks like a long computation, conceptually it consists of just four steps:

Step 1. Set u to be the denominator, and solve for both dx and x in terms of u and du.

Step 2. Substituting in, rewrite the original integral exclusively in terms of u and du; the denominator should now just be u.
Step 3. Separate the fraction into a sum of fractions, and integrate these one at a time.
Step 4. Rewrite your answer in terms of just x.

3. An even more complicated example

Let's try the u-substitution method for a more involved example. Consider

$$\int \frac{10x^3+4}{3x+1}\,dx.$$

Setting $u = 3x+1$, we find (with a bit of algebra) that

$$dx = \frac{1}{3}du \qquad \text{and} \qquad x = \frac{u-1}{3}.$$

Substituting these into our integral yields

$$\begin{aligned}
\int \frac{10x^3+4}{3x+1}\,dx &= \int \frac{\frac{10(u-1)^3}{3^3}+4}{u}\,\frac{1}{3}du \\
&= \frac{1}{3}\int \frac{\frac{10}{27}(u^3-3u^2+3u-1)+4}{u}\,du \\
&= \frac{1}{3}\int \frac{10}{27}u^2 - \frac{10}{9}u + \frac{10}{9} + \frac{98}{27u}\,du \\
&= \frac{1}{3}\left(\frac{10}{27}\cdot\frac{1}{3}u^3 - \frac{5}{9}u^2 + \frac{10}{9}u + \frac{98}{27}\ln|u|\right) + C \\
&= \frac{10}{243}u^3 - \frac{5}{27}u^2 + \frac{10}{27}u + \frac{98}{81}\ln|u| + C \\
&= \frac{10}{243}(3x+1)^3 - \frac{5}{27}(3x+1)^2 + \frac{10}{27}(3x+1) + \frac{98}{81}\ln|3x+1| + C
\end{aligned}$$

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