Instructor: Leo Goldmakher

Name: ______________________________

Section: ____________________________

Williams College
Department of Mathematics and Statistics

MATH 250 : LINEAR ALGEBRA

Problem Set 8 – due Thursday, May 5th

INSTRUCTIONS:
This assignment must be turned in *as a hard copy* to the mailbox of your TA (on your left as soon as you enter Bronfman from Science Quad, labelled by last name), by **9pm** sharp. Assignments turned in later than this, but before 5pm on Friday, will also be graded, but the grade will be reduced by one mark. *Assignments submitted later than Friday at 5pm will be returned without being marked.*

Please print and attach this page as the first page of your submitted problem set.

PROBLEM	**GRADE**
8.1	
8.2	
8.3	
Total	

Please read the following statement and sign below:

I understand that I am not allowed to use the internet to search for problems or solutions. I also understand that I must write down the final version of my assignment without reference to notes copied from anyone else's speech or written text. I pledge to abide by the Williams honor code.

SIGNATURE:____

Problem Set 8

8.1 In class we sketched a proof that any linearly independent set in a finite-dimensional vector space is contained in a basis of that space. The goal of this exercise is to complete this proof. Throughout, let V be a finite-dimensional vector space, and recall that for any finite set $A = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq V$ we define

$$\text{span } A := \Big\{\alpha_1 \vec{v}_1 + \cdots + \alpha_n \vec{v}_n : \alpha_j \in \mathbb{R}\ \forall j\Big\}$$

(i.e. span A is the set of all vectors which can be formed by linearly combining the elements of A).

(a) Suppose $\mathcal{L} \subseteq V$ is linearly independent, and that $\mathcal{S} \subseteq V$ spans V. Prove that if span $\mathcal{L} \supseteq \mathcal{S}$ then $\mathcal{L}$ is a basis of V.

(b) Suppose $\mathcal{L} \subseteq V$ is linearly independent, and that $\exists \vec{v} \in V$ such that $\vec{v} \notin$ span $\mathcal{L}$. Prove that $\mathcal{L} \cup \{\vec{v}\}$ is linearly independent.

(c) Write out a careful proof that any linearly independent set in a finite-dimensional vector space is contained in a basis of that space.

8.2 Consider the set $\mathcal{F}$ of all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the differential equation $f'' + f = 0$. (Here f'' means the second derivative of f. Note that we are implicitly assuming that both f' and f'' exist, since otherwise it would be difficult to satisfy the given differential equation!)

(a) Prove that the $\mathcal{F}$ is a vector space.

(b) What is the dimension of $\mathcal{F}$? Prove it! [*Hint: Differentiate the functions* $g(x) = f(x)\cos x - f'(x)\sin x$ *and* $h(x) = f(x)\sin x + f'(x)\cos x$.]

(c) Consider the function $T : \mathcal{F} \to \mathbb{R}^2$ defined by

$$T(f) := \Big(f(0), f(\pi/2)\Big)$$

Is T a linear map? Either way, justify your answer.

8.3 Given V a finite-dimensional vector space, let $\widehat{V}$ denote the set of all linear maps $T : V \to \mathbb{R}$.

(a) Prove that $\widehat{V}$ is a vector space.

(b) Prove that $\dim V = \dim \widehat{V}$.

(c) Give an explicit example of a non-constant linear map $\varphi : V \to \widehat{\widehat{V}}$. (Here $\widehat{\widehat{V}}$ denotes the set of all linear maps $\widehat{V} \to \mathbb{R}$.)