PROPOSITIONAL AND PREDICATE CALCULUS

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The goal of this essay is to describe two types of logic: Propositional Calculus (also called 0th order logic) and Predicate Calculus (also called 1st order logic). Both work with propositions and logical connectives, but Predicate Calculus is more general than Propositional Calculus: it allows variables, quantifiers, and relations.

1. Propositional Calculus

Given two numbers, we have various ways of combining them: add them, multiply them, etc. We can also take the negative or absolute value or square of a single number, and apply various functions to a given number. In other words, we can perform various operations on both individual numbers and on collections of numbers, and this endows the set of all numbers with a rich structure (e.g. arithmetic).

Can we do the same for mathematical argument? Is there an arithmetic of mathematical assertions?

1.1. **Propositions.**

The first thing to do is to formally define what 'mathematical assertion' means. We shall refer to a mathematical assertion as a *proposition*; the book uses the word *statement* for this concept.

Definition. A *proposition* is a statement that is either true or false, but not both, neither, or sometimes one and sometimes the other.

For example:

(1) *Williams College is located in Williamstown.* is a proposition (because it's true).
(2) *Leo is a frog.* is a propositions (because it's false).
(3) *You are located in Williamstown.* is not a proposition, because it's sometimes true and sometimes false.
(4) *This statement is false.* is not a proposition, because it is neither true nor false.
(5) *Every even number larger than 2 is the sum of two primes.* is a proposition, because it's either true or false. (No one knows which! This is called *Goldbach's conjecture*, and it's been open for at least 270 years.)

The last example is a special type of proposition called a *predicate*, which we'll discuss later. In the meantime, we return to our original question: given two propositions, how can we combine them?

1.2. **Boolean Algebra.**

We use *logical connectives*: and, or, not, thus, etc. These have fancy names and symbols:

(1) 'and' is called *conjunction*, denoted $\wedge$
(2) 'or' is called *disjunction*, denoted $\vee$
(3) 'not' is called *negation*, denoted $\neg$
(4) 'thus' of 'if... then...' is called *conditional*, denoted $\implies$

Now we can study of the "arithmetic" formed by propositions under these (and other) logical connectives. For brevity, set

$$P := \textit{Williams college is in Williamstown} \qquad \text{and} \qquad Q := \textit{Leo is a frog}$$

and consider the following.

(1) $\neg Q$ is true
(2) $P \wedge Q$ is false
(3) $P \vee Q$ is true
(4) $P \implies Q$: true or false?
(5) $P \impliedby Q$: true or false?

The last two are tricky, because it's not clear at all what the connection is between P and Q and how to evaluate the validity of the statement as a whole. Let's abstract these. The first is of the form

$$\textit{true proposition} \implies \textit{false proposition}.$$

Ideally one shouldn't be able to logically deduce a false statement from a true one (at least, this goes against the intuition of what logic is for!). So this hints that we should declare the proposition $P \implies Q$ as false.

The second questionable statement was of the form

$$\textit{false proposition} \implies \textit{true proposition}.$$

Is this something we want to accept? Should it be possible to logically deduce a true statement from a false one? There are several ways to argue why it's reasonable to accept this as a logical argument. First, it's a proposition that's *vacuously true*. To see this, consider the assertion

Every human made of cheese is named Bill.

This is vacuously true—they are all named Bill, because there are no such humans. We can restate this as a conditional statement:

If a human is made of cheese, then the human's name is Bill.

In other words, whenever we have an assertion of the form (*false statement*) $\implies$ (*any statement*) we might consider it true vacuously, because the initial condition is never met so anything can happen.[1] This argument is a bit suspect, though: *the human's name is Bill* isn't a proposition at all!

In a similar vein, one can think about promises made and kept. Suppose I announced:

If you give me a Lamborghini, then I'll give you an A.

If you gave me a Lamborghini and I gave you an A, I was telling the truth. And if you didn't give me a Lamborghini and I didn't give you an A, I was still telling you the truth. What if you didn't give me a Lambo and I gave you an A? I was still telling the truth! The only situation in which I was not telling the truth is if you gave me a Lambo and I didn't give you an A. Thus we're tempted to assert that (True $\implies$ True) is true, (False $\implies$ True) is true, (False $\implies$ False) is true, and (True $\implies$ False) is false. Once again, this argument is suspect, since our example doesn't involve propositions. ("You give me a Lamborghini" is neither true nor false, so it's not a proposition.)

A more convincing argument comes from a mathematical example involving actual propositions. Consider the following logical deduction:

$$2 = 3 \quad \implies \quad \underbrace{0 \times 2}_{0} = \underbrace{0 \times 3}_{0}.$$

In other words, if we assume the 2 really does equal 3, then it logically follows that 0 equals 0. Of course, it turns out the 0 equals 0 even without making this weird assumption! But that isn't relevant here: we've shown that we can deduce a true statement from a false one. Of course, we can also logical deduce a false proposition from a false one:

$$2 = 3 \implies 2 = 3.$$

And we can deduce a true proposition from a true one:

$$0 = 0 \implies 0 = 0.$$

The only thing we cannot accomplish to deduce a false proposition from a true one. Try to deduce $2 = 3$ from the starting point $0 = 0$.

Inspired by this, we define the truth value of $P \implies Q$ using the following 'truth table':

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

[1] This type of argument is summed up in a famous idiom: *If my grandmother had wheels, she'd be a truck.*

Note that the two statements $P \implies Q$ and $Q \implies P$ are logically independent. They might both be true, or one might be true and the other false. (Can you come up with an examples of these?) A visual way to see this is to compare their truth tables:

P	Q	$P \implies Q$	$P \impliedby Q$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Their outputs aren't always the same, so $P \implies Q$ and $P \impliedby Q$ are not logically equivalent. By the way, these have fancy names: given the *conditional statement* $P \implies Q$, the proposition $P \impliedby Q$ is called its *converse*.

We can create similar truth tables for other combinations of propositions, for example for $(\neg P) \impliedby (\neg Q)$:

P	Q	$\neg P$	$\neg Q$	$(\neg P) \impliedby (\neg Q)$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Notice anything? This is the same output as $P \implies Q$! In other words, $(\neg P) \impliedby (\neg Q)$ is logically equivalent to $P \implies Q$; we represent this symbolically as

$$(\neg P) \impliedby (\neg Q) \equiv P \implies Q.$$

The odd-looking proposition on the left hand side is called the *contrapositive* of the conditional $P \implies Q$, and the fact that it's logically equivalent is super useful. In fact, we've already used it in this course: the way we justified that

$$a^2 \text{ is even } \implies a \text{ is even}$$

was by arguing

$$a \text{ is odd } \implies a^2 \text{ is odd},$$

which is the contrapositive of the first statement.

1.3. **Transforming boolean algebra into a genuine arithmetic.** Given a bunch of propositions, we have four operations ($\neg$, $\wedge$, $\vee$, and $\implies$) that we can apply to them (either to a single proposition, or to two propositions) to create new propositions. This is highly reminiscent of arithmetic: given a bunch of numbers, we have four operations ($+, -, \times, \div$) that we can apply to them to create new numbers.

One fun observation is that we can turn Boolean algebra into literal arithmetic. To do this, we assign numerical values to the truth of a given proposition, say

$$\#P := \begin{cases} 1 & \text{if } P \text{ is true, and} \\ 0 & \text{if } P \text{ is false.} \end{cases}$$

Thus $\#(\textit{Leo is a frog}) = 0$. Using this notation, we can rewrite our truth table for $\wedge$:

$\#P$	$\#Q$	$\#(P \wedge Q)$
1	1	1
1	0	0
0	1	0
0	0	0

Note that this is indistinguishable from a (very short) multiplication table. So $\wedge$ is the logical equivalent of multiplication, i.e.

$$\#(P \wedge Q) = \#P \cdot \#Q.$$

A bit of thought shows we can play the same game for $\neg P$:

$$\#(\neg P) = 1 - \#P.$$

There are similar formulas one can derive for the other boolean operations.

1.4. **Biconditionals.** Now we come to a logical connective that will play a fundamental role in our work this semester: the *biconditional*. To motivate it, first consider the following question: what does it mean for a number to be a perfect square? A number is a *perfect square* if it's the square of an integer. Right?

Wrong! This 'definition' tells us that 4 is a perfect square, but it doesn't tell us whether or not 5 is a perfect square. Using 'if and only if' (usually abbreviated iff) resolves the issue, however:

Definition. We say a n is a *perfect square* iff $n = k^2$ for some integer k.

The boolean operator that represents this is called a *biconditional* and is denoted $\iff$. More precisely, we define $\iff$ via the following truth table:

P	Q	$P \iff Q$
T	T	T
T	F	F
F	T	F
F	F	T

Iff captures the idea that P and Q are logically equivalent: each one implies the other. Another way to write this idea down is

$$(P \iff Q) \equiv (P \implies Q) \wedge (P \impliedby Q)$$

a logical equivalence that is easily proved via a truth table. In practice this is usually how we prove biconditional statements. Let's do an example.

Proposition 1.1. $n^2 - 1$ *is a perfect square iff* $n = \pm 1$.

Proof. We prove the two directions separately. We begin with the easier of the two:
($\impliedby$) If $n = \pm 1$, then $n^2 - 1 = 0$ which is a perfect square.
($\implies$) If $n^2 - 1$ is a perfect square, then there exists an integer a such that

$$n^2 - 1 = a^2.$$

Thus, $(n - a)(n + a) = 1$. This implies that both $n + a$ and $n - a$ are 1, or both are -1; in particular, $n + a = n - a$. Thus $a = 0$, whence $n^2 = 1$, so $n = \pm 1$ as claimed. $\square$

2. Predicate Calculus

In practice we often want to make statements like

Conjecture 2.1 (Goldbach's Conjecture)**.** *Every even number larger than 2 is the sum of two primes.*

This is a proposition, but it's a rather fancy type of proposition, because it contains within it multiple propositions: one for each even number larger than 2. This type of statement is called a *predicate*.

Definition. A *predicate* is a sentence written in terms of a finite set of variables that becomes a proposition for any choice of those variables allowed by the sentence.

What are the variables in Goldbach's conjecture?

- the even number, and
- the two primes.

We can rewrite Goldbach's conjecture in a way that makes the variables explicit:

Conjecture 2.2. *For all even integers* $n \geq 4$*, there exist primes* p *and* q *such that* $n = p + q$.

To be propositions, predicates need to have some quantifiers and relations that specify what the variables are allowed to be. For the purposes of this course, we'll require just two quantifiers: 'for all' (aka 'for every' or 'for any') and 'there exist(s)'. We've encountered examples of these already, albeit implicitly:

(1) For all positive integers N we have $1 + 2 + \cdots + N = \frac{N(N+1)}{2}$.
(2) For all integers a, b with $b \neq 0$, we have $\left(\frac{a}{b}\right)^2 \neq 2$.
(3) For every integer $n \geq 2$ there exists a prime p such that n is divisible by p.
(4) For any finite collection of primes, there exists a prime not in our collection.

But these statements have something in common other than the quantifiers: they all contain relations specifying the set where the variables are allowed to live. For example, we can rewrite Goldbach's conjecture in the form

Conjecture 2.3. *For all n belonging to $\{$even integers $\geq 4\}$, there exist p, q belonging to $\{primes\}$ such that $n = p + q$.*

The other predicates we wrote down can be translated into formal predicate form as well:

(1) For all N belonging to $\{$positive integers$\}$ we have $1 + 2 + \cdots + N = \frac{N(N+1)}{2}$.
(2) For all a belonging to $\{$integers$\}$ and all b belonging to $\{$nonzero integers$\}$, $\left(\frac{a}{b}\right)^2 \neq 2$.
(3) For all n belonging to $\{$integers $\geq 2\}$ there exists a prime p such that n is divisible by p.
(4) For all n belonging to $\{$positive integers$\}$ and for all $p_1, p_2, \ldots, p_n$ belonging to $\{$primes$\}$, there exists q belonging to $\{$primes$\}$ such that $q \neq p_i$ for all i belonging to $\{$integers ≥ 1 and $\leq n\}$.

The fact that we're writing the same phrases over and over again motivates introducing some notation!

(1) $\forall$ means 'for all'
(2) $\exists$ means 'there exists'
(3) $\in$ means 'belongs to'

Thus, for example, Goldbach's conjecture becomes

Conjecture 2.4. *$\forall n \in \{$even integers $\geq 4\}$, $\exists p, q \in \{primes\}$ such that $n = p + q$.*

3. A cautionary tale: secret predicates

Consider the conditional statement

If you're in the ocean, then you're not in water.

and its converse

If you're not in water, then you're in the ocean.

Both of these are manifestly false. But a simple truth table shows that $P \implies Q$ and $Q \implies P$ can *never* be simultaneously false! What's going on here?!

The problem is that we are secretly working with predicates, not merely propositions—the use of colloquial English buried the quantifier. To make it explicit, we can rewrite the original conditional above in formal predicate form:

$\forall x \in \{$people$\}$, *if x is in the ocean, then x is not in water.*

A similar rewriting of the second statement reads

$\forall x \in \{$people$\}$, *if x is not in water, then x is in the ocean.*

The former conditional statement is false when applied to a person x who happens to be currently in the ocean, and is therefore false in general; the latter conditional statement is false when x is me right now, writing this document on dry land, and therefore false in general. Thus, both statements are false—*but they fail at different choices of the variable x*. If we specify the same choice of x in both, then one is false and the other is true! For example, let x be me right now. Then the first conditional becomes

If Leo is in the ocean, then Leo is not in water.

which is (vacuously) true—I'm not in the ocean, so the conclusion becomes irrelevant!—while the second conditional statement becomes

If Leo is not in water, then Leo is in the ocean.

which is false.

This illustrates the danger of using colloquial English when working in logic: it's easy to hide quantifiers and relations, in effect making predicates look like propositions when they might not be. This also motivates drawing the following distinction.

3.1. **Propositional Calculus vs Predicate Calculus.** *Propositional calculus* is the study of the boolean algebra of propositions that don't involve predicates (i.e. no variables, quantifiers, or relations). Thus

$$\textit{if } 2 = 3, \textit{ then } 0 = 0$$

is a valid proposition in propositional calculus, but

$$\textit{if } x = 3, \textit{ then } 2x + 5 = 10$$

is not. This is a useful system as a first attempt at codifying rules of logical inference, but is extremely simplistic. For this reason, propositional calculus is sometimes called *0th order logic*.

Once we allow predicates, we get a much more interesting logical system called *predicate calculus*. Note, in particular, that predicate calculus is an extension of propositional calculus: any proposition from propositional calculus continues to be a proposition in predicate calculus, but not conversely. Thus predicate calculus is sometimes called *1st order logic*.

It turns out the story doesn't end here. One can create richer and richer systems of higher-order logic by allowing more operations, for example quantifying relations in addition to variables. Although we won't discuss these further in this course, if you're interested in this topic you should take a few more math courses (abstract algebra and real analysis) and then check out the beautiful book *A Course in Mathematical Logic for Mathematicians* by Yuri Manin.

4. Mathematics: a bird's eye view

Here's how math works. You start with a finite set of axioms; these are propositions that are *defined* to be true. Next, you create theorems, which are true propositions that can be logically deduced from propositions already known to be true. (At the beginning, only the axioms are known to be true; from these you deduce some theorems; from these theorems and the axioms, you deduce other theorems; etc.) A mathematical theory is a finite set of axioms and the set of all propositions that are true with respect to these axioms. Here are some examples of mathematical theories:

(1) Euclidean geometry (generated by 10 axioms)
(2) Non-Euclidean geometry (generated by the same axioms as euclidean geometry, apart from the parallel postulate)
(3) Number theory (generated by the 9 Peano axioms)
(4) Real analysis (generated by 13 axioms)
(5) Set theory (generated by the 9 axioms of ZFC)

Remark. This is all extremely different from science and philosophy, in which *theory* means a conjecture. Broadly speaking, the goal of science is to discover a minimal set of axioms from which explanations for all observed phenomena can be derived. The goal of math, by contrast, is exactly the opposite: to explore the landscape of a given theory by discovering new theorems.

In practice, mathematicians don't take proofs down all the way to the axioms; instead, they reduce down to a bunch of statements that are known consequences of the axioms. We'll do the same in this course, relying on an unnecessarily huge set of axioms: things you learned up through high school. To see honest reductions all the way down to the level of axioms, you should take abstract algebra (355) and real analysis (350).

Here are a few different types of theorems you'll encounter in the wild:

(1) A *Lemma* is a theorem whose primary interest is its utility in the proof of another theorem.
(2) A *Proposition* is a little theorem.
(3) A *Corollary* is a theorem that's a quick consequence of another theorem.
(4) A *Conjecture* is a proposition made in the language of the axioms.

A much less common word (but extremely useful concept) is *Porism*: a theorem that's a quick consequence of the *proof* of another theorem.

5. Final remarks on logic

Mathematical logic is the formal study of mathematical theories. It is a beautiful (albeit difficult) field. Here's a very brief discussion of some of the major breakthroughs from the 20th century. The main thrust is the question of whether a given mathematical theory is nice. Here are some attributes we might wish a nice theory to possess:

(1) *Soundness*: every finite formal deduction from the axioms yields a true proposition. (Colloquially: every valid proof yields a true statement.)
(2) *Completeness*: every true proposition can be attained via a finite sequence of logical deductions from the axioms. (Colloquially: every true statement can be proved.)
(3) *Consistency*: no two theorems derived from the axioms will contradict one another.

Can one prove that a given set of axioms leads to a theory that possesses the above desirable qualities?

The first progress towards this was made by Paul Bernays (in 1918) and Emil Post (in 1921), who proved that propositional calculus (viewed as a mathematical theory itself) is sound and complete. (Colloquially, this says that a proposition is true iff it is a theorem.) This implies that propositional calculus is also consistent.

This is neat, but propositional is extremely simplistic as a model of mathematics. In his 1929 doctoral dissertation, Kurt Gödel proved that predicate calculus – a much richer theory than propositional calculus – is also sound and complete. This result is now known as the *Completeness Theorem*.

Even predicate calculus doesn't quite model the complexity of doing mathematics. What about taking an actually practiced mathematical theory, like number theory? Is it a nice theory? Gödel surprised working mathematicians by proving that *any* theory that's strong enough to do arithmetic in cannot simultaneously be consistent and complete. In other words, number theory is either inconsistent (has contradictory theorems) or incomplete (there exist propositions about numbers that, if true, are not provable from the axioms). This crazy result is now known as *Gödel's first incompleteness theorem*.

The first incompleteness theorem shows that consistency and completeness are incompatible in any sufficiently interesting mathematical theory. But which one fails? Can we check whether or not number theory is consistent, for example? Following up on his first theorem, Gödel then proved a second fundamental limitation on interesting mathematical theories: he proved that any theory that's strong enough to do some arithmetic cannot prove its own consistency (assuming that it is consistent). This is no known as *Gödel's second incompleteness theorem*. It's worth pointing out that we *can* sometimes prove consistency from within a larger framework, e.g. the consistency of the Peano axioms for arithmetic is *provable* in the broader set theory of ZFC.

All of this seems super abstract – largely because it is – but it led to fundamental breakthroughs in computer science as well. One of the most famous is Turing's work on the Halting Problem. The problem is this: does there exist an algorithm that, given as inputs a computer program and an input into the program, predicts whether the program will terminate (halt) after a finite number of steps? (This is desirable, since the alternative is for a program to enter an infinite loop and never produce an output.) Inspired by Gödel's work, Turing proved (in 1936) that such an algorithm cannot exist. This shows the limitations on what we can do with computer science to check its own programs.

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