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MATH 350 : REAL ANALYSIS

Midterm Exam – SOLUTIONS

NAME (PRINT): ____

Last First Nickname

INSTRUCTIONS: This exam will take place in two stages.

STAGE I (55 MINUTES). This phase of the exam is a typical in-class exam: you will work individually, with no communication between you and any other person, and without any external aids or reference materials. At the end of this stage, please **remain seated** until all exams have been collected.

STAGE II (20 MINUTES). Find the other members of your assigned group and sit together (and, as much as possible, apart from the other groups); you are allowed to find some space outside of the exam room, if you like. During this phase, your group will be expected to collaborate on Question **5** and submit a single solution as a group. In other words, your group must arrive at a consensus; you may not submit multiple solutions. At the top of your joint submission, please write all of your names (first + last initial).

Your solutions from Stage I will count as 80% of your exam score; your solutions to Stage II will count 20%. However, *under no circumstance will the collaborative grade lower your individual exam score.*

You are allowed to use any result from class or the book without proving it (unless I specify otherwise in the question).

Please sign below the honor code below prior to starting the exam. Do not open the exam until I tell you to do so.

Best of luck!!

I understand that any breach of academic integrity is a violation of the Honor Code. By signing below, I pledge to abide by the Code.

SIGNATURE:____

1. Prove that $\sqrt{5} \in \mathbb{R}$. In other words, prove that there exists $\alpha \geq 0$ such that $\alpha^2 = 5$.

[*You may not assume Theorem 7.5. Scratchwork will not be graded, so please don't include it here; you may use pages 6–8 for this purpose.*]

Define

$$\alpha := \sup \underbrace{\{x \geq 0 : x^2 \leq 5\}}_{\mathcal{A}}.$$

(A13) guarantees that $\alpha \in \mathbb{R}$, since $\mathcal{A} \neq \varnothing$ (e.g. $1 \in \mathcal{A}$) and is bounded above by 3 (if $y \geq 3$ then $y^2 \geq 9 > 5$, so trichotomy implies $y \notin \mathcal{A}$). Moreover, since α is an upper bound of $\mathcal{A}$ and $1 \in \mathcal{A}$, we see that $\alpha \geq 1 > 0$. To prove the claim it suffices to prove

Claim. $\alpha^2 = 5$.

Proof. We'll prove that $\alpha^2 \not< 5$ and that $\alpha^2 \not> 5$; the claim instantly follows by trichotomy.

Suppose $\alpha^2 < 5$. The Archimedean property implies the existence of $n \in \mathbb{Z}_{\text{pos}}$ such that $n > \frac{2\alpha+1}{5-\alpha^2}$. (Note that the right hand side is a real number, since $5 - \alpha^2 \neq 0$ and therefore has a multiplicative inverse.) Since both n and $5 - \alpha^2$ are positive, we can rearrange this inequality to read

$$5 - \alpha^2 > \frac{2\alpha}{n} + \frac{1}{n}.$$

Since $n \in \mathbb{Z}_{\text{pos}}$, we know $n \geq 1$, whence $\frac{1}{n} \geq \frac{1}{n^2}$. It follows that

$$5 - \alpha^2 > \frac{2\alpha}{n} + \frac{1}{n^2} \qquad \Longrightarrow \qquad 5 > \left(\alpha + \frac{1}{n}\right)^2.$$

Since $\alpha + \frac{1}{n} > \alpha > 0$, we deduce that $\alpha + \frac{1}{n} \in \mathcal{A}$. But this contradicts that α is an upper bound of $\mathcal{A}$! We conclude that $\alpha^2 \not< 5$.

Now suppose instead that $\alpha^2 > 5$. The Archimedean property implies the existence of $n \in \mathbb{Z}_{\text{pos}}$ such that $n > \frac{2\alpha}{\alpha^2 - 5}$. Since n and $\alpha^2 - 5$ are both positive, we can rearrange this inequality to read

$$\alpha^2 - 5 > \frac{2\alpha}{n} \geq \frac{2\alpha}{n} - \frac{1}{n^2},$$

whence $(\alpha - \frac{1}{n})^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > 5$. Since $\alpha \geq 1$ and $n \geq 1$, we have $\alpha - \frac{1}{n} \geq 0$. Thus any $y \geq \alpha - \frac{1}{n}$ must satisfy $y^2 \geq (\alpha - \frac{1}{n})^2 > 5$, hence cannot be an element of $\mathcal{A}$. In other words, any element of $\mathcal{A}$ must be bounded above by $\alpha - \frac{1}{n}$. But this contradicts that α is the *least* upper bound! We conclude that $\alpha^2 \not> 5$. $\square$

2. It turns out that the set $\mathbb{F} := \{a, b, c, d, e, f, g, h, i\}$ with operations $+$ and $\times$ defined by the following tables forms a field:

$+$	a	b	c	d	e	f	g	h	i
a	g	d	e	i	h	a	f	c	b
b	d	e	g	h	f	b	i	a	c
c	e	g	d	f	i	c	h	b	a
d	i	h	f	c	a	d	b	g	e
e	h	f	i	a	b	e	c	d	g
f	a	b	c	d	e	f	g	h	i
g	f	i	h	b	c	g	a	e	d
h	c	a	b	g	d	h	e	i	f
i	b	c	a	e	g	i	d	f	h

$\times$	a	b	c	d	e	f	g	h	i
a	c	i	g	a	h	f	d	b	e
b	i	a	e	b	g	f	h	d	c
c	g	e	d	c	b	f	a	i	h
d	a	b	c	d	e	f	g	h	i
e	h	g	b	e	a	f	i	c	d
f	f	f	f	f	f	f	f	f	f
g	d	h	a	g	i	f	c	e	b
h	b	d	i	h	c	f	e	g	a
i	e	c	h	i	d	f	b	a	g

Answer each of the following. For parts (a)-(d) you do **not** need to justify your answer.

(a) What is the additive identity? Circle one: a b c d e f g h i

(b) What is the multiplicative identity? Circle one: a b c d e f g h i

(c) What is the multiplicative inverse of h? Circle one: a b c d e f g h i

(d) What is $g - h$? Circle one: a b c d e f g h i

(e) Is this an ordered field, i.e., does it satisfy (A12)? Justify your answer with a proof. [*For this part only, an answer without justification receives no credit.*]

No. To see this, suppose $\mathbb{F}$ were ordered. In class we proved that in any ordered field the multiplicative identity is positive; it follows that $c = d + d$ is also positive, whence $f = c + d$ is positive. But this contradicts trichotomy, since the additive identity cannot be positive.

3. Let X be a set of real numbers that has a supremum, and let $a := \sup X$. Prove that $X \cap (a - \epsilon, a] \neq \varnothing$ for every $\epsilon > 0$.

SOLUTION 1. Pick $\epsilon > 0$. Since a is the *least* upper bound, $a - \epsilon$ cannot be an upper bound on X. It follows that $\exists x \in X$ with

$$x > a - \epsilon.$$

On the other hand, a is an upper bound of X, so

$$x \leq a.$$

We conclude that $x \in X \cap (a - \epsilon, a]$, so the set cannot be empty.

SOLUTION 2. Suppose there exists some $\epsilon > 0$ such that $X \cap (a - \epsilon, a] = \varnothing$. Since a is an upper bound of X, we have $X \subseteq (-\infty, a]$. By hypothesis, no elements of X live in $(a - \epsilon, a]$, whence $X \subseteq (-\infty, a - \epsilon]$. But then $a - \epsilon$ would be an upper bound of X, contradicting that a is the least upper bound.

4. State our definition of $\mathbb{Z}_{\text{pos}}$, and prove that 1 is the least element of $\mathbb{Z}_{\text{pos}}$.

Our definition $\mathbb{Z}_{\text{pos}}$ is that it's the intersection of all successor sets.

Claim. *1 is the least element of* $\mathbb{Z}_{\text{pos}}$*.*

Proof. Let $a(n)$ be the assertion $n \geq 1$. Clearly $a(1)$ holds. Now suppose $a(n)$ holds for some $n \in \mathbb{Z}_{\text{pos}}$. Since $1 > 0$, we know $n + 1 > n$, whence $n + 1 > n \geq 1$ and $a(n + 1)$ holds. By induction, $a(m)$ must hold for *all* $m \in \mathbb{Z}_{\text{pos}}$, i.e. $m \geq 1$ for all $m \in \mathbb{Z}_{\text{pos}}$. This concludes the proof. $\square$

5. It's common in optimization problems to try to minimize a quantity with respect to one variable, while maximizing it with respect to another. It turns out that one has to be quite careful, however: minimizing first and then maximizing, vs maximizing first and then minimizing, can lead to different results! The goal of this problem is to make the relationship between these two processes precise.

Consider a function $h : A \times B \to \mathbb{R}$, where A, B are nonempty sets and the image of h is bounded above and below. Define functions $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ by

$$f(x) := \sup\{h(x, b) : b \in B\} \qquad \text{and} \qquad g(x) := \inf\{h(a, x) : a \in A\}.$$

Prove that

$$\sup g(B) \leq \inf f(A),$$

where $f(A)$ and $g(B)$ denote the images of f and g, respectively. [*Informally, f is maximizing h with respect to the second variable, while g is minimizing h with respect to the first variable. Thus our claim is (roughly speaking) that* $\sup_b \inf_a h(a, b) \leq \inf_a \sup_b h(a, b)$.]

First observe that (A13) implies that $f(x)$ is well-defined for any choice of $x \in A$, since the set $\{h(x, b) : b \in B\}$ is nonempty and bounded above. Similarly, $g(x)$ is well-defined for any $x \in B$.

Lemma 1. *$g(\beta) \leq f(\alpha)$ for any choices of $\alpha \in A$ and $\beta \in B$.*

Proof. Pick $\alpha \in A$, $\beta \in B$. Since $g(\beta)$ is a lower bound on the set $\{h(a, \beta) : a \in A\}$, we have

$$g(\beta) \leq h(\alpha, \beta).$$

Similarly, since $f(\alpha)$ is an upper bound on the set $\{h(\alpha, b) : b \in B\}$, we deduce

$$f(\alpha) \geq h(\alpha, \beta).$$

Putting this together yields

$$g(\beta) \leq h(\alpha, \beta) \leq f(\alpha),$$

as claimed. $\square$

Thus armed, we can prove the claim. First observe that $g(B)$ is nonempty and (by our lemma) bounded above; (A13) implies $\sup g(B)$ exists. Moreover, since $f(\alpha)$ is an upper bound of $g(B)$ for any $\alpha \in A$, and the supremum is the *least* upper bound, we deduce

$$\sup g(B) \leq f(\alpha) \qquad \forall \alpha \in A.$$

Flipping this around, we see that the real number $\sup g(B)$ is a lower bound on $f(A)$. Since $f(A) \neq \varnothing$, it has an infimum. By definition, the infimum is at least as big as every lower bound of $f(A)$, whence

$$\inf f(A) \geq \sup g(B),$$

as claimed.