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MATH 350 : REAL ANALYSIS

Solution Set 10

(1) Use the Cauchy criterion to prove that $\sum\limits_{n=1}^{\infty} \frac{1}{2n-1}$ diverges.

Let
$$S_n := \sum_{k \le n} \frac{1}{2k-1}.$$
Then for all $n \in \mathbb{Z}_{\text{pos}}$,
$$S_{2n} - S_n = \underbrace{\frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{4n-1}}_{n \text{ terms}} \ge \frac{n}{4n-1} > \frac{n}{4n} = \frac{1}{4}$$
In particular, (S_n) isn't Cauchy, hence diverges.

(2) Suppose (a_n) is a monotonically decreasing sequence of positive numbers, and that $\sum\limits_{n=1}^{\infty} a_n$ converges. Prove that $\lim\limits_{n\to\infty} na_n = 0$.

Given $\epsilon > 0$. Let
$$S_N := \sum_{n \le N} a_n;$$
since the sequence S_N converges, we know it must be Cauchy. In particular,
$$|S_{2n} - S_n| < \frac{\epsilon}{10}$$
for all n sufficiently large. On the other hand,
$$|S_{2n} - S_n| = \underbrace{a_{n+1} + a_{n+2} + \cdots + a_{2n}}_{n \text{ terms}}$$
$$\ge na_{2n}.$$
Thus, $na_{2n} < \epsilon/10$ for all large n; in particular,
$$2na_{2n} < \epsilon.$$
Since the sequence is monotonic, $a_{2n+1} \le a_{2n}$, whence for all large n
$$(2n+1)a_{2n+1} \le (2n+1)a_{2n} \le 3na_{2n} < \epsilon.$$
Putting this together, we see that $na_n < \epsilon$ for all sufficiently large n.

(3) In class we proved that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges using the Cauchy criterion. Without using this, prove that the partial sums of this series are all bounded above by 2. (The MCT then implies the convergence of the series.)

A straightforward proof by induction shows that $S_N \leq 2 - \frac{1}{N}$ for all N. The claim follows.

(4) Fix $\sigma > 1$, and consider the series $\sum_{n=1}^{\infty} \frac{1}{n^\sigma}$. Let S_N denote the partial sum $\sum_{n \leq N} \frac{1}{n^\sigma}$.

(a) Prove that $S_{2k+1} \leq 1 + 2^{1-\sigma} S_k$ for all $k \in \mathbb{Z}_{\text{pos}}$.

$$S_{2k+1} = 1 + \underbrace{\frac{1}{2^\sigma} + \frac{1}{3^\sigma}}_{\leq 2/2^\sigma} + \underbrace{\frac{1}{4^\sigma} + \frac{1}{5^\sigma}}_{\leq 2/4^\sigma} + \underbrace{\frac{1}{6^\sigma} + \frac{1}{7^\sigma}}_{\leq 2/6^\sigma} + \cdots + \underbrace{\frac{1}{(2k)^\sigma} + \frac{1}{(2k+1)^\sigma}}_{\leq 2/(2k)^\sigma}$$

$$\leq 1 + \frac{2}{2^\sigma}\left(1 + \frac{1}{2^\sigma} + \frac{1}{3^\sigma} + \cdots + \frac{1}{k^\sigma}\right) = 1 + \frac{S_k}{2^{\sigma-1}}$$

(b) Use part (a) to prove that $\sum_{n=1}^{\infty} \frac{1}{n^\sigma}$ converges.

From part (a) we deduce

$$S_{2k+1} \leq 1 + \frac{S_k}{2^{\sigma-1}} \leq 1 + \frac{S_{2k+1}}{2^{\sigma-1}}$$

It follows that $S_{2k+1} \leq \frac{1}{1-2^{1-\sigma}}$. Since (S_N) is monotonically increasing, $0 \leq S_N \leq \frac{1}{1-2^{1-\sigma}}$ for all $N \in \mathbb{Z}_{\text{pos}}$. Finally, because we're assuming $\sigma > 1$, we see that (S_N) is bounded, hence by the MCT converges.

(5) Let $d(n)$ denote the number of digits of n, e.g., $d(13) = 2$ and $d(5784) = 4$. Prove that $\sum_{n=1}^{\infty} \frac{1}{d(n)}$ diverges.

Observe that $d(n) \leq n$ for all $n \in \mathbb{Z}_{\text{pos}}$, whence $\frac{1}{d(n)} \geq \frac{1}{n}$. Applying the comparison test (see below) with the harmonic series $\sum \frac{1}{n}$, we deduce that the series diverges.

(6) The goal of this exercise is to prove a useful result called the *comparison test*. Throughout, suppose (a_n) and (b_n) are sequences satisfying $0 \leq a_n \leq b_n$ for all $n \in \mathbb{Z}_{\text{pos}}$.

(a) Prove that if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$.

Let $A_N := \sum_{n \leq N} a_n$ and $B_N := \sum_{n \leq N} b_n$. Since the sequence (B_N) converges, it must be bounded, so exists $B \in \mathbb{R}$ such that $B_N \leq B$ for all N. Since $a_n \leq b_n$ for all n, we deduce $A_N \leq B$ for all N. Also, $a_n \geq 0$ for all n, so (A_N) is monotonically increasing. The MCT therefore implies that (A_N) converges.

(b) Prove that if $\sum_{n=1}^{\infty} a_n$ diverges, then so does $\sum_{n=1}^{\infty} b_n$. [*Your proof should be extremely short.*]

This is the contrapositive of part (a).

(7) Suppose $a_n \geq 0$ for all n and $\sum_{n=1}^{\infty} \sqrt{a_n}$ converges. Prove that $\sum_{n=1}^{\infty} a_n$ must also converge.

Since $\sum_{n=1}^{\infty} \sqrt{a_n}$ converges, we know that $\sqrt{a_n} \to 0$, whence $a_n \to 0$. In particular, $a_n < 1$ for all large n; it follows that $a_n < \sqrt{a_n}$ for all large n. The comparison test implies that $\sum_{n=1}^{\infty} a_n$ must converge.

(8) Prove the infinite triangle inequality: if $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|$.

Let $A_N := \sum_{n \leq N} |a_n|$ and $S_N := \sum_{n \leq N} a_n$. By definition of absolute convergence, (A_N) converges, whence it must be Cauchy. In particular, given any $\epsilon > 0$ we know that

$$(\dagger) \qquad |A_M - A_N| \leq \epsilon$$

for all large M and N. I claim this forces (S_N) to be Cauchy as well. To see this, pick any $\epsilon > 0$, and pick any M, N. Without loss of generality, say $M \geq N$. Then

$$|S_M - S_N| = \left|\sum_{n=N+1}^{M} a_n\right| \leq \sum_{n=N+1}^{M} |a_n| = A_M - A_N = |A_M - A_N|.$$

In particular, for all large M, N we have $|S_M - S_N| < \epsilon$, whence (S_N) is Cauchy. Thus (S_N) converges.

The rest of the argument is straightforward: by triangle inequality, $|S_N| \leq A_N$, whence

$$\left|\lim_{N\to\infty} S_N\right| = \lim_{N\to\infty} |S_N| \leq \lim_{N\to\infty} A_N.$$

This is precisely the claimed inequality. □

NOTES. Note that the first half of the argument above is necessary; without it, we don't know whether $\lim_{N\to\infty} |S_N|$ converges!

Also, the natural temptation is to apply triangle inequality directly to bound $\left|\sum_{n=1}^{\infty} a_n\right|$, but this isn't rigorous since the triangle inequality only applies to actual sums of numbers (whereas the series is a limit of some sequence, not an actual sum!).

(9) Consider the function $f : \mathbb{Z}_{\text{pos}} \to \mathbb{R}$ defined by $f(n) := (-1)^n + \sqrt{\lfloor \frac{n+3}{2} \rfloor}$. Does the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{f(n)}$ converge or diverge? Prove it. What does this tell you about the Alternating Series Test?

Let S_N denote the partial sum of the first N terms; then

$$\begin{aligned} S_{2N} &= \frac{1}{\sqrt{2}-1} - \frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{3}-1} - \frac{1}{\sqrt{3}+1} + \cdots + \frac{1}{\sqrt{N+1}-1} - \frac{1}{\sqrt{N+1}+1} \\ &= \frac{2}{1} + \frac{2}{2} + \cdots + \frac{2}{N} = 2\sum_{n \leq N} \frac{1}{n}, \end{aligned}$$

which is twice the partial sum of the harmonic series. *continued on next page...*

Since we proved the harmonic series diverges, its partial sums must diverge, hence the sequence (S_{2N}) diverges. The Cauchy criterion implies (S_{2N}) isn't Cauchy, which means that (S_N) cannot be Cauchy. Thus, (S_N) must diverge.

We deduce that monotonicity is a necessary hypothesis in the alternating series test, since that is the only hypothesis not satisfied by the given series.

NOTES. Note that the other hypothesis of the Alternating Series Test is also necessary: if the terms don't tend to 0 then the series cannot converge!

(10) (Meta-analytic) The goal is to play with Dirichlet's trick for evaluating infinite series.

(a) Show that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots = \frac{\pi}{4}$. [*Hint.* $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$ *and* $\tan \frac{\pi}{4} = 1$.]

Following Dirichlet's trick for the alternating harmonic series, we set

$$F(x) := x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots;$$

we're trying to find $F(1)$. Differentiating, we obtain

$$F'(x) = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots = \frac{1}{1+x^2}.$$

Thus,

$$F(1) = \int_0^1 F'(x)\,dx = \int_0^1 \frac{dx}{1+x^2} = \arctan x \Big|_0^1 = \frac{\pi}{4}$$

(b) Show that $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \frac{1}{10} - \frac{1}{11} + \cdots = \frac{\pi}{3\sqrt{3}}$. [*Hint. Recall that* $1 + x + x^2 = \frac{3}{4} + \left(\frac{1}{2} + x\right)^2$, $\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$, *and* $\tan \frac{\pi}{3} = \sqrt{3}$.]

Set

$$F(x) := x - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^5}{5} + \frac{x^7}{7} - \frac{x^8}{8} + \frac{x^{10}}{10} - \frac{x^{11}}{11} + \cdots$$

so that we're looking for $F(1)$. Differentiating, we obtain

$$\begin{aligned} F'(x) &= 1 - x + x^3 - x^4 + x^6 - x^7 + x^9 - x^{10} + \cdots \\ &= (1-x)(1 + x^3 + x^6 + \cdots) \\ &= \frac{1-x}{1-x^3} = \frac{1}{1+x+x^2} \end{aligned}$$

Thus,

$$\begin{aligned} F(1) &= \int_0^1 F'(x)\,dx = \int_0^1 \frac{dx}{1+x+x^2} = \int_{1/2}^{3/2} \frac{du}{\frac{3}{4}+u^2} = \frac{4}{3}\int_{1/2}^{3/2} \frac{du}{1+\left(\frac{2u}{\sqrt{3}}\right)^2} \\ &= \frac{4}{3} \cdot \frac{\sqrt{3}}{2} \arctan \frac{2u}{\sqrt{3}} \Big|_{1/2}^{3/2} = \frac{2}{\sqrt{3}}\left(\arctan \sqrt{3} - \arctan \frac{1}{\sqrt{3}}\right) = \frac{\pi}{3\sqrt{3}} \end{aligned}$$

(11) (**Just for fun—not for submission**) Find a rearrangement of the alternating harmonic series that converges to 1.

Following the idea from class, we write down consecutive positive terms until the partial sum becomes ≥ 1, then consecutive negative terms until the sum becomes < 1, etc. This process yields

$$1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} + \frac{1}{9} - \frac{1}{6} + \frac{1}{11} + \frac{1}{13} - \frac{1}{8} + \frac{1}{15} + \frac{1}{17} - \frac{1}{10} + \frac{1}{19} + \frac{1}{21} - \frac{1}{12} + \cdots$$

Note that if we ignore the first two terms, a nice pattern emerges. It's a fun challenge to prove that the infinite sum with this pattern does, indeed, converge to 1.