Instructor: Leo Goldmakher

NAME: ______________________________

SECTION: ____________________________

Williams College
Department of Mathematics and Statistics

MATH 250 : LINEAR ALGEBRA

Problem Set 4 – due Thursday, March 10th

INSTRUCTIONS:
This assignment must be turned in *as a hard copy* to the mailbox of your TA (on your left as soon as you enter Bronfman from Science Quad, labelled by last name), by **9pm** sharp. Assignments turned in later than this, but before 5pm on Friday, will also be graded, but the grade will be reduced by one mark. *Assignments submitted later than Friday at 5pm will be returned without being marked.*

Please print and attach this page as the first page of your submitted problem set.

PROBLEM	GRADE
4.1	
4.2	
4.3	
4.4	
4.5	
4.6	
4.7	
4.8	
Total	

Please read the following statement and sign below:

I understand that I am not allowed to use the internet to search for problems or solutions. I also understand that I must write down the final version of my assignment without reference to notes copied from anyone else's speech or written text. I pledge to abide by the Williams honor code.

SIGNATURE:____

Problem Set 4

4.1 Suppose $f, g, h : \mathbb{R}^2 \to \mathbb{R}^2$ are linear maps. *Without using matrices*, prove that $f \circ (g+h) = f \circ g + f \circ h$.

4.2 Prove that a singular linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is not invertible.

4.3 The *zero function* is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, i.e., the function mapping $\mathbb{R}^2 \to \mathbb{R}^2$ which outputs $\mathbf{0}$ for all inputs. Now suppose $f \circ g$ is a linear map $\mathbb{R}^2 \to \mathbb{R}^2$, where neither of $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ are the zero function. Must f be linear? If so, prove it. If not, produce a counterexample.

4.4 Given $f : A \to B$ a function and $a \in A$. (Note: f is not necessarily linear!) What can you say about $f\big(f^{-1}(f(a))\big)$? Be as specific as you can, and justify your answer.

4.5 Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a nonsingular linear map with matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(a) *Without using matrices*, prove that $f^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map.

(b) What is the matrix of f^{-1}? Show your work.

4.6 Do not use a computer or calculator for this exercise!

In each of the following examples, determine (i) the matrix of $(f \circ g)$, (ii) the matrix of $(g \circ f)$, (iii) the matrix of f^{-1}. If the matrix of f^{-1} does not exist, carefully explain (with suitable examples) why not.

(a) $f = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $g = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

(b) $f = \begin{pmatrix} 2 & -6 \\ -3 & 9 \end{pmatrix}$, $g = \begin{pmatrix} 2 & 5 \\ 4 & -1 \end{pmatrix}$

(c) $f = \begin{pmatrix} 1 & 2 \\ 0 & -4 \end{pmatrix}$, $g = \begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix}$

(d) $f = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, $g = \begin{pmatrix} k & 0 \\ 0 & \ell \end{pmatrix}$, $ad \neq 0$

4.7 This exercise explores what linear maps do to triangles. Throughout, let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map.

(a) Given $A, B \in \mathbb{R}^2$, the notation $\overline{AB}$ denotes the line segment whose endpoints are A and B. Prove that $f(\overline{AB}) = \overline{f(A)f(B)}$. In other words, a linear function sends line segments to line segments.

(b) Consider a triangle $\triangle ABC$ in the plane. What can you say about the shape of the image of $\triangle ABC$ under f? [*Hint: f might be singular or nonsingular.*]

4.8 Consider the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $ad - bc < 0$, what does this tell you about the geometric effect the matrix has on the plane? Try to describe this as precisely as you can. [*Hint: play around with what the matrix does to a triangle.*]