LINEAR ALGEBRA: LECTURE 8

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Last time we considered (a special case of) the rotation map

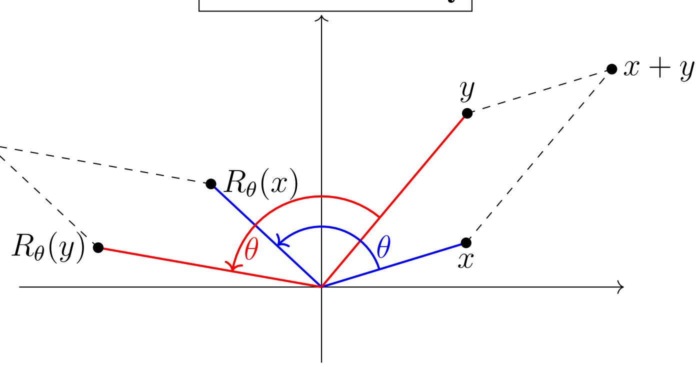
$$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$$

which rotates the plane counterclockwise about the origin by angle θ. Today we explored two questions:

(1) Is R_θ linear?
(2) Can we find a nice formula for $R_\theta(x, y)$?

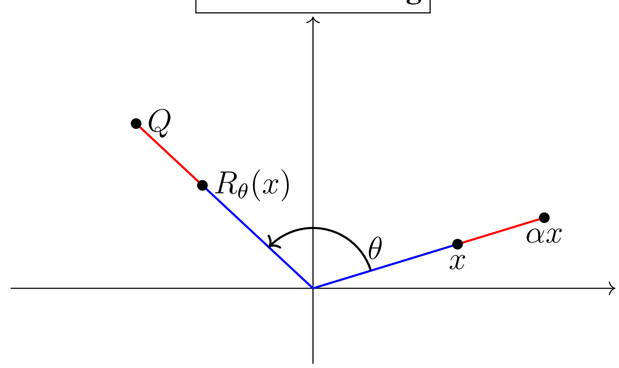
Some of you suggested a geometric approach. The main idea, captured in the illustrations below, is that R_θ is a *rigid motion*: it doesn't affect shapes, and more precisely, moves shapes to congruent shapes. This gives two ways to describe rotated image of points like $x + y$ and αx:

Proof of Additivity



On one hand, $P = R_\theta(x + y)$. On the other, $P = R_\theta(x) + R_\theta(y)$.

Proof of Scaling



On one hand, $Q = R_\theta(\alpha x)$. On the other, $Q = \alpha R_\theta(x)$.

These pictures give an appealing way to think about the problem, not only because they're pretty, but because they gives a sense of *why* R_θ must be additive and scale. However, they're unsatisfying in a different way: it's difficult to tell how rigorous they are. For example, do these pictures represent the general situation? The illustration of additivity does not: what if x and y are collinear with the origin? And how do we know that there aren't some other configurations of x and y which behave differently than the above pictures suggest?

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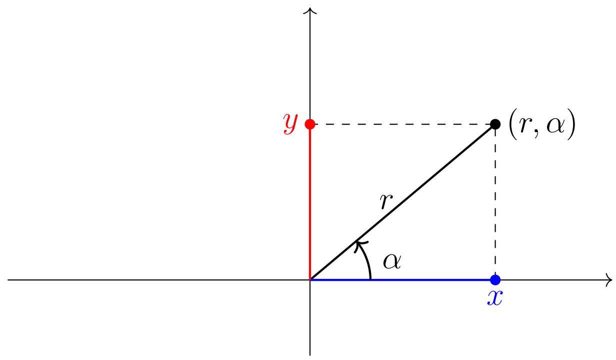
A different approach combines geometry with algebra. Given a point $(x, y) \in \mathbb{R}^2$, write it in polar coordinates as (r, α), say. We can convert back and forth between rectangular and polar coordinates using the following dictionary:

$$x = r\cos\alpha \qquad y = r\sin\alpha$$
$$r = \sqrt{x^2 + y^2} \qquad \alpha = \arctan(y/x)$$

All these formulas can be read off from the following picture:



The point (x, y) labelled in polar coordinates as (r, α).

The advantage of working in polar coordinates is that rotation becomes easy: we have

$$R_\theta(x, y) = (r, \theta + \alpha)$$

where the right hand side is in polar. Translating this back to rectangular coordinates, we find

$$R_\theta(x, y) = \Big(r\cos(\theta + \alpha), r\sin(\theta + \alpha)\Big).$$

Although technically this tells us where $R_\theta(x, y)$ is located, it's not a very satisfying answer because it's not in terms of x and y. This is easy to rectify using trig addition rules:

$$\begin{aligned} R_\theta(x, y) &= \Big(r\cos(\alpha + \theta), r\sin(\alpha + \theta)\Big) \\ &= \Big(r(\cos\alpha\cos\theta - \sin\alpha\sin\theta), r(\sin\alpha\cos\theta + \cos\alpha\sin\theta)\Big) \\ &= \Big(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta\Big) \end{aligned}$$

This formula will allow us to easily prove the linearity of R_θ. More importantly, it will give us a hint about the structure of linear maps from $\mathbb{R}^2 \to \mathbb{R}^2$. We'll pick this up next lecture.