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MATH 250 : LINEAR ALGEBRA

Problem Set 2 – KEY

2.1 Given $x, y, z \in \mathbb{R}^2$. Prove that $x \cdot (y + z) = x \cdot y + x \cdot z$.

Write $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $z = (z_1, z_2)$. Then

$$\begin{aligned} x \cdot (y+z) &= (x_1, x_2) \cdot (y_1 + z_1, y_2 + z_2) \\ &= x_1 y_1 + x_1 z_1 + x_2 y_2 + x_2 z_2 \\ &= (x_1, x_2) \cdot (y_1, y_2) + (x_1, x_2) \cdot (z_1, z_2) \\ &= x \cdot y + x \cdot z \end{aligned}$$

QED

2.2 (a) Prove that for any $x, y \in \mathbb{R}^2$ we have $|x \cdot y| \leq |x||y|$. Moreover, show that equality holds iff x, y, and the origin are collinear (i.e., all lie on a single line)

From class (Lecture Summary 6, Proposition 3 (5)) we know

$$x \cdot y = |x|\,|y| \cos\theta$$

where θ is the angle $\angle x\mathbf{0}y$. Thus

$$|x \cdot y| = |x|\,|y|\,|\cos\theta| \leq |x|\,|y|.$$

[*This is called the Cauchy-Schwarz inequality.*] QED

(b) Prove that for any $x, y \in \mathbb{R}^2$ we have $|x + y| \leq |x| + |y|$. [*Hint: consider $|x+y|^2$ using dot products.*]

From class (Lecture Summary 6, Proposition 3 (3)), we know that $|p|^2 = p \cdot p$ for any $p \in \mathbb{R}^2$. Thus

$$\begin{aligned} |x+y|^2 &= (x+y) \cdot (x+y) \\ &= (x+y) \cdot x + (x+y) \cdot y && \textbf{by 2.1} \\ &= x \cdot (x+y) + y \cdot (x+y) && \text{by LS6 Prop 3(1)} \\ &= x \cdot x + x \cdot y + y \cdot x + y \cdot y \\ &= |x|^2 + 2(x \cdot y) + |y|^2 \\ &\leq |x|^2 + 2|x|\,|y| + |y|^2 && \text{by (a)} \\ &= (|x| + |y|)^2. \end{aligned}$$

The claim follows. [*This is called the triangle inequality.*] QED

2.3 A linear map from $\mathbb{R}^3$ to $\mathbb{R}$ is a function $\mathbb{R}^3 \to \mathbb{R}$ which is additive and scales. Prove that $f : \mathbb{R}^3 \to \mathbb{R}$ is a linear map iff $f(x, y, z) = ax + by + cz$ for some $a, b, c \in \mathbb{R}$.

As usual with if and only if statements, we prove the two directions separately.
($\Rightarrow$) Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is linear. Let

$$a := f(1, 0, 0) \qquad b := f(0, 1, 0) \qquad c := f(0, 0, 1)$$

Then

$$\begin{aligned} f(x, y, z) &= f(x, 0, 0) + f(0, y, 0) + f(0, 0, z) && \text{by additivity} \\ &= xf(1, 0, 0) + yf(0, 1, 0) + zf(0, 0, 1) && \text{by scaling} \\ &= ax + by + cz. \end{aligned}$$

($\Leftarrow$) Given $a, b, c \in \mathbb{R}$, let $f(x, y, z) := ax + by + cz$. To show f is linear, it suffices to prove that f is additive and scales.

- Additivity:

$$\begin{aligned} f(x, y, z) + f(x', y', z') &= ax + by + cz + ax' + by' + cz' \\ &= a(x + x') + b(y + y') + c(z + z') \\ &= f(x + x', y + y', z + z'). \end{aligned}$$

- Scaling

$$f(kx, ky, kz) = akx + bky + ckz = k(ax + by + cz) = k\ f(x, y, z).$$

QED

2.4 Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a linear map such that $f(2, 3) = 2$ and $f(1, 2) = -1$. Determine a formula for $f(x, y)$.

SOLUTION 1. Since f is linear, Theorem 2 from Lecture 5 implies that $\exists a, b \in \mathbb{R}$ such that

$$f(x, y) = ax + by.$$

Plugging in the given points yields a system of equations

$$\begin{aligned} 2a + 3b &= 2 \\ a + 2b &= -1 \end{aligned}$$

Solving the system, we find $a = 7$ and $b = -4$. Thus, $f(x, y) = 7x - 4y$.

SOLUTION 2. By scaling, $f(2, 4) = -2$. By additivity, we have

$$f(0, 1) = f(2, 4) - f(2, 3) = -4.$$

By scaling this implies $f(0, 2) = -8$, and again by additivity we find

$$f(1, 0) = f(1, 2) - f(0, 2) = -1 + 8 = 7.$$

Finally,

$$f(x, y) = xf(1, 0) + yf(0, 1) = 7x - 4y.$$

2.5 Is there a function $f : \mathbb{R}^2 \to \mathbb{R}$ which scales but is not additive? Either give an example of such a function, or prove that no such function exists.

There are many such functions, but some care must be taken that your function is well-defined. For example, the function sending (x, y) to y^2/x almost works, but not quite – it's not defined for $x = 0$. This can be fixed by setting

$$f(x,y) = \begin{cases} \frac{y^2}{x} & \text{if } x \neq 0 \\ 0 & \text{if x = 0} \end{cases}$$

Another good idea which doesn't quite work at first is the function sending (x, y) to $\sqrt{xy}$; this doesn't scale by negative numbers! However, building on this idea gives

$$g(x,y) = \sqrt[3]{x^2 y},$$

which does scale but isn't additive. A third example is

$$h(x,y) = \begin{cases} x & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases}$$

2.6 (a) For any $h \in \mathbb{R}^2$, define a function $M_h : \mathbb{R}^2 \to \mathbb{R}$ by $M_h(x) := h \cdot x$. Prove that M_h is a linear map.

In class we proved that a function $f : \mathbb{R}^2 \to \mathbb{R}$ is linear iff $\exists h \in \mathbb{R}^2$ such that $f(x) = h \cdot x$. Thus, M_h is linear. QED

(b) Let $\mathcal{F}$ denote the set of all linear maps from $\mathbb{R}^2 \to \mathbb{R}$, and consider the function $M : \mathbb{R}^2 \to \mathcal{F}$ defined by $M(h) := M_h$. (Reread the last sentence. The output of the function M is, itself, a function.) Show that M is additive and scales.

Given two function f and g, we define a new function $f + g$ by the condition

$$(f + g)(x) := f(x) + g(x).$$

Given a function f and a real number α, we define a new function αf by the condition

$$(\alpha f)(x) := \alpha f(x).$$

Now, I claim that M is linear, i.e., that it's additive and scales. We verify these:

- Additivity of M: given $h, k \in \mathbb{R}^2$, we have (for arbitrary $x \in \mathbb{R}^2$)

$$\begin{aligned}
\big(M(h + k)\big)(x) &= M_{h+k}(x) \\
&= (h + k) \cdot x \\
&= h \cdot x + k \cdot x \\
&= M_h(x) + M_k(x) \\
&= \big(M(h) + M(k)\big)(x).
\end{aligned}$$

  We deduce that $M(h+k) = M(h)+M(k)$, since we've just checked that these two functions agree on every possible input.

- Scaling of M: given arbitrary $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^2$, we have

$$\big(M(\alpha h)\big)(x) = M_{\alpha h}(x) = (\alpha h) \cdot x = \alpha(h \cdot x) = \big(\alpha M(h)\big)(x)$$

  Since they agree on all x, we deduce that $M(\alpha h) = \alpha M(h)$. QED

2.7 The dot product gives a way of combining two points in $\mathbb{R}^2$ to yield a real number. Suppose $\otimes$ is a different way to combine two points to get a number, satisfying the following properties:

(i) $(1,0) \otimes (0,1) = 1$

(ii) $x \otimes x = 0$ for every $x \in \mathbb{R}^2$

(iii) For any $a \in \mathbb{R}^2$, the functions $L_a : \mathbb{R}^2 \to \mathbb{R}$ and $R_a : \mathbb{R}^2 \to \mathbb{R}$ are both linear maps, where $L_a(x) := a \otimes x$ and $R_a(x) := x \otimes a$.

(a) These properties look complicated, but are actually not so bad once you get past the notation. Build up your intuition by finding the value of $(1,0) \otimes (1,1)$. (Show your work.)

The following will prove to be useful:

Lemma 1. *Given $x, y, z \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$ we have*

$$(x+y) \otimes z = x \otimes z + y \otimes z, \qquad x \otimes (y+z) = x \otimes y + x \otimes z, \tag{*}$$

and

$$(\alpha x) \otimes y = \alpha(x \otimes y) = x \otimes (\alpha y). \tag{†}$$

Proof. We have

$$(x+y) \otimes z = R_z(x+y) = R_z(x) + R_z(y) = x \otimes z + y \otimes z$$

by the additivity of R_z; the second statement of (*) is proved similarly, using the L function instead. We also have

$$(\alpha x) \otimes y = R_y(\alpha x) = \alpha R_y(x) = \alpha(x \otimes y)$$

since R_y scales. The second equality of (†) is proved similarly, using the L function instead. □

Let's see how this applies to our problem, for example. Relation (*) yields

$$\begin{aligned}(1,0) \otimes (1,1) &= (1,0) \otimes \big((1,0) + (0,1)\big) \\ &= (1,0) \otimes (1,0) + (1,0) \otimes (0,1) \\ &= 0 + 1 \\ &= 1\end{aligned}$$

QED

(b) Determine a formula for $(a,b) \otimes (c,d)$. Justify your answer.

Before turning to the main question, we prove a couple of useful results.

Lemma 2. *Given $a, b, x, y \in \mathbb{R}^2$, we have*

$$(a+b) \otimes (x+y) = (a \otimes x) + (a \otimes y) + (b \otimes x) + (b \otimes y).$$

Proof. This follows by repeatedly applying (*) from part (a) and expanding. □

Lemma 3. *For any $x, y \in \mathbb{R}^2$, we have*

$$x \otimes y = -(y \otimes x).$$

Proof. We compute $(x+y)\otimes(x+y)$ in two different ways. On one hand, by property (ii) of $\otimes$, we know

$$(x+y) \otimes (x+y) = 0.$$

On the other hand, by the lemma we have

$$\begin{aligned}(x+y) \otimes (x+y) &= (x \otimes x) + (x \otimes y) + (y \otimes x) + (y \otimes y)\\ &= (x \otimes y) + (y \otimes x).\end{aligned}$$

Putting these two computations together yields the claim. □

Armed with this lemma, I now claim

Proposition 4. $(a,b) \otimes (c,d) = ad - bc$.

Proof. Write

$$(a,b) = (a,0) + (0,b) \qquad \text{and} \qquad (c,d) = (c,0) + (0,d).$$

Using Lemma 2, we find

$$(a,b) \otimes (c,d) = (a,0) \otimes (c,0) + (a,0) \otimes (0,d) + (0,b) \otimes (c,0) + (0,b) \otimes (0,d).$$

Now (†) from part (a) implies

$$(a,0) \otimes (c,0) = ac\big((1,0) \otimes (1,0)\big) = 0.$$

Similarly,

$$(0,b) \otimes (0,d) = 0.$$

It follows that

$$\begin{aligned}(a,b) \otimes (c,d) &= ad\big((1,0) \otimes (0,1)\big) + bc\big((0,1) \otimes (1,0)\big)\\ &= ad - bc\end{aligned}$$

by Lemma 3. QED