LINEAR ALGEBRA: LECTURE 2

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Today we continued exploring which functions $f : \mathbb{R} \to \mathbb{R}$ satisfy

$$f(x+y) = f(x) + f(y) \qquad \forall x, y \in \mathbb{R}. \tag{1}$$

(Recall the notation from class: $\forall$ means *for every*, and $\in$ means *contained in*.) In a sense, we're solving the above equation for the function f. This is quite different from what you've done in previous math courses, where you've tried to solve equations for numbers. This is a theme which will recur throughout linear algebra: where previously numbers were 'nouns' and functions were 'verbs' acting on those nouns, in linear algebra the functions themselves will be our nouns.

Last time we discovered a few nice properties of any f satisfying (1). For example:

Proposition 1. *If f satisfies (1), then $f(-x) = -f(x)\ \forall x \in \mathbb{R}$.*

Proposition 2. *If f satisfies (1), then $f(n) = nf(1)\ \forall n \in \mathbb{Z}$.*

(Here $\mathbb{Z}$, coming from the German word *zahl*, denotes the set of all integers.)

These results, as well as some other examples we tried, led us to guess the following.

Conjecture 3. *If f satisfies (1), then $f(x) = xf(1)\ \forall x \in \mathbb{R}$.*

We began approaching this conjecture by proving a special case of it.

Proposition 4. *If f satisfies (1), then $f\left(\frac{a}{b}\right) = \frac{a}{b}f(1)\ \forall a, b \in \mathbb{Z}$ with $b \neq 0$.*

Proof. If $a = 0$, we're done. Thus, we may assume for the remainder of the proof that neither a nor b is 0.

We split the proof into several cases. First, suppose a and b are both positive. Then we have

$$\begin{aligned} f(1) &= f\Big(\underbrace{\frac{1}{b} + \cdots + \frac{1}{b}}_{b}\Big) \\ &= f\Big(\frac{1}{b}\Big) + f\Big(\underbrace{\frac{1}{b} + \cdots + \frac{1}{b}}_{b-1}\Big) \\ &= f\Big(\frac{1}{b}\Big) + f\Big(\frac{1}{b}\Big) + f\Big(\underbrace{\frac{1}{b} + \cdots + \frac{1}{b}}_{b-2}\Big) \\ &= \cdots \\ &= bf\Big(\frac{1}{b}\Big), \end{aligned}$$

from which it follows that

$$f\Big(\frac{1}{b}\Big) = \frac{1}{b}f(1).$$

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Next, we have

$$\begin{aligned} f\left(\frac{a}{b}\right) &= f\left(\frac{1}{b}\right) + f\left(\frac{a-1}{b}\right) \\ &= f\left(\frac{1}{b}\right) + f\left(\frac{1}{b}\right) + f\left(\frac{a-2}{b}\right) \\ &= \cdots \\ &= af\left(\frac{1}{b}\right) \\ &= \frac{a}{b}f(1) \end{aligned}$$

as claimed. This handles the case when both a and b are positive.

If a and b are both negative, a simple trick saves us a lot of work: using what we just proved above, we would have

$$\begin{aligned} f\left(\frac{a}{b}\right) &= f\left(\frac{-a}{-b}\right) \\ &= \frac{-a}{-b}f(1) \\ &= \frac{a}{b}f(1) \end{aligned}$$

We have therefore proved the theorem in all circumstances but one: that a and b have opposite signs. But then $-a$ and b must have the same sign, whence (by our above proofs) we must have

$$f\left(\frac{-a}{b}\right) = \frac{-a}{b}f(1).$$

Applying Proposition 1 yields

$$f\left(\frac{a}{b}\right) = -f\left(\frac{-a}{b}\right) = -\frac{-a}{b}f(1) = \frac{a}{b}f(1).$$

The proof is now over. □

Note that the square at the end of the proof indicates that the proof is over. Another, more traditional, symbol for this is: QED.

Having warmed up by verifying a special case, we seem to be within striking distance of proving our earlier conjecture. There are some surprises, however. We will discuss these (and other matters) next lecture.