EXISTENCE AND UNIQUENESS OF THE FLOOR FUNCTION

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ABSTRACT. A quick proof of the existence and uniqueness of the floor function.

1. INTRODUCING AND MOTIVATING THE FLOOR FUNCTION

In class, we stated the following result:

Proposition 1.1. *Given $x \in \mathbb{R}$, $\exists! N \in \mathbb{Z}$ and $\exists! \alpha \in [0,1)$ such that $x = N + \alpha$.*

Remark. We call N the *floor* of x, denoted $\lfloor x \rfloor$, and α the *fractional part* of x, denoted $\langle x \rangle$. Intuitively, $\lfloor x \rfloor$ is the largest integer $\leq x$. For example:

$$\begin{aligned} \lfloor 10 \rfloor &= 10 & \langle 10 \rangle &= 0 \\ \lfloor \pi \rfloor &= 3 & \langle \pi \rangle &= 0.1415926\ldots \\ \lfloor -\pi \rfloor &= -4 & \langle -\pi \rangle &= 0.8584\ldots \end{aligned}$$

The floor function appears throughout mathematics, and even though conceptually it is rather simple, there are numerous open problems involving them. For example:

Conjecture 1.2 (Akiyama-Brunotte-Pethő-Steiner, 2006)**.** *Suppose (a_n) is a sequence of integers satisfying the recurrence $a_{n+1} = -\lfloor \lambda a_n \rfloor - a_{n-1}$. Then the sequence (a_n) is periodic, for any $\lambda \in [-2, 2]$.*

This is known to hold for $\lambda = 0, \pm 1$, and the golden ratio $\frac{1+\sqrt{5}}{2}$, but all other choices of λ seem to still be open!

Another place the floor function appears is in the study of *generalized polynomials*, i.e. any function built out of polynomials, addition, multiplication, and floor functions. Generalized polynomials play an important role in dynamical systems, number theory, and arithmetic combinatorics, and their behavior is significantly more complicated than that of ordinary polynomials. For example, pick any irrational $\alpha \in (0,1)$ and any $\beta \in \mathbb{R}$, and set $f(x) := \alpha x + \beta$. It can be proved that the generalized polynomial sequence

$$a_n := \lfloor f(n+1) \rfloor - \lfloor f(n) \rfloor$$

(called a *Sturmian sequence*) only outputs 0 and 1; moreover, the proportion of the time it outputs 1 is precisely α. With more care, it's sometimes possible to specify which outputs are 0 and which are 1; for example, in 2023 Byszewski and Konieczny proved that if $\mathcal{A}$ is any subset of the Fibonacci numbers, then there exists a generalized polynomial the outputs 1 at any input from $\mathcal{A}$ and outputs 0 otherwise. By contrast, in 2022 Konieczny proved that there does *not* exist any generalized polynomial that does this for $\mathcal{A}$ being the set of all powers of 2. Broadly, though, generalized polynomials remain poorly understood. For example, here's a question (posed by Konieczny in his paper *Generalised polynomials and integer powers*) that remains unanswered:

Question. Does there exist a generalized polynomial sequence $a_n \in \mathbb{Z}$ that contains an infinite geometric progression but whose image has density 0?

The fractional part function arises in many contexts as well. For example, Euler discovered that

$$\lim_{N \to \infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \log N \right) = 1 - \int_1^\infty \frac{\langle t \rangle}{t^2} dt.$$

Not much is known about the integral on the right hand side. For example, it is widely conjectured (but still unknown) whether or not it's an irrational number.

2. Proof of Proposition 1.1

Having motivated Proposition 1.1, we now prove it. There were several nice approaches proposed in class. Divij suggested looking at the interval $(x-1, x]$ and letting N be the unique integer in this interval. But how do we know that such an integer exists and is unique? Indeed, this is the heart of what we're trying to prove, so this approach is begging the question! Lily suggested instead that we take $N := \sup\{n \in \mathbb{Z} : n \leq x\}$. The issue with this is a notorious pitfall: the supremum of a set *might not live in the set*. In other words, if we define N this way, we'd have to come up with a separate argument that N is an integer. Instead, we will follow a suggestion from William, who used the well-ordering of $\mathbb{Z}_{\text{pos}}$ to construct N.

Proof. We will only prove the claim for $x \geq 1$, leaving the deduction of the full theorem as an exercise. Fix $x \geq 1$, and consider the set

$$\mathcal{A} := \{n \in \mathbb{Z}_{\text{pos}} : n > x\}.$$

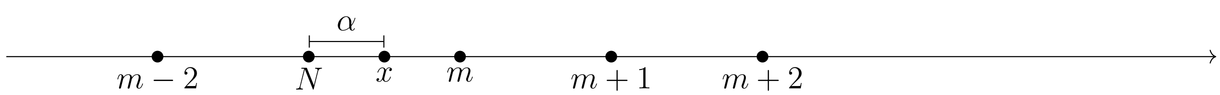
In class we proved:

Lemma 2.1. $\mathcal{A} \neq \varnothing$.

Since $\mathbb{Z}_{\text{pos}}$ is well-ordered, $\mathcal{A}$ must have a least element, say m. Define

$$N := m - 1 \qquad \text{and} \qquad \alpha := x - N.$$

Here's an illustration:



We claim that

(i) $N \in \mathbb{Z}_{\text{pos}}$
(ii) $N \leq x$
(iii) $\alpha \in [0, 1)$

Proof of (i). This almost follows from the following result (proved in Chapter 6 in our textbook):

Lemma 2.2. *If* $m \in \mathbb{Z}_{\text{pos}}, m - 1 \in \mathbb{Z}_{\text{pos}} \cup \{0\}$.

To prove (i), it therefore suffices to show $N \neq 0$. By construction, $m > x$, and by hypothesis, $x \geq 1$. It follows that $m > 1$, whence $N = m - 1 > 0$ so $N \neq 0$. Lemma 2.2 implies $N \in \mathbb{Z}_{\text{pos}}$.

Proof of (ii). Since $N \in \mathbb{Z}_{\text{pos}}$ and $N = m - 1 < m$, we deduce $N \notin \mathcal{A}$. This instantly implies $N \leq x$.

Proof of (iii). By (ii), $\alpha = x - N \geq 0$. On the other hand,

$$\alpha = x - N = \underbrace{x - m}_{<0} + 1 < 1.$$

Putting everything together, we've proved the existence of N and α as in the statement of Proposition 1.1. It remains only to prove uniqueness.

Suppose $x = N + \alpha = M + \beta$, where $M, N \in \mathbb{Z}_{\text{pos}}$ and $\alpha, \beta \in [0, 1)$. Without loss of generality (WLOG), say $M \geq N$. Then

$$M - N = \alpha - \beta < 1.$$

Once again invoking Lemma 2.2, we see that $M - N \in \mathbb{Z}_{\text{pos}} \cup \{0\}$. Since 1 is the least positive integer, we deduce $M - N \notin \mathbb{Z}_{\text{pos}}$, whence $M - N = 0$. This in turn implies $\alpha - \beta = 0$. We've proved that $M = N$ and $\alpha = \beta$, which implies uniqueness! ∎

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