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MATH 250 : LINEAR ALGEBRA

Problem Set 6 – KEY

6.1 Recall (from Lecture 21) the notion of an *equivalence relation*. Decide whether each of the following binary relations is an equivalence relation. If it is, prove it. If not, give an example of how it fails.

(a) $\sim$ (matrix similarity)

This is an equivalence relation, because it satisfies all three of the defining properties:

- Reflexivity. $f = I^{-1}gI$, where I is the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, whence $f \sim f$.
- Symmetry. Suppose $f \sim g$. Then $\exists P$ such that $f = P^{-1}gP$. But this implies that $g = PfP^{-1} = (P^{-1})^{-1}fP^{-1}$ whence $g \sim f$.
- Transitivity. Suppose $f \sim g$ and $g \sim h$. Then $\exists P, Q$ such that $f = P^{-1}gP$ and $g = Q^{-1}hQ$. It follows that $f = P^{-1}Q^{-1}hQP = (QP)^{-1}h(QP)$, whence $f \sim h$.

(b) $\leq$ (less than or equal to)

This is *not* an equivalence relation. Although it is reflexive and transitive, it is not symmetric: $3 \leq 5$ but $5 \not\leq 3$.

(c) $\approx$ (given two sets $A, B \subseteq \mathbb{Z}$ we write $A \approx B$ if and only if A and B differ by finitely many elements. For example, $\{0, 1, 2, 3, \ldots\} \approx \{1, 2, 3, \ldots\}$ since they differ by one element, while $\{1, 2, 3, 4, \ldots\} \not\approx \{2, 4, 6, \ldots\}$ since they differ by infinitely many elements.)

This is an equivalence relation, since it satisfies the three requisite properties. Prior to verifying these, we introduce a useful notation: let

$$A \setminus B := \{x \in A : x \notin B\}$$

so that $A \approx B$ iff both $A \setminus B$ and $B \setminus A$ are finite.

- $A \approx A$, since $A \setminus A$ is empty (hence finite).
- Suppose $A \approx B$, so that $A \setminus B$ and $B \setminus A$ are both finite. It immediately follows that $B \approx A$.
- Suppose $A \approx B$ and $B \approx C$. Then all of

  $$A \setminus B, \qquad B \setminus A, \qquad B \setminus C, \text{ and } \qquad C \setminus B$$

  are finite. I claim that $A \setminus C \subseteq (A \setminus B) \cup (B \setminus C)$. To see this, pick any $x \in A \setminus C$. If $x \in B$, then $x \in B \setminus C$. If $x \notin B$, then $x \in A \setminus B$. Since both $A \setminus B$ and $B \setminus C$ are finite, we conclude that $A \setminus C$ must be finite. In exactly the same way, we see that $C \setminus A$ must be finite. It follows that $A \approx C$.

6.2 Suppose f and g are nonsingular linear maps from $\mathbb{R}^2 \to \mathbb{R}^2$.

(a) Show by example that fg might not equal gf.

There are many examples. One simple one is $R_{\pi/2}\rho \neq \rho R_{\pi/2}$.

(b) Prove that $fg \sim gf$ (matrix similarity).

We have $fg = g^{-1}(gf)g$.

6.3 Suppose P is a nonsingular linear map, and that $f = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1}$.

(a) Prove that λ_1 and λ_2 are eigenvalues of f.

Let $\vec{v}_1 := P(\vec{e}_1)$, $\vec{v}_2 := P(\vec{e}_2)$. Then

$$f(\vec{v}) = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1} P(\vec{e}_1) = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} (\vec{e}_1) = P(\lambda_1 \vec{e}_1) = \lambda_1 \vec{v}_1.$$

Thus λ_1 was an eigenvalue, and the corresponding eigenvector is $P(\vec{e}_1)$.

(b) Find (with proof) an eigenvector corresponding to λ_1?

We just did this above: it's $P(\vec{e}_1)$.

6.4 For each of the following linear functions, (i) determine all eigenvalues, (ii) for each eigenvalue, find a corresponding eigenvector of unit length, and (iii) if possible, write down a spectral decomposition of f.

(a) $f = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$

(a) $\lambda_1 = 3$, $\lambda_2 = 2$

(b) $\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

(c) $f = B \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} B^{-1}$ with $B = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{5}} \end{pmatrix}$.

(b) $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(a) $\lambda_1 = -1$, $\lambda_2 = 1$

(b) $\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(c) $f = B \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} B^{-1}$ with $B = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$.

(c) $h = \begin{pmatrix} 5 & 2 \\ -2 & 1 \end{pmatrix}$

(a) $\lambda = 3$

(b) $\vec{v} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}$

(c) We have only one eigenvector, so the there is no spectral decomposition.

(d) $k = \begin{pmatrix}3 & 0\\0 & 2\end{pmatrix}$

(a) $\lambda_1 = 3$, $\lambda_2 = 2$

(b) $\vec{v}_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}0\\1\end{pmatrix}$

(c) $f = I\begin{pmatrix}3 & 0\\0 & 2\end{pmatrix}I^{-1}$

(e) f^2, where f is the function from part (a) of this question.

(a) $\lambda_1 = 9$, $\lambda_2 = 4$

(b) $\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$, $\vec{v}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix}$

(c) $f = B\begin{pmatrix}9 & 0\\0 & 4\end{pmatrix}B^{-1}$ where $B = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{5}}\\ \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{5}}\end{pmatrix}$.

6.5 Let f_n denote the n^{th} Fibonacci number (with $f_1 = f_2 = 1$).

(a) Determine $\lim_{n\to\infty} \frac{f_{n+1}}{f_n}$ [*Hint: How big is* $\frac{1-\sqrt{5}}{2}$*?*]

From class we know that $f_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}$, where

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \lambda_2 = \frac{1-\sqrt{5}}{2}$$

It follows that

$$\frac{f_{n+1}}{f_n} = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1^n - \lambda_2^n}$$

Note that $|\lambda_2| < 1$. Thus in the limit $\lambda_2^n \to 0$ as $n \to \infty$, whence

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \lambda_1.$$

(b) Evaluate $f_n^2 + f_{n+1}^2$ for $n = 1, 2, 3, 4$. Conjecture a formula.

2, 5, 13, 34. Conjecture: $f_n^2 + f_{n+1}^2 = f_{2n+1}$ for all $n \geq 1$.

(c) Prove your conjectured formula. [*Hint: consider* $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{2n}$]

Calculate $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{2n}$ in two different ways. On one hand, it's

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{2n} = \begin{pmatrix} f_{2n+1} & f_{2n} \\ f_{2n} & f_{2n-1} \end{pmatrix}$$

On the other it's

$$\left(\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n} \right)^2 = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}^2 = \begin{pmatrix} f_n^2 + f_{n+1}^2 & f_n(f_{n-1} + f_{n+1}) \\ f_n(f_{n-1} + f_{n+1}) & f_n^2 + f_{n-1}^2 \end{pmatrix}$$

Comparing the top left entry in these proves the conjecture. $\square$

6.6 (**Bonus**) Prove that any positive integer can be written as the sum of distinct Fibonacci numbers, no two of which are consecutive. For example, $16 = f_4 + f_7$. (In fact, every positive integer has a unique representation in this form!)

Keep playing around with this when you have a chance!