LINEAR ALGEBRA: LECTURE 6

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Last time we began discussing examples of linear maps. In particular, we proved the following:

Proposition 1. *A function $f : \mathbb{R} \to \mathbb{R}$ is linear if and only if there exists $k \in \mathbb{R}$ such that $f(x) = kx$.*

Theorem 2. *A function $f : \mathbb{R}^2 \to \mathbb{R}$ is linear if and only if there exist $a, b \in \mathbb{R}$ such that $f(x, y) = ax + by$.*

These are clearly quite similar. The first thing we do is to rewrite Theorem 2 in a way which makes it look the same as the proposition. To do this, we need to recall the definition of the *dot product*:

Definition. Given two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$, we define their *dot product* to be

$$x \cdot y := x_1 y_1 + x_2 y_2.$$

It's important to keep in mind that $x \cdot y \in \mathbb{R}$; the dot product of two *points* yields a *number*. Using this language, we can now rewrite Theorem 2 in a way which makes it look exactly like the proposition:

Theorem 2. *A function $f : \mathbb{R}^2 \to \mathbb{R}$ is linear if and only if there exists $k \in \mathbb{R}^2$ such that $f(x) = k \cdot x$.*

Note that both k and x are points in $\mathbb{R}^2$ in the above theorem; the output of f is still a number, since dot products output numbers.

The phrases *if and only if*, *there exists*, and *such that* have come up a bunch of times, and will continue to arise. Henceforth we will abbreviate these:

- The symbol *iff* means *if and only if*.
- The symbol $\exists$ means *there exist(s)*.
- The symbol *s.t.* means *such that*.

Thus our theorem reads: $f : \mathbb{R}^2 \to \mathbb{R}$ is linear iff $\exists k \in \mathbb{R}^2$ s.t. $f(x) = k \cdot x$.

Since our theorem is now phrased in terms of dot products, we explored some properties of this product. Here are a few:

Proposition 3 (Properties of the dot product)**.** *Given $x, y, z \in \mathbb{R}^2$ and $\alpha, \beta \in \mathbb{R}$. Let $\mathbf{0}$ denote the origin. Then:*

(1) $x \cdot y = y \cdot x$
(2) $x \cdot (y + z) = x \cdot y + x \cdot z$
(3) $x \cdot x = |x|^2$, *where $|x|$ denotes the distance between x and $\mathbf{0}$.*
(4) $(\alpha x) \cdot (\beta y) = \alpha\beta(x \cdot y)$
(5) $x \cdot y = |x||y| \cos\theta$, *where θ denotes the angle $\angle x\mathbf{0}y$.*
(6) $x \cdot y = 0$ *iff* $\angle x\mathbf{0}y = \frac{\pi}{2}$.

Before we prove these, a few comments.

- The first property allows us to say 'the dot product of x and y' without specifying the order in which we multiply them.
- The quantity $|x|$ is usually called the *magnitude* of x. Note that it's a very natural extension of the absolute value you're familiar with, which measures the distance between a point on the real number line and the number 0.
- When we say *the* angle $\angle x\mathbf{0}y$, we're being a little sloppy: there are two different angles that might refer to. Fortunately, the cosine function agrees on the two choices!

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- We will sometimes refer to θ as 'the angle between x and y'.
- We will always use radians in this course. Radians, which measure a physical quantity (the distance you've walked around the circle), are much more natural than degress (which were invented by people for convenience). Because of this, measuring angles in degrees leads to all sorts of issues; for example, if $f(x) = \sin x$ with x in degrees, then $f'(x) \neq \cos x$!
- By convention, we say the angle between $\mathbf{0}$ and any other point is $\pi/2$. (Note that without this, property (6) would be false as stated.)

Proof. We prove these individually.

(1) Write $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then

$$x \cdot y = x_1 y_1 + x_2 y_2 = y \cdot x.$$

(2) Similar to the first one.

(3) Once again similar to the above, once we note that $|x| = \sqrt{x_1^2 + x_2^2}$ by the Pythagorean theorem. (Make sure you can explain this.)

(4) Once more, similar to the above.

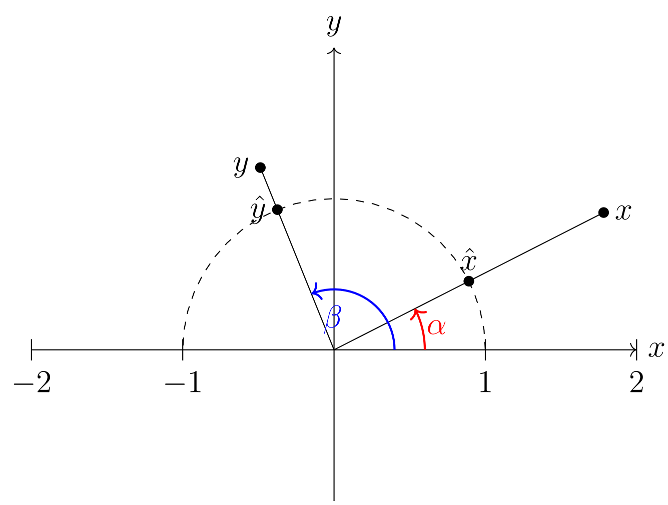
(5) This is the hardest property to prove. The idea is to first renormalize x and y so that they're both on the unit circle, then use properties we already know about the unit circle (in particular, we know a nice relationship between the angle a point forms with the horizontal axis and the coordinates of that point). Finally, we undo the renormalization. Here's a formal proof.

If either x or y are $\mathbf{0}$, the statement trivially holds, so we may assume for the remainder of the proof that neither x nor y are $\mathbf{0}$. Set

$$\hat{x} := \frac{x}{|x|} \qquad \text{and} \qquad \hat{y} := \frac{y}{|y|}$$

and observe that the angle between $\hat{x}$ and $\hat{y}$ is still θ but now $|\hat{x}| = 1 = |\hat{y}|$. Since $\hat{x}$ and $\hat{y}$ lie on the unit circle we may write

$$\hat{x} := (\cos\alpha, \sin\alpha) \qquad \text{and} \qquad \hat{y} := (\cos\beta, \sin\beta).$$



An illustration of the proof

We therefore have

$$\begin{aligned} x \cdot y &= (|x|\,\hat{x}) \cdot (|y|\,\hat{y}) \\ &= |x|\,|y|\,(\hat{x} \cdot \hat{y}) && \text{by (4)} \\ &= |x|\,|y|(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\ &= |x|\,|y|\cos(\alpha - \beta) = |x|\,|y|\cos(-\theta) \\ &= |x|\,|y|\cos\theta && \text{since } \cos \text{ is an even function.} \end{aligned}$$

(6) This follows easily from the previous property.

□

This completes the proofs of the properties. If you've seen polar coordinates before, we can write the proof of Property (5) in a more compact way:

Shorter version of the proof of Property (5). As before, we may assume neither x nor y are $\mathbf{0}$. Write x and y in polar coordinates:

$$x = (|x|, \alpha) \qquad \text{and} \qquad y = (|y|, \beta).$$

Then in rectangular coordinates we have

$$x = (|x| \cos \alpha, |x| \sin \alpha) \qquad \text{and} \qquad y = (|y| \cos \beta, |y| \sin \beta),$$

so

$$\begin{aligned} x \cdot y &= |x|\,|y|(\cos \alpha \cos \beta + \sin \alpha \sin \beta) \\ &= |x|\,|y| \cos |\alpha - \beta|. \end{aligned}$$

Since $|\alpha - \beta|$ is the angle between x and y, we're done. □