LINEAR ALGEBRA: LECTURE 7

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Recall that we have proved that for any linear map $f : \mathbb{R}^2 \to \mathbb{R}$, there exists some point $(a, b) \in \mathbb{R}^2$ such that $f(x, y) = (a, b) \cdot (x, y)$. Thus, in a document about f, we could replace every instance of f by $(a, b)\cdot$ and the meaning wouldn't change; the function f acts like multiplication by (a, b). We will therefore abuse notation and write

$$f = (a, b)$$

Now, this is a very, very bad abuse: the left hand side is a function, while the right hand side is a point, so these two are not only unequal, they are not comparable at all. But this 'identity' highlights that the behavior of f – what it actually does to points – is indistinguishable from the behavior of multiplication (using dot product) by (a, b).

Next, we explored an example of a problem about linear maps. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is linear, and that

$$f(4, 2) = 2 \qquad \text{and} \qquad f(1, 1) = 3.$$

What is $f(x, y)$? We saw two ways to solve this. The first is to use our theorem about linear maps from $\mathbb{R}^2$ to $\mathbb{R}$: we know that there exist $a, b \in \mathbb{R}$ such that

$$f(x, y) = ax + by.$$

Plugging in the points $(4, 2)$ and $(1, 1)$ yields the system of equations

$$\begin{aligned} 4a + 2b &= 2 \\ a + b &= 3 \end{aligned}$$

and solving these yields $a = -2$ and $b = 5$. This tells us that

$$f(x, y) = -2x + 5y.$$

A different approach to the problem, without relying on our theorem, is to use additivity and scaling directly. First, by scaling we see

$$f(4, 2) = 2 \Longrightarrow f(2, 1) = 1.$$

Therefore by additivity we find

$$f(1, 0) = f(2, 1) - f(1, 1) = 1 - 3 = -2.$$

Again by additivity we have

$$f(0, 1) = f(1, 1) - f(1, 0) = 3 + 2 = 5.$$

It follows that

$$f(x, y) = f(x, 0) + f(0, y) = xf(1, 0) + yf(0, 1) = -2x + 5y.$$

This concludes our discussion of linear maps from $\mathbb{R}^2 \to \mathbb{R}$. Next we turn to what will turn out to be a richer subject: linear maps from $\mathbb{R}^2 \to \mathbb{R}^2$. The definition probably won't shock you:

Definition. We call a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ a *linear map* iff f is additive and scales. [Recall that f is additive iff $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}^2$, and f scales iff $f(\alpha x) = \alpha f(x)$ for all $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^2$.]

Here are a few examples of functions mapping $\mathbb{R}^2 \to \mathbb{R}^2$:

(1) $f(x, y) := (2x, 2y)$. This has a nice geometric interpretation: this function stretches the plane out from the origin (by a factor of 2), uniformly in all directions. It is easily seen to be a linear map.

(2) $g(x, y) := (\sin x - y^2, xy)$. Unlike the first example, it's quite hard to describe what g does to the plane geometrically. In any event, it's fairly clear that it's not linear.

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(3) Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be the map which translates the plane to the right by two units. We found a formula for this fairly easily: $f(x, y) = (x + 2, y)$. This function is *not* linear.

(4) Let $R_{\pi/3}(x, y)$ denote the map which rotates the plane $\pi/3$ counterclockwise about the origin. What's a formula for this? Is it linear? The general consensus was affirmative, but how can one prove this? We'll see next time!