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MATH 350 : REAL ANALYSIS

Problem Set 11 – due Friday, December 6th

INSTRUCTIONS:

This assignment is due at 4pm on Friday, to be submitted to the mailbox outside my office. (This week there will be no late penalty for submitting Friday.) However, assignments will not be accepted after 4pm on Friday.

Please type up the following problems in LaTeX: (1), (2), 30.8, 33.4, (7), (8)

(0) Read Chapters 26 & 30–33 (pages 81–86 & 102–111).

(1) Give an ϵ-δ proof that $\lim\limits_{x\to 4} \frac{1}{\sqrt{x}} = \frac{1}{2}$. (No algebra of limits allowed!)

(2) Give concrete examples to show that the following definitions of $\lim\limits_{x\to a} f(x) = L$ don't agree with our intuition about limits (i.e. are bad definitions).

(a) For all $\delta > 0$, there exists $\epsilon > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

(b) For all $\epsilon > 0$, there exists $\delta > 0$ such that if $|f(x) - L| < \epsilon$, then $0 < |x - a| < \delta$.

(3) Textbook problems **30.5, 30.8, 33.2, 33.3, 33.4, 33.5**.

(4) In class, Noam asked whether there exists an uncountable subset of $\mathbb{R}$ without accumulation points.

(a) Give an example of a countable subset of $\mathbb{R}$ with no accumulation points. (No proof necessary.)

(b) Give an example of a countable subset of $\mathbb{R}$ with no isolated points. (No proof necessary.)

(c) Suppose $X \subseteq \mathbb{R}$ such that every point in X is isolated. Prove that X must be countable. [*Hint: Construct an injection from X to $\mathbb{Q}$.*]

(5) Suppose $f : [0, 1] \to \mathbb{R}$ is monotone increasing, i.e. that $f(x) \le f(y)$ whenever $x \le y$.

(a) Show that for any $a \in (0, 1)$, $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ both exist.

(b) Let $\mathcal{D}$ denote the set of all points in $[0, 1]$ at which f is discontinuous. Prove that $\mathcal{D}$ is countable.

(6) The goal of this problem is to explore how continuous functions affect topological properties of sets. (I won't define precisely what I mean by *topological*, but highly recommend taking a course on topology.) Recall that if $\mathcal{A}$ is a subset of the domain of a function f, then $f(\mathcal{A}) := \{f(x) : x \in \mathcal{A}\}$.

(a) If f is continuous on a bounded set $\mathcal{B}$, must $f(\mathcal{B})$ be bounded? Prove or give a counterexample.

(b) If f is continuous on a closed interval $\mathcal{C}$, must $f(\mathcal{C})$ be a closed interval? Prove or give a counterexample.

(c) If f is continuous on an open interval $\mathcal{O}$, must $f(\mathcal{O})$ be an open interval? Prove or give a counterexample.

(7) Consider the following:

Claim. *Given $X \subseteq \mathbb{R}$, $(c_n) \subseteq X$ a Cauchy sequence, and $f : X \to \mathbb{R}$ a continuous function on X. Then the sequence $\big(f(c_n)\big)$ is Cauchy.*

"Proof". Given $\epsilon > 0$. Pick any $a \in X$. Because f is continuous at a, there exists $\delta > 0$ such that

$$|x - a| < \delta \quad \Longrightarrow \quad |f(x) - f(a)| < \epsilon.$$

Since (c_n) is Cauchy, $|c_m - c_n| < \delta$ for all large m, n. Thus $|f(c_n) - f(c_m)| < \epsilon$ for all m, n large. □

Find a counterexample to the claim, and carefully identify the mistake in the alleged proof.

(8) The goal of this problem is to explore the *Cantor set*, a remarkable example of set that tests our intuition about real analysis concepts. Let me first describe the Cantor set informally; a formal definition follows. Start with the closed interval $[0, 1]$. Remove the middle third of this interval, leaving $[0, 1/3] \cup [2/3, 1]$. Remove the middle thirds of each of these two intervals, leaving four closed intervals. Remove the middle thirds of each of these four intervals, leaving eight closed intervals. The set $\mathcal{C}$ of all points that remain after doing this "forever" is called the Cantor set.

To do this more formally, we begin with the open interval $\mathcal{O}_1 := (1/3, 2/3)$. Next, for each $n \geq 1$ define

$$\mathcal{O}_{n+1} := \left(\frac{1}{3} \cdot \mathcal{O}_n\right) \cup \left(\frac{2}{3} + \frac{1}{3} \cdot \mathcal{O}_n\right),$$

where $\alpha \cdot X := \{\alpha x : x \in X\}$ and $\beta + Y := \{\beta + y : y \in Y\}$. Finally, set

$$\mathcal{C} := [0, 1] \setminus \left(\bigcup_{n=1}^{\infty} \mathcal{O}_n\right).$$

It immediately follows that $\mathcal{C}$ is closed and bounded, hence that $\mathcal{C}$ is *compact* by the Heine-Borel theorem (which you'll explore in your essay).

(a) Prove that there doesn't exist any nonempty open interval that's a subset of $\mathcal{C}$. (A topologist would say $\mathcal{C}$ has "empty interior".)

(b) Prove that $\mathcal{C}$ has no isolated points.

(c) The set $\bigcup_{n=1}^{\infty} \mathcal{O}_n$ is the union of disjoint open intervals. Prove that the sum of all the lengths of all these intervals is 1. (In other words, $\mathcal{C}$ has zero length!)

(d) (**Optional! and meta-analytic**) Prove that $x \in \mathcal{C}$ iff x has a ternary (i.e. base 3) expansion that doesn't use the digit 1 anywhere.

(e) (**Optional! and meta-analytic**) Prove that $\mathcal{C}$ is uncountable. [*Note that the set of all endpoints of all the closed intervals in the construction of $\mathcal{C}$ is countable!*]

(f) (**Optional! and meta-analytic**) Given sets $\mathcal{A}$ and $\mathcal{B}$ of real numbers, define their sum and difference to be

$$\mathcal{A} + \mathcal{B} := \{a + b : a \in \mathcal{A}, b \in \mathcal{B}\} \qquad \mathcal{A} - \mathcal{B} := \{a - b : a \in \mathcal{A}, b \in \mathcal{B}\}.$$

Prove that $\mathcal{C} + \mathcal{C} = [0, 2]$ and $\mathcal{C} - \mathcal{C} = [-1, 1]$.

Challenge Define a function $f : \mathbb{R} \to \mathbb{R}$ that's not continuous at any point but satisfies the conclusion of the Intermediate Value Theorem.