Instructor: Leo Goldmakher

Williams College
Department of Mathematics and Statistics

MATH 350 : REAL ANALYSIS

Solution Set 11

(1) Give an ϵ-δ proof that $\lim_{x\to 4} \frac{1}{\sqrt{x}} = \frac{1}{2}$. (No algebra of limits allowed!)

Given $\epsilon > 0$. Pick any x such that

(♣) $$0 < |x-4| < \min\{\epsilon, 3\}.$$

It follows that $|x-4| < 3$, whence $x > 1$. In particular, $\boxed{\sqrt{x} > 1}$ and $\boxed{\sqrt{x}+2 > 3}$. Thus for any x satisfying (♣), we have

$$\left|\frac{1}{\sqrt{x}} - \frac{1}{2}\right| = \left|\frac{\sqrt{x}-2}{2\sqrt{x}}\right| = \left|\frac{x-4}{2\sqrt{x}(\sqrt{x}+2)}\right| < \frac{\epsilon}{2\cdot 3} < \epsilon.$$

(2) Give concrete examples to show that the following definitions of $\lim_{x\to a} f(x) = L$ don't agree with our intuition about limits (i.e. are bad definitions).

(a) For all $\delta > 0$, there exists $\epsilon > 0$ such that if $0 < |x-a| < \delta$, then $|f(x) - L| < \epsilon$.

Claim. *According to this definition, we have* $\lim_{x\to 0} x = 100$.

Proof. Given $\delta > 0$, I claim $\epsilon = \delta + 100$ does the trick. Indeed, for any x satisfying $0 < |x| < \delta$ we have $-\delta - 100 < x - 100 < \delta - 100$, whence $|x - 100| < \epsilon$. □

(b) For all $\epsilon > 0$, there exists $\delta > 0$ such that if $|f(x) - L| < \epsilon$, then $0 < |x-a| < \delta$.

Claim. *Let f be the constant function $x \mapsto 5$. According to this definition,* $\lim_{x\to 0} f(x) \neq 5$.

Proof. Suppose $\lim_{x\to 0} f(x) = 5$. Let $\epsilon = 1$; the definition furnishes a $\delta > 0$ such that

$$|f(x) - 5| < 1 \quad\Longrightarrow\quad 0 < |x| < \delta.$$

But $x = 2\delta$ doesn't satisfy the latter condition, even though $|f(2\delta) - 5| < 1$. □

JP30.5 Prove that $\lim_{x\to 2} \frac{2}{x} = 1$.

Given $\epsilon > 0$. I claim that $\left|\frac{2}{x} - 1\right| < \epsilon$ for every x satisfying $0 < |x-2| < \min\{\epsilon, 1\}$. Indeed, pick any such x. Then $|x-2| < 1$; in particular, $\boxed{x > 1}$. At the same time, we also know $|x-2| < \epsilon$, whence

$$\left|\frac{2}{x} - 1\right| = \frac{|2-x|}{|x|} < \frac{\epsilon}{|x|} < \epsilon.$$

JP30.8 Suppose $\lim_{x \to a} f(x) = L > 0$. Prove that $\exists \delta > 0$ such that if $0 < |x - a| < \delta$ then $f(x) > 0$.

Note that $L/2 > 0$. Thus, by definition, there exists $\delta > 0$ such that

$$0 < |x-a| < \delta \quad \Longrightarrow \quad |f(x)-L| < \frac{L}{2} \quad \Longrightarrow \quad f(x)-L > -\frac{L}{2} \quad \Longrightarrow \quad f(x) > \frac{L}{2} > 0.$$

JP33.2 Let f be defined on $[0,1]$ by the formula

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that f is continuous at no point in $[0,1]$.

It suffices to prove that $\lim_{x \to a} f(x)$ doesn't exist for any $a \in [0,1]$. To see this, pick an $a \in [0,1]$, and suppose $\lim_{x \to a} f(x) = L$. Then there would exist some $\delta > 0$ such that

$$x \in (a-\delta, a) \cup (a, a+\delta) \quad \Longrightarrow \quad |f(x) - L| < \frac{1}{10}.$$

Since both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$, we can find a rational q and an irrational α such that $q, \alpha \in (a-\delta, a) \cup (a, a+\delta)$. Plugging these into our implication above, we deduce

$$|1 - L| < \frac{1}{10} \qquad \text{and} \qquad |0 - L| < \frac{1}{10}.$$

Triangle inequality implies

$$1 = |1 - L + L| \leq |1 - L| + |L| \frac{1}{10} + \frac{1}{10} = \frac{1}{5},$$

a contradiction. Thus the limit cannot exist anywhere in the interval, whence f cannot be continuous at any point in the interval.

JP33.3 Let f be defined on $[0,1]$ by the formula

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that f is continuous only at 0.

There are two things to prove: that f is continuous at 0, and that f is not continuous anywhere else.

Claim. *f is continuous at 0.*

Proof. Given $\epsilon > 0$. For all x within ϵ of 0—that is, all $x \in [0, \epsilon)$—we have

$$|f(x) - f(0)| = f(x) = \left\{ \begin{matrix} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{matrix} \right\} < \epsilon.$$ □

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Claim. *f isn't continuous at any $a \neq 0$.*

Proof. Suppose f is continuous at $a \neq 0$. Then $a > 0$, whence $\exists \delta > 0$ such that

$$x \in (a-\delta, a+\delta) \cap [0,1] \quad \Longrightarrow \quad |f(x) - f(a)| < \frac{a}{2}.$$

We consider two cases:

- If $a \in \mathbb{Q}$, pick an irrational $x \in (a-\delta, a+\delta) \cap [0,1]$. Then
$$\frac{a}{2} > |f(x) - f(a)| = a.$$
- If $a \notin \mathbb{Q}$, pick a rational $x \in (0.99a, a) \cap (a-\delta, a)$. Then
$$\frac{a}{2} > |f(x) - f(a)| = x > 0.99a.$$

Either way, we've reached a contradiction, whence f cannot be continuous at a. □

JP33.4 Let f be defined on $[0,1]$ by the formula

$$f(x) = \begin{cases} {}^1\!/_n & \text{if } x = {}^m\!/_n \text{ is rational in reduced form} \\ 0 & \text{otherwise.} \end{cases}$$

(By convention, we say the reduced form of 0 is $\frac{0}{1}$.) Prove that f is continuous only at the irrational points in $[0,1]$.

We have two things to prove: that f is continuous at irrationals, and that it's discontinuous at rationals. Intuitively, the latter holds because near any rational are a bunch of irrationals, and the former holds because all the rationals extremely close to an irrational have very large denominator. We make these arguments rigorous below.

Claim. *f is discontinuous at rationals.*

Proof. Pick $a \in \mathbb{Q}$; say, $a = {}^m\!/_n$ in reduced terms. If f were continuous at a, then there would exist $\delta > 0$ such that

$$x \in (a-\delta, a+\delta) \cap [0,1] \quad \Longrightarrow \quad |f(x) - f(a)| < \frac{1}{2n}.$$

Pick any irrational $x \in (a-\delta, a+\delta) \cap [0,1]$. We have

$$\frac{1}{2n} > |f(x) - f(a)| = \frac{1}{n},$$

which is a contradiction. Thus f must be discontinuous at a. □

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Claim. *f is continuous at irrationals.*

Proof. Pick $\alpha \notin \mathbb{Q}$. Our goal is to prove that f is continuous at α. Given $\epsilon > 0$, pick $N \in \mathbb{Z}_{\text{pos}}$ such that $\frac{1}{N} < \epsilon$ (such an N exists by Archimedean property). Let

$$Q_N := \{q \in \mathbb{Q} \cap [0,1] : \exists n \leq N \text{ with } n, nq \in \mathbb{Z}_{\text{pos}}\}$$

denote the set of all fractions in the unit interval with denominator smaller than N. Since Q_N is a finite set, the quantity

$$\delta := \min\{|q - \alpha| : q \in Q_N\}.$$

exists. Moreover, $\delta > 0$, since α is irrational. It follows that every rational number in the open interval $(\alpha - \delta, \alpha + \delta)$ has denominator larger than N. Thus for any $x \in (\alpha - \delta, \alpha + \delta)$ we have

$$|f(x) - f(\alpha)| = f(x) = \begin{Bmatrix} 0 & \text{if } x \notin \mathbb{Q} \\ 1/n & \text{if } x = m/n \end{Bmatrix} < \frac{1}{N} < \epsilon.$$

We conclude that f is continuous at α. □

JP33.5 Suppose that f is continuous at every point of $[a,b]$ and $f(x) = 0$ if x is rational. Prove that $f(x) = 0$ for every x in $[a,b]$.

Pick any irrational $\alpha \in [a,b]$. Since f is continuous at α, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$x \in (\alpha - \delta, \alpha + \delta) \implies |f(x) - f(\alpha)| < \epsilon.$$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, for any $\delta > 0$ there exists a rational $x \in (\alpha - \delta, \alpha + \delta)$. Combining this with the above, we deduce that for any $\epsilon > 0$ we have

$$|f(\alpha)| < \epsilon.$$

The only real number satisfying this for every $\epsilon > 0$ is 0, whence $f(\alpha) = 0$. In other words, f vanishes at all irrationals. Since it also vanishes at all rationals, we conclude that $f(x) = 0$ everywhere.

(4) In class, Noam asked whether there exists an uncountable subset of $\mathbb{R}$ without accumulation points.

(a) Give an example of a countable subset of $\mathbb{R}$ with no accumulation points. (No proof necessary.)

$\mathbb{Z}$

(b) Give an example of a countable subset of $\mathbb{R}$ with no isolated points. (No proof necessary.)

$\mathbb{Q}$

(c) Suppose $X \subseteq \mathbb{R}$ such that every point in X is isolated. Prove that X must be countable. [*Hint: Construct an injection from X to $\mathbb{Q}$.*]

If X is finite, we're done, so we henceforth assume X is infinite. For every $x \in X$, let

$$\Delta_x := \inf\left\{|x - y| : y \in X \setminus \{x\}\right\};$$

this exists since the set is nonempty (X is infinite hence contains points other than x) and is bounded below by 0.

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Observe that $\Delta_x \neq 0$ for any $x \in X$, since the alternative would imply that x is an accumulation point of X. Thus

$$\delta_x := \frac{1}{10}\Delta_x > 0.$$

Lemma 1. *If $x, y \in X$ and $x \neq y$, then $(x - \delta_x, x + \delta_x) \cap (y - \delta_y, y + \delta_y) = \varnothing$.*

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $(x - \delta_x, x + \delta_x)$ must contain some rational number q_x. We therefore have a natural mapping $f : X \to \mathbb{Q}$ defined by $x \mapsto q_x$. Note that this is injective, by the lemma. Since X injects into $\mathbb{Q}$ and $\mathbb{Q}$ is countable, we conclude that X must be countable.

Proof of Lemma. Suppose $\alpha \in (x - \delta_x, x + \delta_x) \cap (y - \delta_y, y + \delta_y)$. Then

$$|x - y| \leq |x - \alpha| + |\alpha - y| < \delta_x + \delta_y = \frac{\Delta_x + \Delta_y}{10} \leq \frac{2|x - y|}{10} < |x - y|,$$

a contradiction. □

(5) Suppose $f : [0, 1] \to \mathbb{R}$ is monotone increasing, i.e. that $f(x) \leq f(y)$ whenever $x \leq y$.

(a) Show that for any $a \in (0, 1)$, $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist.

Pick $a \in (0, 1)$, and set

$$L := \sup \underbrace{\{f(x) : 0 < x < a\}}_{\mathcal{A}}.$$

I claim that $\lim_{x \to a^-} f(x) = L$. Given $\epsilon > 0$. There exists $x_0 < a$ such that $f(x_0) > L - \epsilon$, and for all $x \geq x_0$ we have $f(x) > L - \epsilon$ by monotonicity of f. On the other hand, $f(x) \leq L$ for all $x < a$. Thus, we deduce that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < a - x_0$. A substantially similar argument implies the existence of the right-handed limit.

(b) Let $\mathcal{D}$ denote the set of all points in $[0, 1]$ at which f is discontinuous. Prove that $\mathcal{D}$ is countable.

Given $a \in (0, 1)$ at which f is discontinuous, we know from above that $L_a := \lim_{x \to a^-} f(x)$ and $R_a := \lim_{x \to a^+} f(x)$ both exist. Observe that $L_a \leq f(a) \leq R_a$; since f is discontinuous, we must have $L_a \neq R_a$, whence

$$L_a < R_a.$$

Thus we have a map $f : \mathcal{D} \to \mathcal{P}(\mathbb{R})$ defined $f(a) := (L_a, R_a)$. Moreover, f is injective, since $R_a < L_b$ whenever $a < b$. Since $f(a) \neq \varnothing$ and $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists some rational $q_a \in (L_a, R_a)$. Mapping $a \mapsto q_a$ yields an injection of $\mathcal{D}$ into $\mathbb{Q}$, whence $\mathcal{D}$ must be countable.

(6) The goal of this problem is to explore how continuous functions affect topological properties of sets. (I won't define precisely what I mean by *topological*, but highly recommend taking a course on topology.) Recall that if $\mathcal{A}$ is a subset of the domain of a function f, then $f(\mathcal{A}) := \{f(x) : x \in \mathcal{A}\}$.

(a) If f is continuous on a bounded set $\mathcal{B}$, must $f(\mathcal{B})$ be bounded? Prove or give a counterexample.

No: consider $f : (0, 1) \to \mathbb{R}$ defined by $x \mapsto \frac{1}{x}$.

(b) If f is continuous on a closed interval $\mathcal{C}$, must $f(\mathcal{C})$ be a closed interval? Prove or give a counterexample.

This boils down to the Extreme Value Theorem, which we proved in class.

(c) If f is continuous on an open interval $\mathcal{O}$, must $f(\mathcal{O})$ be an open interval? Prove or give a counterexample.

No: consider the constant function $f : (0, 1) \to \mathbb{R}$ defined by $f : x \mapsto 1$.

(7) Consider the following:

Claim. *Given $X \subseteq \mathbb{R}$, $(c_n) \subseteq X$ a Cauchy sequence, and $f : X \to \mathbb{R}$ a continuous function on X. Then the sequence $\big(f(c_n)\big)$ is Cauchy.*

"Proof". Given $\epsilon > 0$. Pick any $a \in X$. Because f is continuous at a, there exists $\delta > 0$ such that

$$|x - a| < \delta \quad \Longrightarrow \quad |f(x) - f(a)| < \epsilon.$$

Since (c_n) is Cauchy, $|c_m - c_n| < \delta$ for all large m, n. Thus $|f(c_n) - f(c_m)| < \epsilon$ for all m, n large. □

Find a counterexample to the claim, and carefully identify the mistake in the alleged proof.

A counterexample is $f : (0, 1) \to \mathbb{R}$ defined by $x \mapsto \frac{1}{x}$ and $c_n := \frac{1}{n}$.

The issue is that in our definition of continuity, δ depends on ϵ *and* on a.

(8) The goal of this problem is to explore the *Cantor set*, a remarkable example of set that tests our intuition about real analysis concepts. Let me first describe the Cantor set informally; a formal definition follows. Start with the closed interval $[0, 1]$. Remove the middle third of this interval, leaving $[0, 1/3] \cup [2/3, 1]$. Remove the middle thirds of each of these two intervals, leaving four closed intervals. Remove the middle thirds of each of these four intervals, leaving eight closed intervals. The set $\mathcal{C}$ of all points that remain after doing this "forever" is called the Cantor set.

To do this more formally, we begin with the open interval $\mathcal{O}_1 := (1/3, 2/3)$. Next, for each $n \geq 1$ define

$$\mathcal{O}_{n+1} := \left(\frac{1}{3} \cdot \mathcal{O}_n\right) \cup \left(\frac{2}{3} + \frac{1}{3} \cdot \mathcal{O}_n\right),$$

where $\alpha \cdot X := \{\alpha x : x \in X\}$ and $\beta + Y := \{\beta + y : y \in Y\}$. Finally, set

$$\mathcal{C} := [0, 1] \setminus \left(\bigcup_{n=1}^{\infty} \mathcal{O}_n\right).$$

It immediately follows that $\mathcal{C}$ is closed and bounded, hence that $\mathcal{C}$ is *compact* by the Heine-Borel theorem (which you'll explore in your essay).

(a) Prove that there doesn't exist any nonempty open interval that's a subset of $\mathcal{C}$. (A topologist would say $\mathcal{C}$ has "empty interior".)

Let

$$\mathcal{C}_m := [0, 1] \setminus \big(\bigcup_{n \leq m} \mathcal{O}_n\big);$$

by definition of the Cantor set, $\mathcal{C}_m \supseteq \mathcal{C}$ for every N. Note that (by induction) $\mathcal{C}_m$ is the disjoint union of 2^m closed intervals, each of length $1/3^m$.

Pick any point $x \in \operatorname{int}(\mathcal{C})$; by definition, there exists $\epsilon > 0$ such that $\mathcal{B}_\epsilon(x) \subseteq \mathcal{C}$, whence

$$\mathcal{B}_\epsilon(x) \subseteq \mathcal{C}_m$$

for every m. But for sufficiently large m we have $\frac{1}{3^m} < \epsilon$, so $\mathcal{C}_m$ *cannot* contain any interval of length ϵ! We conclude that the interior of $\mathcal{C}$ must be empty.

(b) Prove that $\mathcal{C}$ has no isolated points.

We continue using the notation $\mathcal{C}_m$ defined in the previous solution. Recall that $\mathcal{C}_m$ is the disjoint union of 2^m closed intervals, each of length $1/3^m$; moreover, observe that the endpoint of any one of these closed intervals must live in $\mathcal{C}$. This implies that any point of $\mathcal{C}_m$ is within a distance of $1/3^m$ of some point of $\mathcal{C}$. In particular, for any $p \in \mathcal{C}$ and *any* m, we have that p is within a distance of $\frac{1}{3^m}$ of some other point of $\mathcal{C}$. Since $\frac{1}{3^m}$ can be made arbitrarily small, p cannot be isolated.

(c) The set $\bigcup_{n=1}^{\infty} \mathcal{O}_n$ is the union of disjoint open intervals. Prove that the sum of all the lengths of all these intervals is 1. (In other words, $\mathcal{C}$ has zero length!)

Again we use the notation $\mathcal{C}_m$. Since $\mathcal{C}_m$ is the disjoint union of 2^m closed intervals, each of length $1/3^m$, the total length of $\mathcal{C}_m$ is $(2/3)^m$. Since $\mathcal{C}$ is contained in every $\mathcal{C}_m$, its total length must be smaller than $(2/3)^m$ for every m, which shows that it must have length 0.

[Alternative solution.] The total length of intervals composing $\mathcal{O}_n$ is $\frac{1}{3}(\frac{2}{3})^{n-1}$. Since all the $\mathcal{O}_n$'s are disjoint, the total length is $\sum_{n=1}^{\infty} \frac{1}{3}(\frac{2}{3})^{n-1} = \frac{1/3}{1-2/3} = 1$.

(d) (**Optional! and meta-analytic**) Prove that $x \in \mathcal{C}$ iff x has a ternary (i.e. base 3) expansion that doesn't use the digit 1 anywhere.

First description. The first set we remove, $\mathcal{O}_1$, consists of all numbers with ternary expansion of the form $0.1\cdots$. The next set, $\mathcal{O}_2$, consists of remaining numbers whose second ternary digit is a 1. Similarly, $\mathcal{O}_n$ consists of all numbers between 0 and 1 such that the first $n-1$ ternary digits are exclusively 0 and 2, and the n^{th} ternary digit is 1. It follows that any $x \notin \bigcup_{n\geq 1} \mathcal{O}_n$ has a ternary expansion that uses only 0s and 2s.

Second description. Above we defined $\mathcal{C}_m$ to be the m^{th} stage of forming the Cantor set, where we have created 2^m disjoint closed intervals each of length $1/3^m$. Here we develop a convenient nomenclature for the individual closed intervals composing $\mathcal{C}_m$. We will write

$$\mathcal{C}_m = \bigsqcup_{\ell = m\text{-digit binary number}} I_\ell.$$

Thus

$$\begin{aligned}\mathcal{C}_1 &= I_0 \sqcup I_1 \\ \mathcal{C}_2 &= I_{00} \sqcup I_{01} \sqcup I_{10} \sqcup I_{11} \\ &\vdots\end{aligned}$$

For any closed interval I, let $\alpha(I)$ denote the left endpoint of I and $\beta(I)$ denote the right endpoint, i.e. $I = [\alpha(I), \beta(I)]$. We will now define I_ℓ recursively, as follows.

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First, set $I_0 := [0, 1/3]$ and $I_1 := [2/3, 1]$. Next, given an $(m-1)$-digit binary number ℓ, we will define $I_{\ell 0}$ and $I_{\ell 1}$ in terms of the endpoints of the interval I_ℓ:

$$I_{\ell 0} := [\alpha(I_\ell), \alpha(I_\ell) + 1/3^m]$$
$$I_{\ell 1} := [\beta(I_\ell) - 1/3^m, \beta(I_\ell)]$$

A straightforward induction proves our assertion that $\mathcal{C}_m$ is the disjoint union of the closed intervals I_ℓ over all m-digit binary numbers ℓ.

Finally, observe that any $x \in \mathcal{C}_m$ must live in an interval of the form $I_{d_1 d_2 \cdots d_m}$ with each $d_i = 0$ or 1. A final proof by induction shows that

$$x \in I_{d_1 d_2 \cdots d_m} \qquad \iff \qquad x = 0.e_1 e_2 \cdots e_m \ldots \text{ in ternary,}$$

where $e_i := 2d_i$; in particular, the first m ternary digits of x must be 0 or 2. Since $x \in \mathcal{C}$ requires that $x \in \mathcal{C}_m$ for *every* m, we deduce the claim.

(e) (**Optional! and meta-analytic**) Prove that $\mathcal{C}$ is uncountable. [*Note that the set of all endpoints of all the closed intervals in the construction of* $\mathcal{C}$ *is countable!*]

It suffices to prove that $[0,1] \hookrightarrow \mathcal{C}$. Given $x \in [0,1]$, express it in binary; if there are two options for how to do this, pick the option that doesn't end with a tail of all 0's. (For example, we would express $1/2$ in binary as $0.011111\ldots$ rather than as 0.1.) Now multiply each digit by 2 and interpret the string of digits as a ternary expansion. By the previous part, the resulting number $c(x)$ lives in the Cantor set. It's easily verified that c is an injection, thus proving that $c : [0,1] \hookrightarrow \mathcal{C}$. It follows instantly that $\mathcal{C}$ must be uncountable.

(f) (**Optional! and meta-analytic**) Given sets $\mathcal{A}$ and $\mathcal{B}$ of real numbers, define their sum and difference to be

$$\mathcal{A} + \mathcal{B} := \{a + b : a \in \mathcal{A}, b \in \mathcal{B}\} \qquad \mathcal{A} - \mathcal{B} := \{a - b : a \in \mathcal{A}, b \in \mathcal{B}\}.$$

Prove that $\mathcal{C} + \mathcal{C} = [0,2]$ and $\mathcal{C} - \mathcal{C} = [-1,1]$.

Perhaps the easiest approach is to start by proving

$$\frac{1}{2}\mathcal{C} + \frac{1}{2}\mathcal{C} = [0,1]. \tag{1}$$

The $\subseteq$ containment is obvious. To prove the other direction, pick any $x \in [0,1]$ and write its ternary expansion as

$$x = 0.a_1 a_2 a_3 \cdots$$

We can easily write x as a sum of two ternary numbers $0.b_1 b_2 b_3 \cdots$ and $0.c_1 c_2 c_3 \cdots$, all of whose digits are 0 or 1: if $a_k = 0$, set $b_k = c_k = 0$; if $a_k = 1$, set $b_k = 0$ and $c_k = 1$; if $a_k = 2$, set $b_k = c_k = 1$.

From (1), it's immediate that $\mathcal{C} + \mathcal{C} = [0,2]$. To deduce the second claim, observe that $-\mathcal{C} = \mathcal{C} - 1$, whence

$$\mathcal{C} - \mathcal{C} = \mathcal{C} + \mathcal{C} - 1 = [-1,1].$$

Challenge Define a function $f : \mathbb{R} \to \mathbb{R}$ that's not continuous at any point but satisfies the conclusion of the Intermediate Value Theorem.