LINEAR ALGEBRA: LECTURE 12

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Our first goal for today is to prove a result we claimed last time:

Theorem 1. *If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a nonsingular linear map, then f is invertible and $f^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ is linear.*

(For the definitions of *nonsingular* and *invertible*, check out the previous lecture summary.)

Before launching into the proof, we make a useful observation about nonsingular linear maps.

Lemma 2. *Given $f : \mathbb{R}^2 \to \mathbb{R}^2$ a nonsingular linear map. Then $f(p) = \mathbf{0}$ if and only if $p = \mathbf{0}$.*

Proof. Since f is linear, we can write

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Moreover, since f is nonsingular, we know that $ad - bc \neq 0$.

($\Rightarrow$)

Suppose $f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Then

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned}$$

Solving these for x and y (and keeping in mind that $ad - bc \neq 0$) yields $x = 0 = y$.

($\Leftarrow$) $f(\mathbf{0}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}$. □

Armed with this lemma, we can now give a nice (matrix-free!) proof of Theorem 1.

Proof of Theorem 1. Given $f : \mathbb{R}^2 \to \mathbb{R}^2$ a nonsingular linear map. We first prove f is invertible, i.e., that the preimage of any point in the plane consists of precisely one point.

Pick $y \in \mathbb{R}^2$. Since f is nonsingular, we know that $\text{im}\,(f) = \mathbb{R}^2$; in particular, $y \in \text{im}\,(f)$. Thus, we see that $f^{-1}(y) \neq \emptyset$, whence

$$\#f^{-1}(y) \geq 1.$$

Now, pick any two points $p, q \in f^{-1}(y)$. By definition, $f(p) = y = f(q)$, whence by additivity,

$$f(p - q) = \mathbf{0}.$$

Lemma 2 immediately implies that $p = q$. Thus, it's impossible to pick two distinct elements of $f^{-1}(y)$; this shows that

$$\#f^{-1}(y) = 1.$$

It remains only to prove that f^{-1} is linear. I leave this as an exercise to the reader. □

This theorem shows that the vast majority of linear maps are invertible, and that inverting them preserves linearity. Another operation which preserves linearity is composition:

Proposition 3. *The composition $f \circ g$ of any two linear maps $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ is linear.*

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Proof. Given f and g linear, we wish to prove that $f \circ g$ is linear. In other words, we must show that $f \circ g$ is additive and scales.

ADDITIVITY: We have

$$\begin{aligned}(f \circ g)(x+y) &= f\Big(g(x+y)\Big) \\ &= f\Big(g(x)+g(y)\Big) \\ &= f\Big(g(x)\Big) + f\Big(g(y)\Big) \\ &= (f \circ g)(x) + (f \circ g)(y)\end{aligned}$$

SCALING: Similar proof. $\square$

Since $f \circ g$ is linear, we must be able to write it as a matrix. Can we relate this matrix to the matrices of f and g? Sure! Let's say $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $g = \begin{pmatrix} \ell & m \\ n & p \end{pmatrix}$. Then

$$\begin{aligned}(f \circ g)\begin{pmatrix} x \\ y \end{pmatrix} &= f\left(\begin{pmatrix} \ell & m \\ n & p \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right) \\ &= f\begin{pmatrix} \ell x + my \\ nx + py \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \ell x + my \\ nx + py \end{pmatrix} \\ &= \begin{pmatrix} a\ell x + amy + bnx + bpy \\ c\ell x + cmy + dnx + dpy \end{pmatrix} \\ &= \begin{pmatrix} a\ell + bn & am + bp \\ c\ell + dn & cm + dp \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\end{aligned}$$

In other words, we've shown that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \begin{pmatrix} \ell & m \\ n & p \end{pmatrix} = \begin{pmatrix} a\ell + bn & am + bp \\ c\ell + dn & cm + dp \end{pmatrix}$$

This is frequently called *matrix multiplication*, but really this has nothing to do with any familiar multiplication; it's simply the composition of two functions.

Remark. One nice feature of the matrix notation is that it allows you to describe functions and their compositions without specifying the input. By contrast, consider $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ and $g(x) = \log x$. I defy you to express $f \circ g$ without writing down an input!