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MATH 250 : LINEAR ALGEBRA

Problem Set 5 – KEY

5.1 Let $\vec{v} := 2\vec{e}_1 - 3\vec{e}_2$, $\vec{w} := \vec{e}_1 + \vec{e}_2$.

(a) What is the change of basis matrix from $\vec{e}_1, \vec{e}_2$ to $\vec{v}, \vec{w}$?

The change of basis map is $P = \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}$

(b) Use the change-of-basis matrix (as we did in the lecture 17–18 summary) to express the vector $\vec{e}_1 + 2\vec{e}_2$ as a linear combination of $\vec{v}$ and $\vec{w}$. (You should *not* solve a system of equations.)

$$P^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}$$

Thus

$$P^{-1}(1,2) = \frac{1}{5}\begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1/5 \\ 7/5 \end{pmatrix}$$

It follows that $\vec{e}_1 + 2\vec{e}_2 = -\frac{1}{5}\vec{v} + \frac{7}{5}\vec{w}$.

5.2 The goal of this exercise is to explore linear maps which fix the unit circle U. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map with $\det f > 0$ and $f(U) = U$. Write the matrix of f as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(a) Prove that $a^2 + c^2 = 1$ and $b^2 + d^2 = 1$. [*Hint: Consider* $f(1,0)$ *and* $f(0,1)$.]

Since $(1,0) \in U$ and $f(U) = U$, we see that $(a,c) = f(1,0) \in U$. It follows that $a^2 + c^2 = 1$. Similarly, we have $(b,d) = f(0,1) \in U$, whence $b^2 + d^2 = 1$.

(b) Prove that $a^2 + b^2 = c^2 + d^2$. [*Hint: Consider* $f^{-1}(1,0)$ *and* $f^{-1}(0,1)$.]

Since f is invertible and $f(U) = U$, we see that $f^{-1}(U) = U$. We know that $f^{-1} = \frac{1}{\det f}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ whence $\left(\frac{d}{\det f}, \frac{-c}{\det f}\right) = f^{-1}(1,0) \in U$. It follows that

$$c^2 + d^2 = (\det f)^2.$$

Similarly, we have $\left(\frac{-b}{\det f}, \frac{a}{\det f}\right) = f^{-1}(0,1) \in U$, whence

$$a^2 + b^2 = (\det f)^2 = c^2 + d^2.$$

(c) Prove that $\det f = 1$.

From above, we know that $a^2 + b^2 = (\det f)^2 = c^2 + d^2$. We also know that $a^2 + c^2 = 1 = b^2 + d^2$. From all this we deduce that

$$2(\det f)^2 = a^2 + b^2 + c^2 + d^2 = 2,$$

whence $\det f = \pm 1$. Since we're assuming $\det f > 0$, we conclude that $\det f = 1$.

(d) Prove that $f = R_\theta$ for some θ.

First note that, since $a^2 + c^2 = 1$, there must exist some θ such that $a = \cos\theta$ and $c = \sin\theta$. Thus, to conclude the proof it suffices to show the following:

Lemma 1. *$a = d$ and $b = -c$.*

Proof. From (c), we know that

$$ad - bc = 1. \qquad (\dagger)$$

From our proof of (b) we know that $a^2 + b^2 = (\det f)^2$ and $c^2 + d^2 = (\det f)^2$; plugging in ($\dagger$) shows that $a^2 + b^2 = 1 = c^2 + d^2$. From this it follows that

$$(ad - bc)^2 = (a^2 + b^2)(c^2 + d^2).$$

Expanding both sides and simplifying yields

$$(ac + bd)^2 = 0,$$

whence

$$ac + bd = 0. \qquad (\ddagger)$$

Our strategy is to combine ($\dagger$) and ($\ddagger$) to obtain the claimed relationships between a, b, c, d.

- ($\dagger$) and ($\ddagger$) imply
  $$d(ad - bc) + c(ac + bd) = d;$$
  expanding and simplifying the left side yields $a = d$.
- ($\dagger$) and ($\ddagger$) imply
  $$-c(ad - bc) + d(ac + bd) = -c;$$
  expanding and simplifying the left side yields $b = -c$.

We conclude that $a = d$ and $b = -c$ as claimed. □

(e) Now suppose $g : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map such that $\det g < 0$ and $g(U) = U$. Prove that there exists some angle θ such that $g = R_\theta \rho$, where ρ is the reflection across the horizontal axis. [*Hint: If you use part (d), the proof is quite short.*]

Given g as in the problem, set

$$f := g \circ \rho.$$

Note that

$$\det f = (\det g)(\det \rho) = -\det g > 0.$$

Moreover, $f(U) = U$ (see Lemma below). Part (d) therefore implies that $f = R_\theta$. Thus, since ρ^2 is the identity, we have

$$g = f\rho = R_\theta \rho$$

as claimed. It remains only to prove:

Lemma 2. $f(U) = U$

Proof. It suffices to prove that $\rho(U) = U$, since this would imply

$$f(U) = g\rho(U) = g(U) = U.$$

We will do this by showing that $\rho(U)$ is a subset of U, and also that U is a subset of $\rho(U)$; this is only possible if the two are equal. (This is a common trick for proving two sets are equal.)

Pick any point $(a, b) \in U$, so that $a^2 + b^2 = 1$. Then $\rho(a, b) = (a, -b)$ is in U as well, since $a^2 + (-b)^2 = 1$. This shows that $\rho(U)$ is a subset of U, since every point in $\rho(U)$ lives in U. On the other hand, since $\rho = \rho^{-1}$, this implies $\rho^{-1}(U)$ is a subset of U as well. But then U must be a subset of $\rho(U)$. This completes the proof. □

(f) Suppose a linear map $h : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies $h(U) = U$. Prove that either $h = R_\theta$ or $h = R_\theta \rho$. [*Hint: The proof is short, but there is something to check.*]

Observe that h must be nonsingular (since otherwise the image of h would be contained in some line, contradicting that $h(U) = U$). It follows that $\det h$ is either positive or negative. Part (d) implies that if $\det h > 0$, then $h = R_\theta$. Part (e) implies that if $\det h < 0$, then $h = R_\theta \rho$. □

5.3 Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map with $\det f < 0$. Prove that f admits a singular value decomposition. (State a precise theorem, analogous to Theorem 4 from the lectures 17–18 summary.)

Theorem 3. *Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is nonsingular. Then $\exists \alpha, \beta, k, \ell \in \mathbb{R}$ such that*

$$f = R_\alpha \circ \begin{pmatrix} k & 0 \\ 0 & \ell \end{pmatrix} \circ R_\beta.$$

Proof. In class, we proved this in the case $\det f > 0$, so it remains to prove the case $\det f < 0$. Consider the linear map $f \circ \rho$. We have $\det(f \circ \rho) = (\det f)(\det \rho) > 0$, whence (from class) we deduce the existence of $\alpha, \beta, k, \ell \in \mathbb{R}$ such that

$$f \circ \rho = R_\alpha \circ \begin{pmatrix} k & 0 \\ 0 & \ell \end{pmatrix} \circ R_\beta.$$

Composing both sides on the right by ρ, we find

$$f = R_\alpha \circ \begin{pmatrix} k & 0 \\ 0 & \ell \end{pmatrix} \circ R_\beta \circ \rho.$$

By problem **M1–4(a)** from Midterm 1, we know that

$$R_\beta \rho = \rho R_{-\beta}.$$

Thus,

$$\begin{aligned} f &= R_\alpha \circ \begin{pmatrix} k & 0 \\ 0 & \ell \end{pmatrix} \circ \rho \circ R_{-\beta} \\ &= R_\alpha \circ \begin{pmatrix} k & 0 \\ 0 & -\ell \end{pmatrix} \circ R_{-\beta}. \end{aligned}$$

This concludes the the proof. □

5.4 Let $f := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. The goal of this exercise is to determine and apply the singular value decomposition of f. It turns out the SVD of f is intimately linked to the so-called *golden ratio*:

$$\varphi := \frac{1 + \sqrt{5}}{2}$$

As usual, let U denote the unit circle centered at the origin.

(a) As discussed in class, $f(U)$ is an ellipse centered at the origin. Determine the lengths of the major and minor radii of this ellipse. Express your answer in terms of φ. [*Hint: this is a calculus problem.*]

We are trying to maximize and minimize the magnitude of $f(x)$ where $x \in U$. Any point on U has the form $(\cos\theta, \sin\theta)$ for some θ, so we are trying to find the extreme values of $|f(\cos\theta, \sin\theta)|$. The strategy is to use calculus to determine the θ where these extreme values occur. Set

$$d(\theta) := |f(\cos\theta, \sin\theta)|^2 = (\cos\theta + \sin\theta)^2 + \cos^2\theta.$$

A bit of double-angle formula magic gives

$$d(\theta) = \frac{3}{2} + \sin 2\theta + \frac{1}{2}\cos 2\theta.$$

Note that $\sqrt{d(\theta)}$ is maximized (or minimized) iff $d(\theta)$ is maximized (or minimized). We therefore set out to determine all θ for which $d(\theta)$ has an extreme value. To do this, we find where $d'(\theta) = 0$. We have

$$d'(\theta) = 2\cos 2\theta - \sin 2\theta = 0.$$

This immediately implies that $\cos 2\theta \neq 0$. Dividing both sides by $\cos 2\theta$ and rearranging yields

$$\tan 2\theta = 2.$$

We have therefore found that $|f(\cos\theta, \sin\theta)|$ has an extreme value at $\theta_m = \frac{1}{2}\tan^{-1} 2$. Now that we know $\tan 2\theta_m = 2$, it's fairly straightforward to determine the radii. Simple geometric arguments give

$$\sin 2\theta_m = \frac{2}{\sqrt{5}} \qquad \text{and} \qquad \cos 2\theta_m = \frac{1}{\sqrt{5}}$$

This allows us to compute $d(\theta_m)$; some simple algebra gives

$$d(\theta_m) = \frac{3 + \sqrt{5}}{2}$$

Set

$$\vec{v} := f(\cos\theta_m, \sin\theta_m).$$

Since $\theta_m > 0$, we see that $\vec{v}$ is the radius of the ellipse located in the first quadrant. The magnitude of the radius is

$$|\vec{v}| = \sqrt{d(\theta_m)} = \varphi.$$

Now set

$$\vec{w} := f(\cos(\theta_m + \frac{\pi}{2}), \sin(\theta_m + \frac{\pi}{2}));$$

I claim $\vec{w}$ is perpendicular to $\vec{v}$ (and must therefore be the second radius of the ellipse). To see this, note that

$$\begin{aligned}
\vec{v} \cdot \vec{w} &= f(\cos\theta_m, \sin\theta_m) \cdot f(\cos(\theta_m + \frac{\pi}{2}), \sin(\theta_m + \frac{\pi}{2})) \\
&= f(\cos\theta_m, \sin\theta_m) \cdot f(-\sin\theta_m, \cos\theta_m) \\
&= (\cos\theta_m + \sin\theta_m, \cos\theta_m) \cdot (\cos\theta_m - \sin\theta_m, -\sin\theta_m) \\
&= \frac{1}{2}(\cos 2\theta_m)(2 - \tan 2\theta_m) \\
&= 0.
\end{aligned}$$

A straightforward calculation gives

$$|\vec{w}| = \sqrt{d(\theta_m + \pi/2)} = \varphi - 1.$$

Thus $\vec{v}$ is the major radius (with magnitude φ) and $\vec{w}$ is the minor radius (with magnitude $\varphi - 1$).

(b) Let α denote the tilt of the ellipse $f(U)$, i.e., the angle formed by the positive horizontal axis and the radius of the ellipse in the first quadrant. Prove that $\tan\alpha = \varphi - 1$.

In part (a) we proved that the major radius of the ellipse is

$$\vec{v} = f(\cos\theta_m, \sin\theta_m) = (\cos\theta_m + \sin\theta_m, \cos\theta_m).$$

By the half-angle formula,

$$\tan\theta_m = \frac{\sin 2\theta_m}{1 + \cos 2\theta_m} = \varphi - 1.$$

It follows that

$$\begin{aligned}\tan\alpha &= \frac{\cos\theta_m}{\cos\theta_m + \sin\theta_m} \\ &= \frac{1}{1 + \tan\theta_m} \\ &= \frac{1}{\varphi} \\ &= \varphi - 1.\end{aligned}$$

Note that $\tan\alpha = \tan\theta_m$, whence

$$\alpha = \theta_m.$$

(c) As discussed in lecture, there exists a square grid which gets mapped by f to a rectangular grid. Describe (as precisely as possible) these two grids for $f = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. [*Note:* $\det f < 0$!]

We saw above that $f(\cos\theta_m, \sin\theta_m)$ is the major radius of the ellipse. Let

$$\mathcal{S} := \{(R_{\theta_m}(x\vec{e}_1 + y\vec{e}_2) : x, y \in \mathbb{Z}\}.$$

In other words, $\mathcal{S}$ is the canonical square grid, tilted by an angle of θ_m counterclockwise. Our work above shows that $\mathcal{R} := f(\mathcal{S})$ is the rectangular grid

$$\mathcal{R} = \{x\vec{v} + y\vec{w} : x, y \in \mathbb{Z}\}.$$

By our work above, $\vec{v}$ is the vector of length φ at an angle θ_m, and $\vec{w}$ is a vector perpendicular to $\vec{v}$ of length $\varphi - 1$. In other words, $\mathcal{R}$ is the rectangular grid built out of $\varphi \times (\varphi - 1)$ rectangles, tilted at an angle θ_m.

(d) Determine the singular value decomposition of f, i.e., determine α, β, k, ℓ such that

$$f = R_\alpha \begin{pmatrix} k & 0 \\ 0 & \ell \end{pmatrix} R_\beta.$$

By our work in problem **5.3** and the above parts, we know that

$$f = R_{\theta_m} \begin{pmatrix} \varphi & 0 \\ 0 & 1-\varphi \end{pmatrix} R_\beta$$

for some angle β. To determine this angle β, note that

$$\vec{v} = R_{\theta_m}(\varphi \vec{e}_1) = R_{\theta_m} \begin{pmatrix} \varphi & 0 \\ 0 & 1-\varphi \end{pmatrix} \vec{e}_1$$

On the other hand,

$$f^{-1}(\vec{v}) = R_{\theta_m} \vec{e}_1,$$

whence

$$\vec{v} = f(R_{\theta_m} \vec{e}_1) = R_{\theta_m} \begin{pmatrix} \varphi & 0 \\ 0 & 1-\varphi \end{pmatrix} R_\beta R_{\theta_m} \vec{e}_1$$

Putting the above together gives

$$R_\beta R_{\theta_m} \vec{e}_1 = \vec{e}_1$$

whence $\beta = -\theta_m$. Thus, we conclude that the Singular Value Decomposition of $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = R_{\theta_m} \begin{pmatrix} \varphi & 0 \\ 0 & 1-\varphi \end{pmatrix} R_{-\theta_m}$$

where $\theta_m = \tan^{-1}(\varphi - 1)$.