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NAME: ________________________________

Williams College
Department of Mathematics and Statistics

MATH 250 : LINEAR ALGEBRA

Problem Set 2 – due Thursday, February 25th

INSTRUCTIONS:
This assignment must be turned in *as a hard copy* to the mailbox of your TA (on your left as soon as you enter Bronfman from Science Quad, labelled by last name), by **9pm** sharp. Assignments turned in later than this, but before 5pm on Friday, will also be graded, but the grade will be reduced by one mark. *Assignments submitted later than Friday at 5pm will be returned without being marked.*

Please print and attach this page as the first page of your submitted problem set.

PROBLEM	GRADE
2.1	
2.2	
2.3	
2.4	
2.5	
2.6	
2.7	
Total	

Please read the following statement and sign below:

I understand that I am not allowed to use the internet to search for problems or solutions. I also understand that I must write down the final version of my assignment without reference to notes copied from anyone else's speech or written text. I pledge to abide by the Williams honor code.

SIGNATURE:____

Problem Set 2

2.1 Given $x, y, z \in \mathbb{R}^2$. Prove that $x \cdot (y + z) = x \cdot y + x \cdot z$.

2.2 (a) Prove that for any $x, y \in \mathbb{R}^2$ we have $|x \cdot y| \leq |x||y|$. Moreover, show that equality holds iff x, y, and the origin are collinear (i.e., all lie on a single line)

(b) Prove that for any $x, y \in \mathbb{R}^2$ we have $|x + y| \leq |x| + |y|$. [*Hint: consider* $|x + y|^2$ *using dot products.*]

2.3 A linear map from $\mathbb{R}^3$ to $\mathbb{R}$ is a function $\mathbb{R}^3 \to \mathbb{R}$ which is additive and scales. Prove that $f : \mathbb{R}^3 \to \mathbb{R}$ is a linear map iff $f(x, y, z) = ax + by + cz$ for some $a, b, c \in \mathbb{R}$.

2.4 Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a linear map such that $f(2, 3) = 2$ and $f(1, 2) = -1$. Determine a formula for $f(x, y)$.

2.5 Is there a function $f : \mathbb{R}^2 \to \mathbb{R}$ which scales but is not additive? Either give an example of such a function, or prove that no such function exists.

2.6 (a) For any $h \in \mathbb{R}^2$, define a function $M_h : \mathbb{R}^2 \to \mathbb{R}$ by $M_h(x) := h \cdot x$. Prove that M_h is a linear map.

(b) Let $\mathcal{F}$ denote the set of all linear maps from $\mathbb{R}^2 \to \mathbb{R}$, and consider the function $M : \mathbb{R}^2 \to \mathcal{F}$ defined by $M(h) := M_h$. (Reread the last sentence. The output of the function M is, itself, a function.) Show that M is additive and scales. [*Hint: To do this, you will have to figure out what the words* additive *and* scale *mean in this context. Don't look them up – the point of the problem is to show you that you can arrive at a natural definition yourself.*]

2.7 The dot product gives a way of combining two points in $\mathbb{R}^2$ to yield a real number. Suppose $\otimes$ is a different way to combine two points to get a number, satisfying the following properties:

(i) $(1, 0) \otimes (0, 1) = 1$

(ii) $x \otimes x = 0$ for every $x \in \mathbb{R}^2$

(iii) For any $a \in \mathbb{R}^2$, the functions $L_a : \mathbb{R}^2 \to \mathbb{R}$ and $R_a : \mathbb{R}^2 \to \mathbb{R}$ are both linear maps, where $L_a(x) := a \otimes x$ and $R_a(x) := x \otimes a$.

(a) These properties look complicated, but are actually not so bad once you get past the notation. Build up your intuition by finding the value of $(1, 0) \otimes (1, 1)$. (Show your work.)

(b) Determine a formula for $(a, b) \otimes (c, d)$. Justify your answer.