LINEAR ALGEBRA: LECTURE 3

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Today we continued exploring which functions $f : \mathbb{R} \to \mathbb{R}$ satisfy

$$f(x+y) = f(x) + f(y) \qquad \forall x, y \in \mathbb{R}. \tag{1}$$

Last time we proved the following:

Theorem 1. *If $f : \mathbb{R} \to \mathbb{R}$ satisfies (1), then $f(\alpha) = \alpha f(1)$ $\forall \alpha \in \mathbb{Q}$.*

Here $\mathbb{Q}$ denotes the set of all fractions. We can express this using only symbols:

$$\mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z},\ b \neq 0 \right\}.$$

The $\{\cdots\}$ means a set (or collection). What are we collecting in our set? All objects of the form $\frac{a}{b}$ which satisfy the properties following the colon.

It may strike you as strange to define $\mathbb{Q}$, since we already have the symbol $\mathbb{R}$ for all numbers. It turns out that many real numbers – indeed, the vast majority of real numbers – are not in $\mathbb{Q}$. Numbers which live in $\mathbb{R}$ but not in $\mathbb{Q}$ are called *irrational*, because they made no sense to ancient Greek mathematicians. (Elements of $\mathbb{Q}$, by contrast, are called *rational*.) Here's a classical example of an irrational number:

Theorem 2. $\sqrt{2} \notin \mathbb{Q}$.

Before we prove this theorem, it's worth pointing out that this is impossible to verify by computation. Indeed, even using the most powerful supercomputers available, one would only be able to compare $\sqrt{2}$ to 0% of all possible fractions! We're going to be able to prove the above with our bare hands.

Proof. Suppose $\sqrt{2}$ were rational. Then we would be able to write $\sqrt{2} = \frac{a}{b}$ for some integers a, b with $b \neq 0$. Squaring both sides and clearing denominators would give

$$a^2 = 2b^2. \tag{2}$$

In particular, this means a^2 is *even* (i.e. divisible by 2). It follows that a itself must be even; see Lemma 3 below. Thus, we can write $a = 2m$ for some integer m. Plugging this back into equation (2) and simplifying, we see that

$$b^2 = 2m^2.$$

Therefore, b^2 is even, so (again by Lemma 3, below) b must be even.

Let's review our proof thus far. We've shown that if $\sqrt{2} = \frac{a}{b}$, then both a and b must be even. This is a bit odd, but not impossible – there are plenty of fractions where both numerator and denominator are even (for example, $\frac{2}{4}$). Of course, it's silly to write a fraction this way; since both top and bottom are even, we may as well divide both by 2. Thus, if $\sqrt{2} = \frac{a}{b}$, then $\sqrt{2} = \frac{a/2}{b/2}$, where $a/2$ is an integer and $b/2$ is a natural number.

Now comes the crucial point: *we've shown that* whenever $\sqrt{2}$ *is a fraction, both its numerator and denominator must be even!* Thus, both $a/2$ and $b/2$ must be even. This means that $a/4$ and $b/4$ are both integers; since $\sqrt{2} = \frac{a/4}{b/4}$, we see that each of these must be even, so that $a/8$ and $b/8$ are both integers. Proceeding in this way, we see that $a/2^n$ would have to be an even integer for every $n \in \mathbb{N}$. Now note that when n is large enough, we'll have $0 \leq a/2^n < 1$. There's only one even integer between 0 and 1, namely, 0 itself. Thus, a must have been zero. But this is a contradiction, since $\sqrt{2} \neq 0$. Therefore, our initial assumption that $\sqrt{2} \in \mathbb{Q}$ must have been incorrect. The theorem is proved. $\square$

Date: February 10, 2016.

A different way to write down the proof, which is cleaner in some ways, is given below. But first, we quickly verify the following useful fact (which we used twice during our proof):

Lemma 3. *Given $n \in \mathbb{Z}$. If n^2 is even, then n must be even.*

Proof. Given n^2 an even integer. Since every integer must be either even or odd, it suffices to show that n cannot be odd. If n were odd, then we could write $n = 2k + 1$ for some $k \in \mathbb{Z}$. But this would imply

$$n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1,$$

which contradicts our assumption that n^2 is even! Thus, n cannot be odd, hence must be even. $\square$

It's worth pointing out that it's also true that if n is even, then n^2 must be even. However, this fact is *independent* of the above lemma – it tells us nothing about whether or not the lemma is true! We'll return to this point in a future lecture.

The above proof of Theorem 2 is perfectly acceptable as a proof, but (with hindsight) can be cleaned up a bit by adding a harmless assumption at the start of the proof. Here's an alternative, cleaned-up version of the proof; I've highlighted the addition assumption.

Tidy proof of Theorem 2. Suppose $\sqrt{2}$ were rational. Then we would be able to write $\sqrt{2} = \frac{a}{b}$ for some integers a, b with $b \neq 0$. **We can safely assume that $a \neq 0$, and that $\frac{a}{b}$ is a reduced fraction.** (If it isn't reduced, reduce it, and start the proof over!) Squaring both sides and clearing denominators would give

$$a^2 = 2b^2. \tag{2}$$

In particular, this means a^2 is *even* (i.e. divisible by 2). It follows from Lemma 3 that a itself must be even. Thus, we can write $a = 2m$ for some integer m. Plugging this back into equation (2) and simplifying, we see that

$$b^2 = 2m^2.$$

Therefore, b^2 is even, so Lemma 3 tells us that b must be even. We've now shown that if $\sqrt{2} = \frac{a}{b}$ (a reduced nonzero fraction), then both a and b must be even. But this is patently impossible, since the fraction must be reduced! The only possible conclusion we can make is that our assumption was incorrect; we deduce that $\sqrt{2}$ must be irrational. $\square$

Now that we know about the existence of irrational numbers, we can consider how a function f satisfying (1) behaves at irrational inputs. We will take this up next lecture.