Instructor: Leo Goldmakher

NAME: ______________________________

SECTION: ____________________________

Williams College
Department of Mathematics and Statistics

MATH 250 : LINEAR ALGEBRA

Problem Set 5 – due Thursday, April 14th

INSTRUCTIONS:

This assignment must be turned in *as a hard copy* to the mailbox of your TA (on your left as soon as you enter Bronfman from Science Quad, labelled by last name), by **9pm** sharp. Assignments turned in later than this, but before 5pm on Friday, will also be graded, but the grade will be reduced by one mark. *Assignments submitted later than Friday at 5pm will be returned without being marked.*

Please print and attach this page as the first page of your submitted problem set.

PROBLEM	GRADE
5.1	
5.2	
5.3	
5.4	
Total	

Please read the following statement and sign below:

I understand that I am not allowed to use the internet to search for problems or solutions. I also understand that I must write down the final version of my assignment without reference to notes copied from anyone else's speech or written text. I pledge to abide by the Williams honor code.

SIGNATURE:____

Problem Set 5

5.1 Let $\vec{v} := 2\vec{e}_1 - 3\vec{e}_2$, $\vec{w} := \vec{e}_1 + \vec{e}_2$.

(a) What is the change of basis matrix from $\vec{e}_1, \vec{e}_2$ to $\vec{v}, \vec{w}$?

(b) Use the change-of-basis matrix (as we did in the lecture 17–18 summary) to express the vector $\vec{e}_1 + 2\vec{e}_2$ as a linear combination of $\vec{v}$ and $\vec{w}$. (You should *not* solve a system of equations.)

5.2 The goal of this exercise is to explore linear maps which fix the unit circle U. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map with $\det f > 0$ and $f(U) = U$. Write the matrix of f as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(a) Prove that $a^2 + c^2 = 1$ and $b^2 + d^2 = 1$. [*Hint: Consider* $f(1,0)$ *and* $f(0,1)$.]

(b) Prove that $a^2 + b^2 = c^2 + d^2$. [*Hint: Consider* $f^{-1}(1,0)$ *and* $f^{-1}(0,1)$.]

(c) Prove that $\det f = 1$.

(d) Prove that $f = R_\theta$ for some θ.

(e) Now suppose $g : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map such that $\det g < 0$ and $g(U) = U$. Prove that there exists some angle θ such that $g = R_\theta \rho$, where ρ is the reflection across the horizontal axis. [*Hint: If you use part (d), the proof is quite short.*]

(f) Suppose a linear map $h : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies $h(U) = U$. Prove that either $h = R_\theta$ or $h = R_\theta \rho$. [*Hint: The proof is short, but there is something to check.*]

5.3 Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map with $\det f < 0$. Prove that f admits a singular value decomposition. (State a precise theorem, analogous to Theorem 4 from the lectures 17–18 summary.)

5.4 Let $f := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. The goal of this exercise is to determine and apply the singular value decomposition of f. It turns out the SVD of f is intimately linked to the so-called *golden ratio*:

$$\varphi := \frac{1 + \sqrt{5}}{2}$$

As usual, let U denote the unit circle centered at the origin.

(a) As discussed in class, $f(U)$ is an ellipse centered at the origin. Determine the lengths of the major and minor radii of this ellipse. Express your answer in terms of φ. [*Hint: this is a calculus problem.*]

(b) Let α denote the tilt of the ellipse $f(U)$, i.e., the angle formed by the positive horizontal axis and the radius of the ellipse in the first quadrant. Prove that $\tan \alpha = \varphi - 1$.

(c) As discussed in lecture, there exists a square grid which gets mapped by f to a rectangular grid. Describe (as precisely as possible) these two grids for $f = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. [*Note:* $\det f < 0$!]

(d) Determine the singular value decomposition of f, i.e., determine α, β, k, ℓ such that

$$f = R_\alpha \begin{pmatrix} k & 0 \\ 0 & \ell \end{pmatrix} R_\beta.$$