Instructor: Leo Goldmakher

Williams College
Department of Mathematics and Statistics

MATH 374 : TOPOLOGY

EXPOSITORY ESSAY

OUTLINE due at 10pm EST on Monday, November 25th
FINAL DRAFT due at 10pm EST on Monday, December 9th

Instructions. The purpose of this assignment is to write a clear, polished, self-contained, and engaging exposition on a topic not covered in our class (to be selected from the list below). Your final draft should read like a chapter from a textbook, written for an audience of your classmates in topology who don't yet know anything about your topic, and should contain all necessary definitions, statements of fundamental properties, relevant intuition and heuristic explanations, complete proofs of main theorems, and proof sketches of auxiliary results. Perhaps most importantly, your essay should motivate the topic. What question is your main theorem answering? What historical problems were people studying that led them to pose that question? What key insights allowed people to prove the main results of your essay? What has been the impact of the work on future research?

Clarity of exposition is paramount in mathematical writing. Here are a few suggestions on how to achieve it:

1. FIND A TOY MODEL. When thinking about the proof of a general theorem, start by trying to go through the proof of a special case. Is there a simplified, more concrete version of theorem whose proof contains all the creative steps of the original, but far fewer technicalities? Perhaps rather than giving the most general proof, you could give the simplified one and then sketch what types of changes must be made for the general case.

2. SEPARATE IDEA FROM TECHNIQUE. Every proof is a combination of creative steps and technical steps. The latter might include unpacking or reinterpreting a definition, a straightforward computation, or even an involved computation whose individual steps are all well-known. On first reading, it can be very difficult to distinguish between the creative steps and the technical ones; as an author, delineating these is one of your primary objectives!

3. SUPPRESS SOME–BUT NOT ALL–DETAILS. The overall conceptual idea of a proof is fundamental to understand (and highlight) as an author. That said, the devil is in the details, and it's very important to work through those as well. Too many details, however, can obscure the overall structure of the argument—the ideal is not to explain every step, but to explain just enough that the idea comes across clearly and that the interested reader can sit down and fill in all the missing details.

Length. Your essay should be at least four pages of text (not counting bibliography), but should not exceed ten pages (single-spaced, 10-12pt font).

Text editor. The assignment should be written using LaTeX, the text editor used by the overwhelming majority of mathematicians (and theoretical physicists, and computer scientists, and anyone who writes technical papers involving math). I've posted instructions on how to set up and use LaTeX on your computer or online, as well as information on tutorials and example files, to the course website. There are also a few dedicated student tutors on campus whom you can get LaTeX assistance from for free; for more information on how to arrange this, please write to me directly.

References. You may freely consult any references you find (textbooks, articles, course notes, blogs, wikipedia, online forums, etc); Schow library has a rich collection of math books. You are expected to explicitly cite sources, in particular including a thorough bibliography. (*The one exception is AI sources; see next paragraph.*) While it's occasionally unavoidable to employ a direct quote, this should be extremely rare—almost all of your descriptions and arguments should be paraphrased, and proofs should be digested and then presented in an order that makes the most sense to you (which is likely a totally different order than given in your source material!).

You are allowed to use ChatGPT and other AI tools, which can be useful as a starting point but are sometimes very misleading (confidently asserting incorrect statements with incorrect attributions). Ultimately, it is your responsibility to verify the correctness of everything you describe in your essay. Using AI to inspire you where to look is fine, but *it cannot be your final resource on any topic*; in particular, you are expected to cite all your other sources, but you may **not** cite any AI.

Deadlines. You should submit an outline (written in LaTeX) by email to me no later than **10pm EST on Monday, November 25th**. The outline should contain a preliminary list of sources, a very rough idea of the introduction, and a sketch of the topics you intend to cover and in which order they might appear. Of course, this is allowed to change as your essay evolves! The essay itself will receive a maximum of a B+, with the rest of the grade on the project determined by the outline. Late outlines will be accepted until 12pm EST on Thursday, November 28th, but will be worth a maximum of 1/3 of a letter grade (as opposed to 2/3).

The deadline for the final draft of the essay is **10pm EST on Monday, December 9th**, to be submitted by email to me. Late submissions will be accepted, but the overall grade will be reduced by 1/3 of a letter grade each subsequent day after the deadline.

Best wishes, and please don't hesitate to reach out with questions!

-Leo

TOPICS

Pick one of the topics below to explore and write about!

- Prove that Brouwer's fixed point theorem in 2 dimensions is equivalent to the Hex theorem (that a game of Hex never ends in a draw). [*In class, we'll give a different proof of Brouwer's fixed point theorem.*]
- Prove that existence of Nash equilibrium is equivalent to Brouwer's fixed point theorem. [*You don't have to prove the latter; see comment above.*]
- Describe some consequences of fixed point theorems for differential equations, e.g. the Cauchy-Peano theorem.
- Prove Urysohn's Lemma and sketch some of its consequences (e.g. Urysohn's metrization theorem or the Tietze Extension theorem).
- Describe the notion of topological dimension, in particular, the Lebesgue covering dimension. Compare to other notions of dimension.
- Prove the notorious Jordan Curve theorem (any non-self-intersecting loop separates the plane into an "inside" and an "outside"—sounds obvious, but is remarkably difficult to prove).
- Give rigorous examples of *space-filling curves*, e.g. the Peano curve and the Lebesgue curve. Also discuss the Hahn-Mazurkiewicz theorem.