Instructor: Leo Goldmakher

NAME: ______________________________

SECTION: ____________________________

Williams College
Department of Mathematics and Statistics

MATH 250 : LINEAR ALGEBRA

Midterm Exam 1 – due Thursday, March 17th

INSTRUCTIONS:
This midterm must be turned in *as a hard copy* to the box outside my office by **10pm** on Thursday. *Any midterms not in my box by Friday morning will receive a zero. No exceptions.*

Please print and attach this page as the first page of your submitted problem set.

PROBLEM	GRADE
1	
2	
3	
4	
5	
Total	

Please read the following statement and sign below:

I understand that the only aids I may use on this exam are my notes from class, my graded problem sets, the solution sets from the problem sets, and the posted lecture summaries. ***No other aids are permitted.*** *In particular, I may not use the internet or books to assist me in any way on this exam, nor may I consult any person (apart from the instructor). I pledge to abide by the Williams honor code.*

SIGNATURE:____

Midterm 1

M1–1 Consider $g : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$g(x, y) := (x, y^2),$$

and let $\mathcal{L}$ be the line segment connecting $(0, 0)$ to $(2, 1)$. What is the image $g(\mathcal{L})$? Sketch a picture, and give as precise a mathematical description as you can.

M1–2 Carefully explain why $f\big(f^{-1}(x)\big) = x$ for any $x \in \text{im}\ (f)$. What happens if $x \notin \text{im}\ (f)$?

M1–3 Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map. In class we showed that the image of the unit square whose lower left vertex is at the origin has area $\det f$. Prove that this is true for an arbitrary unit square in the plane.

M1–4 In class we've considered several times the linear map $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ which reflects across the horizontal axis. In this problem we explore the more general reflection $\sigma_{\mathcal{L}} : \mathbb{R}^2 \to \mathbb{R}^2$ across a given line $\mathcal{L}$.

(a) Prove that $R_\theta \circ \rho = \rho \circ R_{-\theta}$.

(b) Prove that if $\mathcal{L}$ is a line passing through the origin, then $\sigma_{\mathcal{L}}$ is linear. [*Hint: Write $\sigma_{\mathcal{L}}$ as a composition of linear maps.*]

(c) Suppose $\mathcal{L}$ and $\mathcal{L}'$ are two distinct lines in the plane, both passing through the origin. Describe $\sigma_{\mathcal{L}} \circ \sigma_{\mathcal{L}'}$ geometrically, with justification. [*Hint: Use parts (a) and (b).*]

M1–5 We say a function $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ is *distance-preserving* iff

$$|\phi(x) - \phi(y)| = |x - y| \qquad \forall x, y \in \mathbb{R}^2.$$

In other words, the distance between the images of any two points is the same as the distance between the two points themselves.

(a) Give an example of a distance-preserving function which is not a linear map.

(b) Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is distance-preserving and satisfies $f(\mathbf{0}) = \mathbf{0}$. Prove that $|f(x)| = |x|$ for all $x \in \mathbb{R}^2$.

(c) Suppose f is as in (b). Prove that $f(x) \cdot f(y) = x \cdot y$ for all $x, y \in \mathbb{R}^2$. [*Hint: Start with the distance-preserving relation* $|f(x) - f(y)| = |x - y|$.]

(d) Suppose f is as in (b). Prove that f must be linear. [*Hint: First prove that* $|f(\alpha x) - \alpha f(x)| = 0$.]

(e) Suppose f is as in (b). Prove that there exists $\theta \in \mathbb{R}$ such that either $f = R_\theta$ or $f = R_\theta \circ \rho$. Here $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection across the horizontal axis, i.e., $\rho(x, y) := (x, -y)$. [*Hint: What can you say about $f(1, 0)$? What about $f(0, 1)$? Use the previous parts of this problem!*]

(f) Prove that any distance-preserving map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ can be written as the composition of a translation, a rotation, and (possibly) a reflection. [A *translation* is a map $T_k : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_k(x) := x + k$. I'm asking you to prove that either $\phi = T_k \circ R_\theta$ or $\phi = T_k \circ R_\theta \circ \rho$.]